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FRACTAL n-HEDRAL TILINGS OF $\mathbb{R}^d$

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Abstract

An n-hedral tiling of $\mathbb{R}^d$ is a tiling with each tile congruent to one of the n distinct sets. In this paper, we use the iterated function systems (IFS) to generate n-hedral tilings of $\mathbb{R}^d$. Each tile in the tiling is similar to the attractor of the IFS. These tiles are fractals and their boundaries have the Hausdorff dimension less than d. Our results generalize a result of Bandt.

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1. Introduction

Denote all positive integers by $\mathbb{N}$. Let $\mathscr{T} = \{T_i : i \in \mathbb{N}\}$ be a family of closed sets in $\mathbb{R}^d$. If $\bigcup_{i\in\mathbb{N}} T_i = \mathbb{R}^d$ and the interiors of the sets T_i are pairwise disjoint, then $\mathscr{T}$ is called a *tiling* of $\mathbb{R}^d$ and T_i are called the *tiles* of $\mathscr{T}$. If every tile in $\mathscr{T}$ is congruent to one fixed T, then T is called the *prototile* of $\mathscr{T}$ and $\mathscr{T}$ is a *monohedral tiling*. In general, if there are n distinct tiles $T_{i_1}, \dots, T_{i_n}$ such that every tile in $\mathscr{T}$ is congruent to one of them, then $\mathscr{T}$ is called an *n-hedral tiling* and $T_{i_1}, \dots, T_{i_n}$ are *n prototiles* of $\mathscr{T}$. We also say that $\{T_{i_1}, \dots, T_{i_n}\}$ *admits* the tiling $\mathscr{T}$. For more concepts about tilings, see Grünbaum and Shephard [8].

We use $\mathbb{M}_d(\mathbb{E})$ to denote all the $d \times d$ matrices whose entries are in the space $\mathbb{E}$. A matrix $B \in \mathbb{M}_d(\mathbb{R})$ is *expanding* if all of its eigenvalues satisfy $|\lambda_i| > 1$. A *lattice* in $\mathbb{R}^d$ can be defined as the set $L := \{(n_1, \dots, n_d)(\mathbf{v}_1, \dots, \mathbf{v}_d)' : n_i \in \mathbb{Z}, 1 \le i \le d\}$, where $\mathbf{v}_1, \dots, \mathbf{v}_d$ are d linearly independent vectors in $\mathbb{R}^d$ and $(\cdot)'$ denotes the transposition. Throughout this paper, we will use $\mathscr{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_d\}$ as the base for $\mathbb{R}^d$. For a set $K \subset \mathbb{R}^d$, its Lebesgue measure is denoted by $\mu(K)$. The interior is K°, and the

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boundary is $\partial K = K \backslash K^\circ$. Let $B \in \mathbb{M}_d(\mathbb{R})$ be expanding with $|\det B| = N$. Let $\mathscr{D} = \{\mathbf{d}_1, \ldots, \mathbf{d}_N\} \subset \mathbb{R}^d$. Then there exists a unique non-empty compact set K such that $K = \bigcup_{i=1}^N B^{-1}(K + \mathbf{d}_i)$. In fact, K is the attractor of the iterated function system (IFS) $\{f_i : f_i(\mathbf{x}) = B^{-1}(\mathbf{x} + \mathbf{d}_i)\}_{i=1}^N$. If $\mu(K) > 0$, then K is a *self-affine tile*. It is well known that $\mu(K) > 0$ is equivalent to $K^\circ \neq \emptyset$, and such K can tile $\mathbb{R}^d$ by translation ([11, 13]). In case that $B \in \mathbb{M}_d(\mathbb{Z})$ and $\mathscr{D} \subset L$, Bandt [1] gave a sufficient condition for $K^\circ \neq \emptyset$:

$$L = \bigcup_{i=1}^{N} (\mathbf{d}_i + B(L)). \tag{1.1}$$

Notice that the conditions $B \in \mathbb{M}_d(\mathbb{Z})$ and $\mathscr{D} \subset L$ are necessary for (1.1) to hold. If rotations and reflections of a tile in the tiling are considered, then more interesting tiles can be generated. Let $B \in \mathbb{M}_d(\mathbb{Z})$ and $\mathbb{W} \subset \mathbb{M}_d(\mathbb{Z})$ be a finite group with determinants ± 1. If $B\mathbb{W} = \mathbb{W}B$, then $\mathbb{W}$ is called a *symmetry group* of B. Bandt [1] proved that if B is expanding with $|\det B| = N$, and

