Can. J. Math., Vol. XXII, No. 6, 1970, pp. 1168–1175

IDEAL EXTENSIONS OF TOPOLOGICAL SEMIGROUPS

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1. Introduction. In the study of compact semigroups the constructive method rather than the representational method is usually the better plan of attack. As it was pointed out by Hofmann and Mostert in the introduction to their book [**10**] this method is more productive than searching for a representation theory. Hofmann and Mostert described a constructive method called the Hormos and showed that any irreducible compact semigroup is obtained by the Hormos construction. Many of the important examples of irreducible semigroups which motivated their work were obtained by Hunter [**11**; **12**; **13**; **14**].

In this paper, we apply the constructive method of ideal extensions [**5**] in algebraic semigroups to topological semigroups which are not necessarily compact. Many of Hunter's examples and examples of the Hormos technique can also be obtained by our method of topological ideal extensions. The topological ideal extension method, however, is, in general, a different type of construction technique. Cohen and Wade [**6**] have actually implicitly used this technique to describe the structure of a clan with zero on an interval. Clifford [**4**] has used a particular case of the topological ideal extension method to study connected, ordered topological semigroups with idempotent endpoints. For an account of the algebraic theory of ideal extensions see [**5**].

A topological semigroup is a Hausdorff space with a continuous, associative multiplication. The set of idempotents of a topological semigroup S will be denoted by $E(S)$ and the closure of a subset A of a topological space X by $\mathrm{Cl}(A)$. Convergent nets will be denoted by Greek subscripts, e.g. $x_\alpha \to x$ and convergent sequences by italic subscripts, e.g. $y_i \to y$. The term iseomorphism will mean topological isomorphism.

Acknowledgement. I would like to thank P. S. Mostert and K. H. Hofmann for valuable conversations on the topic of extensions and especially for suggesting the possible use of product topologies in this investigation.

2. Topological ideal extensions determined by continuous partial homomorphisms. In the algebraic theory of ideal extensions [**5**] it is shown that a partial homomorphism determines an ideal extension and that every extension of a semigroup with identity is obtained by a partial homomorphism. We now apply these results to topological semigroups.

Received February 3, 1970 and in revised form, July 20, 1970.

(2.1) *Definition.* Let S and T be disjoint topological semigroups, T having a zero 0. A topological semigroup H is a *topological* (*ideal*) *extension* of S by T if and only if H contains an ideal S' which is iseomorphic to S such that H/S' is iseomorphic to T.

(2.2) *Definition.* A *partial homomorphism* [**5**] of a subset B of a semigroup M into a subset C of a semigroup N is a function f of B into C such that if $x, y \in B$ and $xy \in B$, then $f(x)f(y) \in C$ and $f(x)f(y) = f(xy)$.

If T is a topological semigroup with zero 0, let $T^* = T - \{0\}$.

(2.3) THEOREM. *Let S and T be topological semigroups, T having a zero* 0. *Let f: $T^* \to S$ be a continuous partial homomorphism such that* $\mathrm{Cl}(f(T^*))$ *is compact, and let $\bar{T}^* = \{(t, f(t)) : t \in T^*\}$. Then $H = \bar{T}^* \cup (0 \times S)$ is a topological extension of S by T, where H is given the relative product topology and coordinatewise multiplication. Furthermore, H is closed in $T \times S$.*

Proof. It is easy to check that H is a closed subsemigroup of $T \times S$. Also, $S' = 0 \times S$ is iseomorphic to S. To show that H/S' is iseomorphic to T, define a continuous function g: $H/S' \to T$ by $g(t, f(t)) = t$ for $(t, f(t)) \in \bar{T}^*$ and $g(S') = 0$. To prove that g^{-1} is continuous, let $t_\alpha \to t$ in T. If $t \neq 0$, then we can assume that $t_\alpha \neq 0$ for each α and thus

$$g^{-1}(t_\alpha) = (t_\alpha, f(t_\alpha)) \to (t, f(t)) = g^{-1}(t).$$

If $t = 0$ and $\{g^{-1}(t_\alpha)\}$ does not converge to $g^{-1}(0) = S'$, then there exists a subnet $\{t_\beta\}$ of $\{t_\alpha\}$ such that $t_\beta \neq 0$ for each β and no subnet of $\{g^{-1}(t_\beta)\}$ converges to S'. By the compactness of $\mathrm{Cl}(f(T^*))$ there exists a subnet $\{t_\gamma\}$ of $\{t_\beta\}$ such that $f(t_\gamma) \to s$ for some $s \in \mathrm{Cl}(f(T^*))$. Then we have that

$$(t_\gamma, f(t_\gamma)) \to (0, s)$$

in $T \times S$ and hence $(t_\gamma, f(t_\gamma)) \to S'$ in H. This contradiction shows that g^{-1} is continuous, and hence g is an iseomorphism. Therefore H is a topological extension of S by T.

