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CHARACTERIZATIONS OF SEMISIMPLE CLASSES

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Abstract

Two characterizations of semisimple classes of associative and alternative rings (and semigroups with 0) are given:
(i) A class is a semisimple class if and only if it is hereditary, closed under extensions and subdirect sums;
(ii) A class is a semisimple class if and only if it is hereditary, closed under extensions, and has the co-inductive property.

The first characterization sharpens Armendariz's (1968) result proved for associative rings, the second one is categorically dual to a characterization of radical classes due to Amitsur (1954).

1. Introduction

The purpose of this paper is to characterize the semisimple classes of associative, alternative rings and semigroups with 0, respectively, as hereditary classes being closed under subdirect sums and extensions. Here the requirement of being subdirectly closed can be replaced by the co-inductive property. Semisimple classes of groups have been characterized similarly by Trần Văn Hạo (1962) as early as 1962. A *semisimple class* is a class consisting of rings (semigroups) having zero radical with respect to an appropriate Kurosh–Amitsur radical. A class **S** is called *hereditary*, whenever $I \triangleleft A \in \mathbf{S}$ implies $I \in \mathbf{S}$, further, **S** is said to be *closed under subdirect sums*, if any subdirect sum of **S**-rings (subdirect product of **S**-semigroups) is again in **S**. We say that **S** is *closed under extensions*, if $B \triangleleft A$, $B \in \mathbf{S}$ and $A/B \in \mathbf{S}$ implies $A \in \mathbf{S}$.

For details of the theory of radical and semisimple classes we refer to Wiegandt (1974). Concerning axiomatizations of semisimple classes of associative rings we refer to the papers Armendariz (1968), Divinsky (1973) and van Leeuwen (to appear). In the proofs we use methods developed by

Anderson, Divinsky and Suliński (1965). In the case of alternative rings the proof is based on a lemma which may be applicable in other contexts, too.

In this paper we shall frequently refer to the following three conditions:

(i) the class **S** is hereditary;

(ii) the class **S** is closed under subdirect sums;

(iii) the class **S** is closed under extensions.

2. Associative rings

In this section we shall prove

THEOREM 1. *A class* **S** *of associative rings is a semisimple class if and only if* **S** *satisfies conditions* (i), (ii) *and* (iii).

PROOF. Armendariz (1968) (see also Wiegandt (1974) Theorem 30.1) has proved that a class **S** is a semisimple class if and only if **S** satisfies conditions (i), (ii), (iii) and

(iv) If $K \lhd I \lhd A$ such that $O \neq I/K \in \mathbf{S}$, then there exists an ideal L of A such that $L \subseteq K$ and $I/L \in \mathbf{S}$.

Hence all that we have to show is that conditions (i), (ii) and (iii) imply condition (iv).

Let I be an ideal of a ring A as demanded in condition (iv), and consider the ideal $J = \bigcap_\alpha (K_\alpha \lhd I \mid I/K_\alpha \in \mathbf{S})$ of I. By the hypothesis $J \neq I$. Since **S** satisfies (ii), it follows that $O \neq I/J \in \mathbf{S}$. We want to show that J is an ideal of the ring A. Having proved this and putting $L = J$ we obtain the requirement of condition (iv).

Choose an element $a \in A$ and define the mapping

$$\varphi : J \to (aJ + J)/J$$

by $\varphi(x) = ax + J$ for all $x \in J$. By Proposition 5.1 of Wiegandt (1974) φ maps J homomorphically onto the ideal $(aJ + J)/J$ of I/J. Since $I/J \in \mathbf{S}$, by condition (i) we get

$$J/\mathrm{Ker}\,\varphi \cong (aJ + J)/J \in \mathbf{S}$$

where $\mathrm{Ker}\,\varphi = \{y \in J \mid ay \in J\}$. We claim that $\mathrm{Ker}\,\varphi$ is an ideal in I. Suppose $y \in \mathrm{Ker}\,\varphi$ and $i \in I$. Then $a(iy) = (ai)y \in J$ and $a(yi) = (ay)i \in J$, since $y \in \mathrm{Ker}\,\varphi$. Since $a(iy), a(yi) \in J$, it follows that $iy, yi \in \mathrm{Ker}\,\varphi$. Thus $\mathrm{Ker}\,\varphi$ is an ideal of I. Now