$$L = \bigcup_{i=1}^{N} w_i^{-1} (\mathbf{d}_i + B(L)) \text{ with } w_i \in \mathbb{W}, \tag{1.2}$$

then the attractor K of $\{f_i : f_i(\mathbf{x}) = w_i B^{-1}(\mathbf{x}) + \mathbf{d}_i\}_{i=1}^N$ has non-empty interior. Note that if $\widetilde{K}$ is the attractor of $\{f_i : f_i(\mathbf{x}) = (w_i^{-1} B)^{-1}(\mathbf{x} + w_i^{-1}\mathbf{d}_i)\}_{i=1}^N$, then $K = B\widetilde{K}$. A lot of work has been done about self-affine tiles ([1–3, 11, 13, 19, 20]). However, not much is known about the attractor of the IFS involving different matrices.

Here we prove the following generalization of Bandt's result:

THEOREM 1.1. *Let $B \in \mathbb{M}_d(\mathbb{Z})$ be expanding and $\mathbb{W} \subset \mathbb{M}_d(\mathbb{Z})$ be a symmetry group of B. Let $\{\mathbf{d}_1, \ldots, \mathbf{d}_N\} \subset L$. If $B_i = w_i B^{m_i}$ with $w_i \in \mathbb{W}$, $m_i \in \mathbb{N}$, $i = 1, \ldots, N$, and $L = \bigcup_{i=1}^N (\mathbf{d}_i + B_i(L))$, then the attractor K of $\{f_i\}_{i=1}^N$ has non-empty interior, where $f_i(\mathbf{x}) = B_i^{-1}(\mathbf{x} + \mathbf{d}_i)$.*

In the above theorem, the case $\sum_{i=1}^N |\det B_i|^{-1} = 1$ is of particular interest because the attractor K is then essentially non-overlapping, that is, the intersection of $f_i(K)$ and $f_j(K)$ has Lebesgue measure zero if $i \neq j$. We show that in such a case the Hausdorff dimension of the boundary of K is strictly less than d. Indeed we prove the following more general result:

THEOREM 1.2. *Let $\{f_i : f_i(\mathbf{x}) = A_i\mathbf{x} + \mathbf{a}_i, A_i \in \mathbb{M}_d(\mathbb{R}), \mathbf{a}_i \in \mathbb{R}^d\}_{i=1}^N$ be a family of functions which satisfies the weak contractivity condition. If the attractor K has non-empty interior and $\sum_{i=1}^N |\det A_i| = 1$, then the Hausdorff dimension of ∂K is strictly less than d.*

The concept of weak contractivity condition in the above theorem is defined in Section 2.

Though the self-affine tile generated by the IFS involving one matrix admits a monohedral tiling of $\mathbb{R}^d$, if we allow different expanding matrices in IFS, then in general the attractor cannot tile $\mathbb{R}^d$ as a single prototile. In Section 6, we prove that for the IFS $\{f_i : f_i(\mathbf{x}) = A_i\mathbf{x} + \mathbf{a}_i, A_i \in \mathbb{M}_d(\mathbb{R}), \mathbf{a}_i \in \mathbb{R}^d\}_{i=1}^N$, if the attractor K has non-empty interior and $\sum_{i=1}^N |\det A_i| = 1$, then $\mathbb{R}^d$ can be tiled by affine copies of K of approximately the same size, where an affine copy of K is an image of K under an affine transformation. For the special case in Theorem 1.1 with $\sum_{i=1}^N |\det B_i|^{-1} = 1$, we show that the attractor K gives rise to an n-hedral tiling of $\mathbb{R}^d$ for some positive integer n.

Some other methods can also generate tilings by tiles of the same shape but different sizes. See [6, 8, 10, 12, 15].

We prove Theorem 1.1 and Theorem 1.2 in Section 3. In Section 4, we consider the open set condition related to the function systems in Theorem 1.1. In Section 5, we give some examples to show how to generate fractal tiles from Theorem 1.1.