(2.4) *Definition.* A topological extension $(H, \circ)$ of S by T with iseomorphism k: $T \to H/S'$ is a *product extension* of S by T if and only if $H \subseteq T \times S$ with the relative product topology, $S' = 0 \times S$, $p_1 p^{-1}(k(t)) = t$ for each $t \in T^*$, where p_1 is the projection into the first coordinate and p: $H \to H/S'$ is the natural projection, and $(t_1, s_1) \circ (t_2, s_2) = (t_1 t_2, s_1 s_2)$ for $(t_1, s_1), (t_2, s_2) \in H$ with $t_1 t_2 \neq 0$.

The topological extension constructed in (2.3) is a product extension. We have the following converse.

(2.5) THEOREM. *Let S and T be disjoint topological semigroups, S with identity and T with zero* 0. *Then every product extension of S by T is obtained by the process in* (2.3). (*It is not necessary that the associated homomorphism f be such that* $\mathrm{Cl}(f(T^*))$ *is compact.*)

https://doi.org/10.4153/CJM-1970-134-0 Published online by Cambridge University Press

Proof. Let $(H, \circ)$ be a product extension with iseomorphism $k: T \to H/S'$. The notation in (2.4) is used. Let $p_2: H \to H$ be the projection into the second coordinate. Define $f: T^* \to S$ by

$$f(t) = p_2(p^{-1}(k(t)) \circ (0, 1)).$$

We can prove in a straightforward manner that f is a continuous partial homomorphism and determines a topological extension which is $(H, \circ)$. We omit the details except for the following which is typical in proving that the extension determined by f is $(H, \circ)$. If $t \in T^*$, $s \in S$, then we have

$$\begin{aligned}(t, p_2(p^{-1}k(t))) \circ (0, s) &= ((t, p_2(p^{-1}(k(t)))) \circ (0, 1))(0, s)\\ &= (p^{-1}(k(t)) \circ (0, 1))(0, s)\\ &= (t, f(t))(0, s)\\ &= (0, f(t)s).\end{aligned}$$

Some examples are now given which show how a topological extension is obtained by a continuous partial homomorphism. Let R be the reals under addition, R^+ the non-negative reals under addition, $U = R^+ \cup \{\infty\}$ the one-point compactification of R^+ with the binary operation extended to U by defining $a + \infty = \infty + a = \infty$ for $a \in R^+$, and let C be the circle group considered as the boundary of the complex unit disk.

(2.6) *Example.* Let S be any topological semigroup with $E(S)$ non-empty and T any topological semigroup with zero. Fix $e \in E(S)$ and define $f_e: T^* \to S$ by $f_e(t) = e$ for each $t \in T^*$.

(2.7) *Example.* Let S be the non-negative reals under multiplication and T the reals with multiplication defined by

$$t_1 \circ t_2 = \begin{cases} t_1 t_2, & \text{if } t_1, t_2 \geqq 0,\\ 0, & \text{if } t_1 \leqq 0, t_2 \geqq 0 \text{ or } t_1 \geqq 0, t_2 \leqq 0,\\ -t_1 t_2, & \text{if } t_1, t_2 \leqq 0.\end{cases}$$

Define $f: T^* \to S$ by $f(t) = 1$ if $t \in (0, \infty)$ and $f(t) = 0$ if $t \in (-\infty, 0)$.

(2.8) *Example.* Let $T = U$, S a solenoidal group (i.e. compact with dense one-parameter group), and $f: T^* \to S$ a dense one-parameter subsemigroup. Hofmann and Mostert [**10**] have shown that any solenoidal semigroup (i.e. a compact semigroup with a dense one-parameter subsemigroup) is either a solenoidal group or is H/R_r, where $0 < r \leqq \infty$ and $((a_1, b_1)(a_2, b_2)) \in R_r$ if and only if $b_1 = b_2$ and $a_1, a_2 \geqq r$.