$$I/\mathrm{Ker}\,\varphi \Big/ J/\mathrm{Ker}\,\varphi \cong I/J \in \mathbf{S}$$

holds and since $J/\mathrm{Ker}\,\varphi \in \mathbf{S}$ condition (iii) implies $I/\mathrm{Ker}\,\varphi \in \mathbf{S}$. Hence $J = \bigcap_\alpha K_\alpha \subseteq \mathrm{Ker}\,\varphi$ and it follows that $(aJ + J)/J \cong J/\mathrm{Ker}\,\varphi = 0$. Thus $aJ \subseteq J$

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for any $a \in A$. The case $Ja \subseteq J$ can be handled analogously and so J is an ideal in A.

Remarks. If the class **S** consists of semiprime rings, then the implication (i), (ii), (iii) $\Rightarrow$ (iv) is nearly trivial.

Conditions (i), (ii) and (iii) are independent as exhibited by the following examples.

1) Consider the class $\mathbf{Z}_p$ of all zero-rings on elementary p-groups (i.e. groups A with $pA = 0$ where p is a fixed prime). Now $\mathbf{Z}_p$ is subring-hereditary (which is much more than (i)), it is subdirectly closed, but it fails to have (iii), since the zero-ring $Z(p^2)$ on the cyclic group of order p^2 is not contained in $\mathbf{Z}_p$.

2) Let **C** denote the class of all rings but a single simple ring A (and all isomorphic copies of A). The class **C** is not hereditary, but satisfies conditions (ii) and (iii).

3) Let **P** be an arbitrary (not necessarily hereditary) radical class. Then **P** is homomorphically closed. If, in addition, **P** is closed under subdirect sums, then **P** is inherited by subrings (cf. Wiegandt (1974) Theorem 31.4), in particular **P** is hereditary. So if a radical class **P** is closed under subdirect sums, then **P** satisfies (i), (ii) and (iii). Hence a radical class **P** is closed under subdirect sums if and only if it is a semisimple class (cf. Armendariz (1968) Theorem 4.5). So any hereditary radical class which is not a semisimple class, satisfies (i) and (iii) but not (ii). In particular, the class **J** of all Jacobson radical rings is such a class, moreover, **J** is homomorphically closed, inherited by one-sided ideals (this is more than (i)) and closed under taking discrete and complete direct sums and even inverse limits (this is definitely less than (ii)). It is worth mentioning that if the Jacobson radical rings are considered as algebras with addition, multiplication and circle operation, then **J** is a variety, so it also satisfies (ii).

3. Alternative rings

A not necessarily associative ring A is said to be an *alternative ring*, if

$$(xx)y = x(xy) \qquad \text{and} \qquad y(xx) = (yx)x$$

for every x and y of A. The *associator* (x, y, z) is defined as

$$(x, y, z) = (xy)z - x(yz)$$

and for alternative rings we have the equalities

(1) $(x, y, z) = (y, z, x) = (z, x, y) = -(x, z, y) = -(z, y, x) = -(y, x, z)$.

We shall use also

(2) $(u, rv, s) + (v, ru, s) = v(u, r, s) + u(v, r, s)$.
Further, if $J \lhd I \lhd A$, then
(3) $A(JJ) \subseteq J$,
(4) $J + aJ \lhd I$ for every $a \in A$,
(5) $(II)(AJ) \subseteq J$,
(6) $[(aJ)(aJ)][(aJ)(aJ)] \subseteq J$ for every $a \in A$.
These assertions have been proved in Anderson, Divinsky and Suliński (1965) Lemma 4.