2. Notations and preliminaries

Let $K \subset \mathbb{R}^d$. The Hausdorff dimension and the t-dimensional Hausdorff measure of K are denoted by $\dim_H(K)$ and $\mathscr{H}^t(K)$ respectively [7]. We use $\|\cdot\|$ to denote the Euclidean norm on $\mathbb{R}^d$. The norm of a matrix $B \in \mathbb{M}_d(\mathbb{R})$ is

$$\|B\| = \max\left\{\frac{\|B\mathbf{x}\|}{\|\mathbf{x}\|} : \mathbf{x} \in \mathbb{R}^d, \|\mathbf{x}\| \neq 0\right\}.$$

A function $f : \mathbb{R}^d \to \mathbb{R}^d$ is a *contraction* if $\|f(\mathbf{x}) - f(\mathbf{y})\| \leq r\|\mathbf{x} - \mathbf{y}\|$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$, where $r < 1$ is a constant. If equality holds, then f is called a *similarity* and r is called the *contracting ratio* of f. An *iterated function system* is a family of contractions $\{f_i\}_{i=1}^N$, see [4]. We use $\mathbb{S} = \{1, 2, \ldots, N\}$ to denote the set of indices of the functions. Define

$$\mathbb{S}^n = \underbrace{\mathbb{S} \times \cdots \times \mathbb{S}}_{n} \quad \text{and} \quad \mathbb{S}^* = \bigcup_{n \in \mathbb{N}} \mathbb{S}^n.$$

An element $\mathbf{s}$ of $\mathbb{S}^n$ can be written as $\mathbf{s} = s_1 \cdots s_n$, where $s_i \in \mathbb{S}$, $1 \leq i \leq n$. Define $f_\mathbf{s} = f_{s_1} \circ f_{s_2} \circ \cdots \circ f_{s_n}$. It is well known, [9], that for any IFS $\{f_i\}_{i=1}^N$, there is a unique non-empty compact set K such that $K = \bigcup_{i=1}^N f_i(K)$. The set K is called the *attractor* of $\{f_i\}_{i=1}^N$. In fact, let Ω be the space of all non-empty compact sets of $\mathbb{R}^d$ with the Hausdorff metric. Define $F : \Omega \to \Omega$ as follows:

$$F(Q) := \bigcup_{i=1}^N f_i(Q) \quad \text{for } Q \in \Omega.$$

Then F is a contraction on Ω and K is the fixed point of F. Moreover, the contractivity condition of f_i can be relaxed. A family of functions $\{f_i : i \in \mathbb{S}\}$ is said to satisfy the *weak contractivity condition* if there is $n \in \mathbb{N}$ such that $f_{\mathbf{s}}$ are contractions for all $\mathbf{s} \in \mathbb{S}^n$. In such case, the function F^n, and hence F, has a fixed point. We also call such a system an IFS.

THEOREM 2.1 ([1]). *If $\{f_i\}_{i=1}^N$ satisfies the weak contractivity condition, then for any compact $Q_0 \neq \emptyset$, the sequence $Q_k = F(Q_{k-1})$, $k = 1, 2, \dots$, converges in Hausdorff metric to the unique non-empty compact K with $K = \bigcup_{i=1}^N f_i(K)$.*

3. Proofs of Theorem 1.1 and Theorem 1.2

We first prove the following:

LEMMA 3.1. *Let $\{f_i\}_{i=1}^N$ be as in Theorem 1.1. Then $\{f_i\}_{i=1}^N$ satisfies the weak contractivity condition.*

PROOF. Let $\mathbb{S} = \{1, 2, \dots, N\}$. Then $\|f_{\mathbf{s}}(\mathbf{x}) - f_{\mathbf{s}}(\mathbf{y})\| \leq \|B_{s_1}^{-1} \cdots B_{s_n}^{-1}\| \|\mathbf{x} - \mathbf{y}\|$ for any $\mathbf{s} \in \mathbb{S}^n$. Since $B\mathbb{W} = \mathbb{W}B$, it is easy to get that $B_{s_1}^{-1} \cdots B_{s_n}^{-1} = wB^{-M}$ for some $w \in \mathbb{W}$, where $M = \sum_{i=1}^n m_{s_i}$. So $\|B_{s_1}^{-1} \cdots B_{s_n}^{-1}\| \leq (\max_{w \in \mathbb{W}} \|w\|) \cdot \|B^{-M}\|$. By the spectral radius formula [16, Theorem 10.13], we have $\lim_{M \to \infty} \|B^{-M}\|^{1/M} = |\lambda_{\min}|^{-1} < 1$, where $\lambda_{\min}$ is the eigenvalue of B with the smallest module. Hence if n is sufficiently large, $(\max_{w \in \mathbb{W}} \|w\|) \cdot \|B^{-M}\| < 1$, and $f_{\mathbf{s}}$ are contractions for all $\mathbf{s} \in \mathbb{S}^n$. So $\{f_i\}_{i=1}^N$ satisfies the weak contractivity condition. □

Now we use the basic idea in [1] to prove Theorem 1.1.