(2.9) *Example.* Let $T = U$, S the non-negative reals under multiplication, and let $f: T^* \to S$ be defined by $t \to 1/e^t$, $t \in T^*$.

(2.10) *Example.* Let $T = (U \times C)/(0 \times C)$, $S = C \times R^+$, and let $f: T^* \to S$ be defined by $f(u, c) = \exp(2\pi(-1)^{1/2}u)$. This is the familiar example of a tube winding down on a circle.

We now show how the extension process in (2.3) is useful in characterizing clans with zero on an interval. (A clan is a compact, connected topological semigroup with identity.) All pertinent definitions may be found in [**6**]. Cohen and Wade [**6**] have characterized all clans with zero on an interval. From their work we see that any clan with zero on an interval is obtained from "standard" or familiar clans on an interval by the process in the following theorem.

(2.11) THEOREM. *Let S and T be clans with zeros on intervals $[c, d]$ and $[a, b]$, respectively, such that*

(a) *S and T are matched and*

(b) *either S is full, T is standard, or T is left trivial.*

Define f: $T^* \to S$ by $f(t) = d$ for $t \in (0, b]$ and $f(t) = c$ for $t \in [a, 0)$. Then f is a continuous partial homomorphism and hence determines a topological extension E of S by T. Conversely, every topological extension of S by T which is a clan is determined in this fashion.

3. The translational hull and topological extensions.

The translational hull of a topological semigroup will now be topologized and used to study topological extensions.

The following definitions may be found in [**5**]. A *right translation* of a semigroup S is a transformation r of S such that $r(ab) = a(r(b))$ for all $a, b \in S$. A *left translation* is a transformation w of S such that $w(ab) = (w(a))b$ for all $a, b \in S$. A right translation r and a left translation w are *linked* if $a(w(b)) = (r(a))b$ for all $a, b \in S$. The set of left (right) translations L (P) is a subsemigroup of the full semigroup of translations of S. The set of inner left (right) translations, i.e., the left (right) multiplications by elements of S, is a subsemigroup of L (P). For $s \in S$ let r_s (w_s) denote the right (left) inner translation determined by s. The *translational hull* $\bar{S}$ of a semigroup S is the set of all pairs (w, r) of linked left and right translations w and r of S. A semigroup structure is given to $\bar{S}$ by defining $(w_1, r_1)(w_2, r_2) = (w_1w_2, r_2r_1)$. A *weakly reductive* semigroup is a semigroup S such that $as = bs$ and $sa = sb$ for all $s \in S$ implies that $a = b$.

(3.1) LEMMA [**5**]. *The set $\bar{S}_0 = \{(w_s, r_s): s \in S\}$ is a subsemigroup of $\bar{S}$ and the function $s \to (w_s, r_s)$ is an algebraic isomorphism if and only if S is weakly reductive.*

The next definition is useful in extending Lemma 3.1 to topological semigroups.

(3.2) *Definition.* A topological semigroup S is *net reductive* if and only if $\{s_\alpha\}$ a net in S, $s \in S$, such that $s_\beta' s_\alpha \to s's$ and $s_\alpha s_\beta' \to ss'$ for all $s_\beta' \to s'$ in S imply $s_\alpha \to s$.

A common example of a net reductive semigroup is a topological semigroup with identity. Another common example is provided by the following easily proven result.

(3.3) PROPOSITION. *A compact, weakly reductive topological semigroup S is net reductive.*

Define a topology for the translational hull $\bar{S}$ of a topological semigroup S by $(w_\alpha, r_\alpha) \to (w, r)$ in $\bar{S}$ if and only if $w_\alpha s_\beta \to ws$ and $r_\alpha s_\beta \to rs$ for all $s_\beta \to s$ in S. We assume from now on that S is such that this type of convergence defines a Hausdorff topology.