Our goal is to prove

THEOREM 2. *A class* $\mathbf{S}$ *of alternative rings is a semisimple class if and only if* $\mathbf{S}$ *satisfies conditions* (i), (ii) *and* (iii).

PROOF. Armendariz (1968) has considered only associative rings, so we cannot follow the scheme of the proof of Theorem 1.

Let $\mathbf{S}$ be a semisimple class of a radical class $\mathbf{R}$ of alternative rings. By Corollary 2 of Theorem 2 of Anderson, Divinsky and Suliński (1965) the class $\mathbf{S}$ is hereditary. Let A be a subdirect sum of $\mathbf{S}$-rings. Then A contains ideals A_α such that $A/A_\alpha \in \mathbf{S}$ and $\bigcap_\alpha A_\alpha = 0$. Now $A/A_\alpha \in \mathbf{S}$ implies that $\mathbf{R}(A) \subseteq A_\alpha$. Hence $\mathbf{R}(A) \subseteq \bigcap_\alpha A_\alpha = 0$. Thus $A \in \mathbf{S}$ and (ii) has been established. Consider a ring A such that $A/B \cong C$ and $B, C \in \mathbf{S}$. Again, $A/B \in \mathbf{S}$ implies that $\mathbf{R}(A) \subseteq B$. Since $B \in \mathbf{S}$, it follows that $\mathbf{R}(A) \subseteq \mathbf{R}(B) = 0$. Hence $A \in \mathbf{S}$ and (iii) has been proved.

Conversely, suppose that $\mathbf{S}$ satisfies conditions (i), (ii) and (iii). We claim that $\mathbf{S}$ is the semisimple class $\mathscr{S}\mathscr{U}\mathbf{S}$ of the upper radical class $\mathscr{U}\mathbf{S}$ determined by the hereditary class $\mathbf{S}$. Obviously $\mathbf{S} \subseteq \mathscr{S}\mathscr{U}\mathbf{S}$. All that we have to show is that $\mathscr{S}\mathscr{U}\mathbf{S} \subseteq \mathbf{S}$. To this end define the operator $\mathbf{T}$ associating to any ring A the ideal $\mathbf{T}(A) = \bigcap_\alpha (I_\alpha \lhd A \mid A/I_\alpha \in \mathbf{S})$. By the definition of the upper radical, a ring A is a $\mathscr{U}\mathbf{S}$-ring if and only if $\mathbf{T}(A) = A$. Moreover, it follows from (ii) that $A/\mathbf{T}(A) \in \mathbf{S}$ and similarly that $\mathbf{T}(A)/\mathbf{T}(\mathbf{T}(A)) \in \mathbf{S}$. The crucial point of the proof is to show that $\mathbf{T}(\mathbf{T}(A)) \lhd A$. Having proved this, we can complete the proof as follows. Consider the isomorphism

$$\frac{A/\mathbf{T}(\mathbf{T}(A))}{\mathbf{T}(A)/\mathbf{T}(\mathbf{T}(A))} \cong A/\mathbf{T}(A) \in \mathbf{S}$$

and apply (iii); we get $A/\mathbf{T}(\mathbf{T}(A)) \in \mathbf{S}$. Hence by the definition of $\mathbf{T}(A)$ it follows that $\mathbf{T}(A) \subseteq \mathbf{T}(\mathbf{T}(A))$, whence $\mathbf{T}(A) = \mathbf{T}(\mathbf{T}(A))$. Thus $\mathbf{T}(A)$ is a $\mathscr{U}\mathbf{S}$-ideal of A and consequently $\mathbf{T}(A) \subseteq \mathscr{U}\mathbf{S}(A)$. The inclusion $\mathbf{T}(A) \supseteq \mathscr{U}\mathbf{S}(A)$ is a trivial consequence of the definition of the operator $\mathbf{T}$. Thus we have the equality $\mathbf{T}(A) = \mathscr{U}\mathbf{S}(A)$ for every alternative ring A. In particular, if $A \in \mathscr{S}\mathscr{U}\mathbf{S}$, then $\mathbf{T}(A) = 0$ and (ii) implies that $A \in \mathbf{S}$. Thus $\mathscr{S}\mathscr{U}\mathbf{S} \subseteq \mathbf{S}$.