PROOF OF THEOREM 1.1. From Lemma 3.1, we know that there exists $n \in \mathbb{N}$ such that $f_{\mathbf{s}}$ are contractions for all $\mathbf{s} \in \mathbb{S}^n$. Let $\mathbf{x}_0$ be the fixed point of f_1^n. It is easy to show that $\mathbf{x}_0$ is also the fixed point of f_1. Now let $Q_0 = \{\mathbf{x}_0\}$ and define $Q_k = F(Q_{k-1})$, $k = 1, 2, \dots$. Then

$$Q_k = \left\{B_{s_1}^{-1}\mathbf{d}_{s_1} + \cdots + B_{s_1}^{-1} \cdots B_{s_k}^{-1}\mathbf{d}_{s_k} + B_{s_1}^{-1} \cdots B_{s_k}^{-1}\mathbf{x}_0 : \mathbf{s} = s_1 \cdots s_k \in \mathbb{S}^k\right\}.$$

And $Q_0 \subset Q_1 \subset \cdots$ is an increasing sequence. So $K = \overline{\bigcup_{i \in \mathbb{N}} Q_i}$. Since $L = \bigcup_{i=1}^N (\mathbf{d}_i + B_i(L))$, for any $\mathbf{z} \in L$ and any $k \in \mathbb{N}$, there are $\mathbf{s} \in \mathbb{S}^k$ and $\{\mathbf{z}_1, \dots, \mathbf{z}_k\} \subset L$ such that $\mathbf{z} = \mathbf{d}_{s_1} + B_{s_1}\mathbf{z}_1 = \mathbf{d}_{s_1} + B_{s_1}\mathbf{d}_{s_2} + B_{s_1}B_{s_2}\mathbf{z}_2 = \cdots = \mathbf{d}_{s_1} + B_{s_1}\mathbf{d}_{s_2} + \cdots + B_{s_1} \cdots B_{s_{k-1}}\mathbf{d}_{s_k} + B_{s_1} \cdots B_{s_k}\mathbf{z}_k$. Hence

$$\begin{aligned}\mathbf{z} + \mathbf{x}_0 &= B_{s_1} \cdots B_{s_k}(\mathbf{z}_k + B_{s_k}^{-1}\mathbf{d}_{s_k} + \cdots + B_{s_k}^{-1} \cdots B_{s_1}^{-1}\mathbf{d}_{s_1} + B_{s_k}^{-1} \cdots B_{s_1}^{-1}\mathbf{x}_0) \\ &\in B_{s_1} \cdots B_{s_k}(\mathbf{z}_k + K).\end{aligned}$$

We have $B_{s_1}\cdots B_{s_k} = B^q w$ for some $w \in \mathbb{W}$, where $q = \sum_{i=1}^k m_{s_i}$. Let $m = \max_{1\le i\le N} m_i$. For $n \in \mathbb{N}$, pick k such that $n < q \le n+m$. Let $p = q-n$. Then $p \in \{1,\dots,m\}$ is an integer. We have

$$B^{-n}(\mathbf{z}+\mathbf{x}_0) \in B^p w(\mathbf{z}_k + K).$$

It follows that

$$B^{-n}(L+\mathbf{x}_0) \subset \bigcup_{p=1}^{m}\bigcup_{w\in\mathbb{W}} B^p w(L+K).$$

Since K is compact, K is contained in a ball $U_a = \{\mathbf{x}\in\mathbb{R}^d : \|\mathbf{x}\| \le a\}$. Let D be the maximum of the diameters of $B^p wK$ ($1 \le p \le m$, $w \in \mathbb{W}$). Let

$$G = \left\{\mathbf{z}\in L : \|B^p w(\mathbf{z})\| \le (D+a) \text{ for some } 1\le p\le m \text{ and some } w\in\mathbb{W}\right\}.$$