(3.4) THEOREM. *Let S be a weakly reductive, net reductive topological semigroup. Then the translational hull $\bar{S}$ is a topological semigroup, S is iseomorphic to $\bar{S}_0$, $\bar{S}_0$ is an ideal in $\bar{S}$, and identifying S with $\bar{S}_0$, we have $(w, r)s = ws$ and $s(w, r) = rs$ for all $s \in S$, $(w, r) \in \bar{S}$.*

Proof. It is easy to check that multiplication is continuous in $\bar{S}$. By (3.1), the continuous function $f: S \to \bar{S}$ defined by $f(s) = (w_s, r_s)$ is an algebraic isomorphism into $\bar{S}$. Let $(w_{s_\alpha}, r_{s_\alpha}) \to (w_s, r_s)$ in $\bar{S}_0$. If $s_\beta' \to s'$ in S, then $s_\alpha s_\beta' \to ss'$ and $s_\beta' s_\alpha \to s's$. Now S net reductive implies $s_\alpha \to s$ and hence f^{-1} is continuous. Therefore f is an iseomorphism. The remaining algebraic results have been proved in [**5**, p. 12].

The next result shows how the translational hull is useful in studying topological extensions. If S is weakly reductive and net reductive, then identify S and $\bar{S}_0$. If E is a topological extension of S by T, then in the notation $E = (0 \times S) \cup \{(t, g(t)): t \in T^*\}, g(t)$ means the second coordinate of the element of E in which t occurs. It is not assumed that g is a partial homomorphism.

(3.5) THEOREM. *Let S be a weakly reductive, net reductive topological semigroup such that $\bar{S}$ is compact and let*

$$(E, \circ) = ((0 \times S) \cup \{(t, g(t)): t \in T^*\}, \circ)$$

be a product extension of S by a topological semigroup T with zero 0. *Then there exists a product extension*

$$(\bar{E}, *) = ((0 \times \bar{S}) \cup \{(t, \theta(t)): t \in T^*\}, *)$$

of $\bar{S}$ by T such that

$$(0, s) * (t, \theta(t)) = (0, s) \circ (t, g(t))$$

for $s \in S$, $t \in T^*$ and such that

$$(t_1, \theta(t_1)) * (t_2, \theta(t_2)) = (t_1, g(t_1)) \circ (t_2, g(t_2))$$

for $t_1, t_2 \in T^*$ with $t_1 t_2 = 0$. Conversely, if

$$(\bar{E}, *) = ((0 \times \bar{S}) \cup \{(t, m(t)): t \in T^*\}, *)$$

is a product extension of $\bar{S}$ by T, then

$$(E, *) = ((0 \times \bar{S}_0) \cup \{(t, m(t)): t \in T^*\}, *) = (\bar{E} - (0 \times (\bar{S} - \bar{S}_0)), *)$$

is a product extension of S by T if $(E, *)$ is a subsemigroup of $(\bar{E}, *)$.

Proof. Assume that

$$(E, \circ) = ((0 \times S) \cup \{(t, g(t)): t \in T^*\}, \circ)$$

is a product extension of S by T with iseomorphism $k: T \to E/S'$. Let $i: S \to (0 \times S)$ be the homeomorphism defined by $i(s) = (0, s)$, p_1 and p_2 the projections of $T \times S$ onto the first and second coordinates, respectively, and $p: E \to E/S'$ the natural projection. Define $\theta: T^* \to \bar{S}$ by

$$\theta(t) = (p_2 w_{dk(t)} i, p_2 r_{dk(t)} i)$$

where $d = p^{-1}$, $w_{dk(t)}: 0 \times S \to 0 \times S$ is defined by

$$w_{dk(t)}(0, s) = dk(t) \circ (0, s)$$

and $r_{dk(t)}: 0 \times S \to 0 \times S$ is defined by

$$r_{dk(t)}(0, s) = (0, s) \circ dk(t).$$

One can show that θ is a continuous partial homomorphism. By (2.3), θ determines a product extension

$$(\bar{E}, *) = (0 \times \bar{S}) \cup \{(t, \theta(t)): t \in T^*\}, *)$$

of $\bar{S}$ by T. If $s \in S$ and $t \in T^*$, then

$$\begin{aligned}(0, s) * (t, \theta(t)) &= (0, (w_s, r_s)) * (t, \theta(t)) = (0, (w_s, r_s))(t, (p_2 w_{dk(t)} i, p_2 r_{dk(t)} i)) \\ &= (0, (w_{p_2 r_{dk(t)} i(s)}, r_{p_2 r_{dk(t)} i(s)})) = (0, p_2 r_{dk(t)} i(s)) = (0, p_2((0, s) \circ dk(t))) \\ &= (0, s) \circ dk(t) = (0, s) \circ (t, g(t)).\end{aligned}$$

In a similar fashion one can show that if $t_1, t_2 \in T^*$ such that $t_1 t_2 = 0$, then

$$(t_1, \theta(t_1)) * (t_2, \theta(t_2)) = (t_1, g(t_1)) \circ (t_2, g(t_2)).$$

The converse follows from arguments similar to the ones used in the first part of the theorem.