The proof will be complete if we exhibit $\mathbf{T}(\mathbf{T}(A)) \lhd A$. For this purpose we shall prove the following lemma which may be useful in other contexts too.

LEMMA. *Let* $\mathbf{Q}$ *be a class of alternative rings satisfying conditions* (i) *and* (iii). *Suppose* $I \lhd A$, $J = \bigcap_\alpha (K_\alpha \lhd I \mid I/K_\alpha \in \mathbf{Q})$ *and* $I/J \in \mathbf{Q}$. *Then* $J \lhd A$.

PROOF. If J is not an ideal of A, then there exists an element $a \in A$ such that $aJ \not\subseteq J$ or $Ja \not\subseteq J$. Assume that $aJ \not\subseteq J$. Now, define the mapping

$$\varphi : J \to (aJ + J)/J$$

by $\varphi(y) = ay + J$ for all $y \in J$. We claim that φ maps J homomorphically onto $(aJ + J)J$. Obviously φ is surjective and preserves addition. Applying (3) we get

$$\varphi(y_1y_2) = a(y_1y_2) + J = J$$

for each $y_1, y_2 \in J$. Further

$$\varphi(y_1)\varphi(y_2) = (ay_1)(ay_2) + J.$$

So φ will preserve the multiplication if and only if

$$(*) \qquad (aJ)(aJ) \subseteq J$$

holds. To prove (*), suppose that $(aJ)(aJ) \not\subseteq J$. Then there exists an element $x \in J$ such that $(ax)(aJ) \not\subseteq J$. Define the mapping

$$\psi : J \to [(ax)(aJ) + J]/J$$

by $\psi(z) = (ax)(az) + J$ for all $z \in J$. Clearly ψ preserves addition and maps J onto $[(ax)(aJ) + J]/J$. Further, using (3) we obtain

$$\psi(z_1z_2) = (ax)[a(z_1z_2)] + J = J$$

and applying (6) we get

$$\psi(z_1)\psi(z_2) = [(ax)(az_1)][(ax)(az_2)] + J = J.$$

Thus ψ is a homomorphism. Next, consider

$$\operatorname{Ker} \psi = \{z \in J \mid (ax)(az) \in J\}.$$

Now we are going to show that $\operatorname{Ker}\psi \lhd I$. To do so, take elements $i \in I$, $z \in \operatorname{Ker}\psi$. We have to show that iz, $zi \in \operatorname{Ker}\psi$. $z \in \operatorname{Ker}\psi$ implies that $(ax)(az) \in J$. Firstly we show that $iz \in \operatorname{Ker}\psi$. This means that $(ax)[a(iz)] \in J$. Using the definition of the associator and (1) we have

$$(**)\qquad (ax)[a(iz)] = (ax)[(ai)z - (a,i,z)] = (ax)[(ai)z] - (ax)(a,i,z)$$
$$= [(ax)(ai)]z - (ax, ai, z) + (ax)(i,a,z).$$

Applying (2) we get

$$(ax)(i,a,z) = (ax, ai, z) + (i, a(ax), z) - i(ax, a, z).$$

Substituting this into (**) we obtain

$$(ax)[a(iz)] = [(ax)(ai)]z + (i, a(ax), z) - i(ax, a, z)$$
$$= [(ax)(ai)]z + [i(a(ax))]z - i[(a(ax))z] - i[((ax)a)z - (ax)(az)].$$

Here the first three terms are obviously in J and since $(ax)(az) \in J$ also the last one is in J. Thus $iz \in \operatorname{Ker}\psi$; that is $\operatorname{Ker}\psi$ is a left ideal of I.