Then G is finite and if $\mathbf{z}\in L\backslash G$, then $B^p w(\mathbf{z}+K)\cap U_a = \emptyset$ for all $1\le p\le m$ and all $w\in\mathbb{W}$. So

$$B^{-n}(L+\mathbf{x}_0)\cap U_a \subseteq \bigcup_{p=1}^{m}\bigcup_{w\in\mathbb{W}} B^p w(G+K),$$

which holds for all $n\in\mathbb{N}$. We claim that $\bigcup_{n\in\mathbb{N}} B^{-n}(L+\mathbf{x}_0)$ is dense in $\mathbb{R}^d$. Let $\delta > 0$ be such that all $\mathbf{x}\in\mathbb{R}^d$ has distance less than δ from the lattice L. Let $\mathbf{u}\in\mathbb{R}^d$. For any $k\in\mathbb{N}$, there is $\mathbf{v}_k\in L$ such that $\|\mathbf{v}_k - B^k\mathbf{u}\| \le \delta$. So $\|B^{-k}(\mathbf{v}_k+\mathbf{x}_0)-\mathbf{u}\| = \|B^{-k}(\mathbf{v}_k+\mathbf{x}_0 - B^k\mathbf{u})\| \le \|B^{-k}\|(\delta+\|\mathbf{x}_0\|) \to 0$ when $k\to\infty$. This proves the claim. Now it follows that

$$\bigcup_{p=1}^{m}\bigcup_{w\in\mathbb{W}} B^p w(G+K) = \overline{\bigcup_{p=1}^{m}\bigcup_{w\in\mathbb{W}} B^p w(G+K)} \supseteq \overline{\bigcup_{n\in\mathbb{N}}\left(B^{-n}(L+\mathbf{x}_0)\cap U_a\right)} = U_a.$$

So by Baire category theorem, K has non-empty interior. □

Next we prove Theorem 1.2.

Proof of Theorem 1.2. Let $\left\{\widetilde{f}_i : \widetilde{f}_i(\mathbf{x}) = \widetilde{A}_i\mathbf{x}+\widetilde{\mathbf{a}}_i, \widetilde{A}_i \in \mathbb{M}_d(\mathbb{R})\right\}_{i=1}^N$ be a family of contractions with attractor $\widetilde{K}$. In [14], it was proved that if $\widetilde{K}$ has non-empty interior and $\sum_{i=1}^N\left|\det\widetilde{A}_i\right| = 1$, then $\dim_H(\partial\widetilde{K}) < d$. Now let $\mathbb{S} = \{1,2,\dots,N\}$. Since $\{f_i\}_{i=1}^N$ satisfies the weak contractivity condition, there exists $n\in\mathbb{N}$ such that $\{f_{\mathbf{s}} : \mathbf{s}\in\mathbb{S}^n\}$ is a family of contractions on $\mathbb{R}^d$. We have $K = \bigcup_{i\in\mathbb{S}} f_i(K) = \bigcup_{\mathbf{s}\in\mathbb{S}^n} f_{\mathbf{s}}(K)$. Regarding $f_{\mathbf{s}}$ as $\widetilde{f}_i$, we get the theorem immediately. □

From Lemma 3.1 and Theorem 1.2, the following result follows directly.

Corollary 3.2. *In Theorem* 1.1, *if* $\sum_{i=1}^N |\det B_i|^{-1} = 1$, *then* $\dim_H(\partial K) < d$.

4. IFS and open set condition

An IFS $\{f_i\}_{i=1}^N$ is said to satisfy the *open set condition* (OSC) if there exists a bounded open set V such that $V \supset \bigcup_{i=1}^N f_i(V)$ and $f_i(V) \cap f_j(V) = \emptyset$ for $i \neq j$ [9]. We call V an *OSC set* of $\{f_i\}_{i=1}^N$. Obviously if $\{f_i\}_{i=1}^N$ satisfies OSC with the OSC set V, then removing some of f_i, the IFS of the remaining functions still satisfies OSC with the same OSC set V. If f_i are all similarities with contracting ratios r_i, then it is well known [17] that OSC is equivalent to $0 < \mathscr{H}^t(K) < \infty$, where t is the *similarity dimension* of $\{f_i\}_{i=1}^N$ defined as the unique number satisfying $\sum_{i=1}^N r_i^t = 1$. For such K, $\mathscr{H}^t(f_i(K) \cap f_j(K)) = 0$, or more precisely, $\dim_H(f_i(K) \cap f_j(K)) < t$ for $i \neq j$ [14]. So $f_i(K)$ $(1 \leq i \leq N)$ are essentially disjoint.

Let $B \in \mathbb{M}_d(\mathbb{Z})$. A set $\{\mathbf{d}_1, \dots, \mathbf{d}_N\} \subset L$ is called a *residue system* for B if $L = \bigcup_{i=1}^N (\mathbf{d}_i + B(L))$ and $(\mathbf{d}_i + B(L)) \cap (\mathbf{d}_j + B(L)) = \emptyset$ for $i \neq j$.