4. σ-topological ideal extensions. In § 2 it was shown how a continuous partial homomorphism produced a topological extension. This procedure constructed a new topological semigroup from two topological semigroups. This method can be repeated a finite number of times to produce a larger collection of topological semigroups. In this section we explicitly describe how the construction method of (2.3) can be repeated a countable number of times. If S, T, and f are as in (2.3), then we will denote the resulting extension by $H = 0 \times S \cup \{(f(t), t): t \in T^*\}$. This differs from the notation in (2.3) in as far as the first and second coordinates are reversed. Let Z^+ denote the set of positive integers.

For each $i \in Z^+$ let (S, T_i, f_i) be a triple with S a topological semigroup, T_i a topological semigroup with zero 0_i, and f_i a continuous partial homomorphism with

$$f_i: T_i^* \to (S \times 0(i-1, i-1)) \cup ((f_1(T_1^*), T_1^*) \times 0(i-1, i-2)) \cup ((f_2(T_2^*), T_2^*) \times 0(i-1, i-3)) \cup \ldots \cup ((f_{i-2}(T_{i-2}^*), T_{i-2}^*) \times 0(i-1, 1)) \cup (f_{i-1}(T_{i-1}^*), T_{i-1}^*)$$

such that $\mathrm{Cl}(f_i(T_i^*))$ is compact where

$$0(i,j) = (0_{i-j+1}, 0_{i-j+2}, \ldots, 0_i) \qquad \text{for } i \geqq j \geqq 1, \quad i, j \in Z^+,$$

$0(i,j) = \emptyset$ for $j \leqq 0$, $0(0,0) \times S = S$, and

$$(f_i(T_i^*), T_i^*) = \{(f_i(t), t): t \in T_i^*\} \text{ for } i \in Z^+,$$

$(f_i(T_i^*), T_i^*) = \emptyset$ for $i \leqq 0$. Let $E(S, T_i, f_i)$ be the subsemigroup

$$(S, 0_1, 0_2, \ldots) \cup ((f_1(T_1^*), T_1^*), 0_2, 0_3, \ldots) \cup ((f_2(T_2^*), T_2^*), 0_3, 0_4, \ldots) \cup \ldots \cup ((f_i(T_i^*), T_i^*), 0_{i+1}, 0_{i+2}, \ldots)$$

of $S \times (\prod_{i=1}^{\infty} T_i)$. We call $E(S, T_i, f_i)$ the *σ-topological ideal extension* over (S, T_i, f_i).

(4.1) *Example.* Let $S = C$, $T_i = U$ and let f_1 be defined by

$$f_1(u) = \exp(2\pi(-1)^{1/2}u)$$

for $u \in U$, and f_i defined by $f_i(u) = (f_1(u), \infty, \ldots, \infty)$ for $u \in U$, $i = 2, 3, \ldots$.

(4.2) *Example.* Let $S = C$ with identity 1, $T_i = (U \times C)/(\infty \times C)$ for each $i \in Z^+$, and let f_1 be defined by $f_1(u, c) = \exp(2\pi(-1)^{1/2}u)$ for $(u, c) \in T_1^*$, and f_i defined by

$$f_i(u, c) = ((1, 0)_{i-1}, \exp(2\pi(-1)^{1/2}u))$$

where $(u, c) \in T_i^*$ and $(1, 0)_i = (1, 0, 1, 0, \ldots, 1, 0)$ with 1 and 0 each occurring i times. This example resembles a semigroup constructed by Hunter in [**11**].

We can construct the well-known semigroup of an arc winding on the unit disk as a topological extension by a process similar to the one in (2.10). Using this semigroup and constant homomorphisms, we can construct an example of Hunter [**12**, Example 2] which is a compact, connected semigroup with no arc at its identity element.

Using the topological extension method of (2.6) in conjunction with σ-topological ideal extensions, we can produce many of the topological semigroups which are constructed by the idea of tangency.

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