Interchanging z and i we obtain

$$(ax)[a(zi)] = [(ax)(az)]i + [z(a(ax))]i$$
$$- z[(a(ax))i] - z[((ax)a)i - (ax)(ai)].$$

The first term of the right hand side is in J since $(ax)(az) \in J$. The other terms are obviously in J. Hence $zi \in \operatorname{Ker}\psi$ and $\operatorname{Ker}\psi$ is a right ideal of I too. This proves that $\operatorname{Ker}\psi \lhd I$. We proceed by showing that $(ax)(aJ) + J \lhd I$. To this end it suffices to prove

$$\text{(a)}\qquad I[(ax)(aJ)] \subseteq (ax)(aJ) + J$$

and

$$\text{(b)}\qquad [(ax)(aJ)]I \subseteq (ax)(aJ) + J.$$

To see (a), apply in order (1), (5) and (4):

$$I[(ax)(aJ)] = [I(ax)](aJ) + (I, ax, aJ) = [I(ax)](aJ) + (ax, I, aJ)$$
$$\subseteq (II)(aJ) + [(ax)I](aJ) + (ax)[I(aJ)]$$
$$\subseteq J + (II)(aJ) + (ax)(aJ + J) \subseteq J + (ax)(aJ) + (ax)J \subseteq (ax)(aJ) + J.$$

To obtain (b), apply (4):

$$[(ax)(aJ)]I = (ax)[(aJ)I] + (ax, aJ, I) \subseteq (ax)(aJ + J) + (ax, I, aJ).$$

Here the first term is in $(ax)(aJ) + J$ and the second one as well (cf. the proof of part (a)). Thus $(ax)(aJ) + J$ is indeed an ideal of I.

Taking into account the hereditariness of $\mathbf{Q}$ and the isomorphism

$$J/\operatorname{Ker}\psi \cong [(ax)(aJ) + J]/J \lhd I/J \in \mathbf{Q}$$

we obtain that $J/\operatorname{Ker}\psi \in \mathbf{Q}$. Furthermore, by (iii) and by the isomorphism

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$$\frac{I/\operatorname{Ker}\psi}{J/\operatorname{Ker}\psi} \cong I/J \in \mathbf{Q}$$

we have $I/\operatorname{Ker}\psi \in \mathbf{Q}$. Hence by the definition of J we obtain that $J \supseteq \operatorname{Ker}\psi \supseteq J$, or $\operatorname{Ker}\psi = J$. Consequently $(ax)(aJ) \subseteq J$. This contradiction proves (*), and hence φ is a homomorphism.

Next, consider

$$\operatorname{Ker}\varphi = \{x \in J \mid ax \in J\}.$$

We exhibit that $\operatorname{Ker}\varphi \lhd I$. Let $i \in I$, $x \in \operatorname{Ker}\varphi$. We claim that ix, $xi \in \operatorname{Ker}\varphi$. We have $ax \in J$ and we have to show that $a(ix)$, $a(xi) \in J$. We get

$$\begin{aligned} a(xi) &= (ax)i - (a, x, i) = (ax)i - (i, a, x) \\ &= (ax)i - (ia)x + i(ax) \in JI + IJ + IJ \subseteq J. \end{aligned}$$

Hence $xi \in \operatorname{Ker}\varphi$ holds. Taking into account $xi \in \operatorname{Ker}\varphi$ we have

$$a(ix) = (ai)x - (a, i, x) \in IJ + J = J.$$

Hence $\operatorname{Ker}\varphi$ is an ideal of I. By (4) we have that $aJ + J \lhd I$ and so

$$J/\operatorname{Ker}\varphi \cong (aJ + J)/J \lhd I/J \in \mathbf{Q}.$$

By condition (i) we get $J/\operatorname{Ker}\varphi \in \mathbf{Q}$. Considering the isomorphism

$$\frac{I/\operatorname{Ker}\varphi}{J/\operatorname{Ker}\varphi} \cong I/J \in \mathbf{Q}$$

condition (iii) is applicable which yields $I/\operatorname{Ker}\varphi \in \mathbf{Q}$. Consequently $\operatorname{Ker}\varphi \supseteq J$, that is $aJ \subseteq J$ (a contradiction), so J is a left ideal of A.