LEMMA 4.1. *Let $B_i \in \mathbb{M}_d(\mathbb{Z})$ be non-singular, $1 \leq i \leq N$, and $\{\mathbf{d}_1, \dots, \mathbf{d}_N\} \subset L$. Then any two of the following three conditions imply the other.*

(i) $L = \bigcup_{i=1}^N (\mathbf{d}_i + B_i(L))$.

(ii) $(\mathbf{d}_i + B_i(L)) \cap (\mathbf{d}_j + B_j(L)) = \emptyset$ *for all* $1 \leq i, j \leq N$ *with* $i \neq j$.

(iii) $\sum_{i=1}^N |\det B_i|^{-1} = 1$.

PROOF. Let $n \in \mathbb{N}$ and $C = [0, n-1] \times \cdots \times [0, n-1]$ d times. Then C contains n^d lattice points of L.

(i) and (ii) implies (iii). Among the n^d lattice points of C, the number of points from $\mathbf{d}_i + B_i(L)$ is $n^d/|\det B_i| + o(n^d)$. By (ii) these points are all different. So using (i), we have $\sum_{i=1}^N \left(n^d/|\det B_i| + o(n^d)\right) = n^d$. Let $n \to \infty$. We get immediately $\sum_{i=1}^N |\det B_i|^{-1} = 1$.

(i) and (iii) implies (ii). Suppose that there exist $i \neq j$ with $(\mathbf{d}_i + B_i(L)) \cap (\mathbf{d}_j + B_j(L)) \neq \emptyset$. Then there are $\mathbf{d}, \mathbf{p}, \mathbf{q} \in L$ such that $\mathbf{d} = \mathbf{d}_i + B_i\mathbf{p} = \mathbf{d}_j + B_j\mathbf{q}$. We have $B_i^{-1} = 1/\det B_i \, B_i^{adj}$, where $\det B_i \in \mathbb{Z}$ and $B_i^{adj} \in \mathbb{M}_d(\mathbb{Z})$ is the adjoint of B_i. Let $\widetilde{B} = B_i B_i^{adj} B_j = (\det B_i) B_j$. For any $\mathbf{z} \in L$,

$$\mathbf{d} + \widetilde{B}\mathbf{z} = \mathbf{d}_i + B_i\mathbf{p} + B_i B_i^{adj} B_j \mathbf{z} \in \mathbf{d}_i + B_i(L),$$

and

$$\mathbf{d} + \widetilde{B}\mathbf{z} = \mathbf{d}_j + B_j\mathbf{q} + (\det B_i) B_j \mathbf{z} \in \mathbf{d}_j + B_j(L).$$

So $\mathbf{d} + \widetilde{B}(L) \subset (\mathbf{d}_i + B_i(L)) \cap (\mathbf{d}_j + B_j(L))$. It follows that the total number of distinct points of $\bigcup_{k=1}^N \left(\mathbf{d}_k + B_k(L)\right)$ in C is at most $\sum_{i=1}^N \left(n^d/\left|\det B_i\right| + o(n^d)\right) - \left(n^d/\left|\det \widetilde{B}\right| + o(n^d)\right) = \left(1 - \left|\det \widetilde{B}\right|^{-1}\right) n^d + o(n^d)$, which is less than n^d when n is large enough. But this contradicts (i).

(ii) and (iii) implies (i). For each i, let $\{\mathbf{d}_i, \mathbf{d}_{i,2}, \dots, \mathbf{d}_{i,n_i}\}$ be a residue system for B_i. Suppose for the contrary that $L \neq \bigcup_{i=1}^N (\mathbf{d}_i + B_i(L))$. Then there exists $\mathbf{d} \in L \setminus \bigcup_{i=1}^N (\mathbf{d}_i + B_i(L))$. So for each $1 \le i \le N$ there exists $2 \le j_i \le n_i$ with $\mathbf{d} \in \mathbf{d}_{i,j_i} + B_i(L)$. Hence $\mathbf{d} \in \bigcap_{i=1}^N (\mathbf{d}_{i,j_i} + B_i(L))$. Now using the same arguments as in the proof of the last part inductively, we see that there is $\widetilde{B} \in \mathbb{M}_d(\mathbb{Z})$ such that

$$\mathbf{d}+\widetilde{B}(L) \subset \bigcap_{i=1}^N \left(\mathbf{d}_{i,j_i} + B_i(L)\right) \subset L \setminus \bigcup_{i=1}^N \left(\mathbf{d}_i + B_i(L)\right).$$