By similar arguments we get that $Ja \subseteq J$. Thus J is a two-sided ideal of A and the Lemma is proved.

Putting $I = \mathbf{T}(A)$ and $J = \mathbf{T}(\mathbf{T}(A))$ the Lemma yields the desired relation $\mathbf{T}(\mathbf{T}(A)) \lhd A$. Thus the proof of Theorem 2 has been completed.

REMARK. Theorem 2 cannot be extended to arbitrary not necessarily associative rings. Leavitt and Armendariz (1967) have given examples for non-hereditary semisimple classes in the class of all not necessarily associative rings.

4. Semigroups with 0

One possible way of defining Kurosh–Amitsur radicals for semigroups with 0, is the following. A class $\mathbf{R}$ of semigroups with 0 is called a radical class if

(I) $\mathbf{R}$ is closed under Rees factor semigroups;
(II) for any semigroup A, $\mathbf{R}(A) = \bigcup_\alpha (I_\alpha \lhd A \mid I_\alpha \in \mathbf{R}) \in \mathbf{R}$;
(III) $\mathbf{R}(A/\mathbf{R}(A)) = 0$ where $A/\mathbf{R}(A)$ denotes the Rees factor semigroup of A by $\mathbf{R}(A)$.

Investigations concerning such radicals can be found in Eqbal Ahmed and Wiegandt (1973), Grigor (1971 and 1973) and Wiegandt (1972). An analogous proof to that of Theorem 1 (considering Rees factor semigroups, 0-disjoint unions and subdirect products for homomorphic images, sums and subdirect sums, respectively) yields the following statement:

A class $\mathbf{S}$ *of semigroups with* 0 *is a semisimple class if and only if* $\mathbf{S}$ *is hereditary, closed under subdirect products and closed under extensions.*

5. Characterization by co-inductive property

We say that a class $\mathbf{S}$ of associative or alternative rings (semigroups with 0) has the *co-inductive property*, if $\mathbf{S}$ satisfies condition

(iv) if $B_1 \supseteq B_2 \supseteq \cdots B_\gamma \supseteq \cdots$ is a descending chain of ideals of any ring (semigroup) A such that $A/B_\gamma \in \mathbf{S}$ for all γ and $B = \bigcap_\gamma B_\gamma$, then also $A/B \in \mathbf{S}$.

THEOREM 3. *A class* $\mathbf{S}$ *of associative or alternative rings* (*semigroups*) *is a semisimple class if and only if conditions* (i), (iii), *and* (iv) *are satisfied.*

PROOF. Since any semisimple class satisfies (i) and (iii), we only show that $\mathbf{S}$ satisfies (iv), if $\mathbf{S}$ is a semisimple class. Let I/B denote the $\mathscr{U}\mathbf{S}$-radical of A/B. We have

$$\frac{A/B}{B_\gamma/B} \cong A/B_\gamma \in \mathbf{S}$$

for all γ, whence $B_\gamma/B \supseteq I/B$. Hence $B_\gamma \supseteq I$ for all γ implying $B \supseteq I$. Consequently $B = I$. Thus $A/B \in \mathbf{S}$.

Now, suppose that $\mathbf{S}$ satisfies (i), (iii) and (iv). Taking into account our previous results, it suffices to show that $\mathbf{S}$ satisfies condition (ii). In fact, $\mathbf{S}$ will be proved to satisfy the following stronger condition:
(v) for any ring (semigroup) A, $A/\bigcap_\alpha B_\alpha \in \mathbf{S}$ holds where B_α runs through all ideals of A satisfying $A/B_\alpha \in \mathbf{S}$.
From (v) it readily follows that $\mathbf{S}$ is closed under subdirect sums (products). It follows from (iv) and Zorn's lemma that any ring (semigroup) A contains an ideal B such that $A/B \in \mathbf{S}$ and which is minimal with respect to this property. Up till now B is not necessarily unique, but we proceed by showing that

actually this is the case. Let C be any ideal of A such that $A/C \in \mathbf{S}$. Consider the isomorphism

$$A/B \cong \frac{A/(B \cap C)}{B/(B \cap C)}.$$

We have

$$B/(B \cap C) \cong (B + C)/C \lhd A/C.$$

Since $A/C \in \mathbf{S}$ and $\mathbf{S}$ is hereditary, we conclude $B/(B \cap C) \in \mathbf{S}$. Also $A/B \in \mathbf{S}$. Hence by using condition (iii) we obtain that $A/(B \cap C) \in \mathbf{S}$. By the minimality of B this implies that $B \subseteq B \cap C$, whence $B \subseteq C$. This proves that B is the smallest ideal of A satisfying $A/B \in \mathbf{S}$. Consequently the intersection $\bigcap_\alpha B_\alpha$ equals B. So $\mathbf{S}$ satisfies (v) and the proof is complete.

Let us remark that the statement of Theorem 3 is dual to the following characterization of radical classes due to Amitsur (1954):

A class $\mathbf{R}$ of rings is a radical class if and only if

(i*) $\mathbf{R}$ is homomorphically closed,

(iii*) $\mathbf{R}$ is closed under extensions,

(iv*) $\mathbf{R}$ has the inductive property: if $B_1 \subseteq B_2 \subseteq \cdots \subseteq B_\gamma \subseteq \cdots$ is a chain of $\mathbf{R}$-ideals of any ring A, then $\bigcup_\gamma B_\gamma \in \mathbf{R}$.

Finally we present several characterizations of semisimple classes. Recall that a subring (subsemigroup) B of a ring (semigroup) A is said to be an *accessible subring* (*subsemigroup*), if there are finitely many subrings (subsemigroups) $B_1, \cdots, B_n$ of A such that $B = B_1 \lhd B_2 \lhd \cdots \lhd B_n = A$.

For a class $\mathbf{S}$ *of associative or alternative rings* (*semigroups with* 0) *the following conditions* (I), (II), (III), (IV) *and* (V) *are equivalent*:

(I) $\mathbf{S}$ *is a semisimple class of an appropriate radical class*;

(II) $\mathbf{S}$ *satisfies*

(A) *If* $A \in \mathbf{S}$, *then every non-zero ideal of* A *has a non-zero homomorphic image* (*Rees factor semigroup*) *in* $\mathbf{S}$,

(B) *If every non-zero ideal of* A *has a non-zero homomorphic image* (*Rees factor semigroup*) *in* $\mathbf{S}$, *then* $A \in \mathbf{S}$;

(III) $\mathbf{S}$ *satisfies*

(A_0) *If* $A \in \mathbf{S}$, *then every non-zero accessible subring* (*subsemigroup*) *of* A *has a non-zero homomorphic image* (*Rees factor semigroup*) *in* $\mathbf{S}$,

(B_0) *If every non-zero accessible subring* (*subsemigroup*) *of* A *has a non-zero homomorphic image* (*Rees factor semigroup*) *in* $\mathbf{S}$, *then* $A \in \mathbf{S}$;

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(IV) **S** *is a hereditary class closed under subdirect sums (products) and extensions*;

(V) **S** *is a hereditary, co-inductive class being closed under extensions.*

The equivalence of (I) and (II) is well known. Divinsky (1973) has proved the equivalence of (II) and (III) for associative rings. Though in the lecture notes by Wiegandt (1974) only associative rings are considered, there the proof of the equivalence of (II) and (III) is based on methods of Leavitt and Yu-lee Lee (1969), Yu-lee Lee (1969) and Watters (1969) and it works for alternative rings too. For semigroups with 0 the proofs are analogous to those for associative rings. Finally, the equivalence of (I), (IV) and (V) has been established in this paper.

Added in Proof. For associative rings A. D. Sands has proved the same statement in his paper "Strong upper radicals", *Quart. J. Math. Oxford*, **27** (1976), 21–24.

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