Counting the lattice points in C, we have $\sum_{i=1}^N \left(n^d/|\det B_i| + o(n^d)\right) + n^d/\left|\det \widetilde{B}\right| + o(n^d) \le n^d$. Hence $\sum_{i=1}^N |\det B_i|^{-1} < 1$, which contradicts (iii). $\square$

In Theorem 1.1, if $\sum_{i=1}^N |\det B_i|^{-1} = 1$, which by Lemma 4.1 is equivalent to $(\mathbf{d}_i + B_i(L)) \cap (\mathbf{d}_j + B_j(L)) = \emptyset$ for all $i \neq j$, then it can be shown easily that $\mu(f_i(K) \cap f_j(K)) = 0$ for $i \neq j$. So K° is an OSC set for $\{f_i\}_{i=1}^N$. In case $L \neq \bigcup_{i=1}^N (\mathbf{d}_i + B_i(L))$, we have the following:

PROPOSITION 4.2. *In Theorem* 1.1, *if* $L \supset \bigcup_{i=1}^N (\mathbf{d}_i + B_i(L))$ *and* $(\mathbf{d}_i + B_i(L)) \cap (\mathbf{d}_j + B_j(L)) = \emptyset$ *for* $i \neq j$, *then* $\{f_i\}_{i=1}^N$ *satisfies OSC.*

PROOF. Let $m = \max_{1 \le i \le N} m_i$. Let $\{\mathbf{a}_1, \dots, \mathbf{a}_p\}$ be a residue system for B, and $\{\mathbf{b}_1, \dots, \mathbf{b}_q\}$ a residue system for B^m. We observe that for any $\mathbf{z} \in L$ and any $\mathbf{b}_i \in \{\mathbf{b}_1, \dots, \mathbf{b}_q\}$, either $(\mathbf{z} + B^m(L)) \cap (\mathbf{b}_i + B^m(L)) = (\mathbf{b}_i + B^m(L))$, or $(\mathbf{z} + B^m(L)) \cap (\mathbf{b}_i + B^m(L)) = \emptyset$. For any $w \in \mathbb{W}$, $w(L) \subset L$. Since $w^{-1} \in \mathbb{W} \subset \mathbb{M}_d(\mathbb{Z})$, $L = ww^{-1}(L) \subset w(L)$. So $L = w(L)$. Now pick any $B_i = w_i B^{m_i}$. If $m_i = m$, then $\mathbf{d}_i + B_i(L) = \mathbf{d}_i + B^m \widetilde{w}(L) = \mathbf{b}_j + B^m(L)$ for some $j \in \{1, \dots, q\}$, where $\widetilde{w} \in \mathbb{W}$. If $m_i < m$, then by

$$w_i B^{m_i}(L) = w_i B^{m_i} \left(\bigcup_{j=1}^p (\mathbf{a}_j + B(L)) \right) = \bigcup_{j=1}^p \left(w_i B^{m_i} \mathbf{a}_j + w_i B^{m_i+1}(L) \right),$$

We can break down $w_i B^{m_i}(L)$ into smaller subspaces. This can be done further until we get $\mathbf{d}_i + w_i B^{m_i}(L) = \bigcup_{j=1}^{M_i} (\mathbf{b}_{i_j} + B^m(L))$, where $\{\mathbf{b}_{i_j}\}_{j=1}^{M_i} \subset \{\mathbf{b}_1, \dots, \mathbf{b}_q\}$. Applying this process to all B_i, we get

$$\bigcup_{i=1}^N (\mathbf{d}_i + B_i(L)) = \bigcup_{i=1}^N \bigcup_{j=1}^{M_i} \left(\mathbf{b}_{i_j} + B^m(L)\right).$$

Since $\mathbf{d}_i + B_i(L)$ are disjoint, all $\mathbf{b}_{i_j}$ are different. Without loss of generality, we may assume $\bigcup_{i=1}^N \bigcup_{j=1}^{M_i} \{\mathbf{b}_{i_j}\} = \{\mathbf{b}_1, \dots, \mathbf{b}_{q'}\}$, where $q' < q$. Then

$$L = \bigcup_{i=1}^N (\mathbf{d}_i + B_i(L)) \bigcup \bigcup_{i=q'+1}^q (\mathbf{b}_i + B^m(L)).$$

For $i = N+1, \dots, N+q-q'$, define $f_i(\mathbf{x}) = B^{-m}(\mathbf{x} + \mathbf{b}_i)$. Then $\{f_i\}_{i=1}^{N+q-q'}$ satisfies OSC by Theorem 1.1 and Lemma 4.1. Therefore $\{f_i\}_{i=1}^{N}$ satisfies OSC. □

5. Construction and examples

Theorem 1.1 provides a systematic method to construct fractals consisting of essentially disjoint pieces of different sizes, but each similar to the original. The conditions in Theorem 1.1 can be satisfied easily by using (1.1) and (1.2). We show here how to do this through several examples. For simplicity, we assume that the base $\mathscr{B}$ is orthonormal. In these examples, all f_i are similarities and OSC is satisfied. We find that some attractors look quite nice and intriguing, which we did not see before.

EXAMPLE 5.1. We have $\mathbb{Z} = 2\mathbb{Z} \cup (2\mathbb{Z}+1)$. Iterating the right hand side partially we get $\mathbb{Z} = 2\mathbb{Z} \cup (2(2\mathbb{Z} \cup (2\mathbb{Z}+1)) + 1) = 2\mathbb{Z} \cup (4\mathbb{Z}+1) \cup (4\mathbb{Z}+3)$.So the corresponding IFS is: $f_1(x) = x/2$, $f_2(x) = (x+1)/4$ and $f_3(x) = (x+3)/4$. Let $K = \bigcup_{i=0}^{\infty} \left[(2^{2i}-1)/2^{2i}, (2^{2i+1}-1)/2^{2i+1}\right] \cup \{1\}$. Then K is compact and it can be verified that $K = \bigcup_{i=1}^{3} f_i(K)$. By the uniqueness of such a set, K is the attractor of $\{f_i\}_{i=1}^{3}$. We see that K consists of three pieces $f_1(K)$, $f_2(K)$ and $f_3(K)$, where $f_2(K)$ and $f_3(K)$ have the same size, while $f_1(K)$ has a different size. This process of iteration can be continued to get more complicated tiles.

EXAMPLE 5.2. Let

$$B = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad \mathbf{d} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{h} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Let

$$w_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad w_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Note that $w_1, w_2 \in \mathbb{W}$, where

$$\mathbb{W} = \left\{ \begin{pmatrix} m & 0 \\ 0 & n \end{pmatrix}, \begin{pmatrix} 0 & m \\ n & 0 \end{pmatrix} : m = \pm 1,\ n = \pm 1 \right\}$$

is a symmetry group of B. The following is well known. (i) $\mathbb{Z}^2 = B\mathbb{Z}^2 \cup (B\mathbb{Z}^2 + \mathbf{d})$ gives twindragon; (ii) $\mathbb{Z}^2 = B\mathbb{Z}^2 \cup (w_1 B\mathbb{Z}^2 + \mathbf{v})$ gives Levy dragon; (iii) $\mathbb{Z}^2 = B\mathbb{Z}^2 \cup (w_2 B\mathbb{Z}^2 + \mathbf{h})$ gives Heighway dragon. By (ii) we have

$$\begin{aligned} \mathbb{Z}^2 &= B\big(B\mathbb{Z}^2 \cup (w_1 B\mathbb{Z}^2 + \mathbf{v})\big) \cup \big(w_1 B\mathbb{Z}^2 + \mathbf{v}\big) \\ &= B^2\mathbb{Z}^2 \cup \big(Bw_1 B\mathbb{Z}^2 + B\mathbf{v}\big) \cup \big(w_1 B\mathbb{Z}^2 + \mathbf{v}\big). \end{aligned}$$

Note that $Bw_1B = wB^2$ for some $w \in \mathbb{W}$. The corresponding attractor is shown in Figure 1. Also we have

$$\begin{aligned}\mathbb{Z}^2 &= B\mathbb{Z}^2 \cup \big(w_1B\big(B\mathbb{Z}^2 \cup \big(w_1B\mathbb{Z}^2 + \mathbf{v}\big)\big) + \mathbf{v}\big)\\ &= B\mathbb{Z}^2 \cup \big(\big(w_1B^2\mathbb{Z}^2 + \mathbf{v}\big) \cup \big(w_1Bw_1B\mathbb{Z}^2 + w_1B\mathbf{v} + \mathbf{v}\big)\big).\end{aligned}$$

The corresponding IFS is: $f_1(\mathbf{x}) = B^{-1}(\mathbf{x})$, $f_2(\mathbf{x}) = (w_1B^2)^{-1}(\mathbf{x} + \mathbf{v})$, $f_3(\mathbf{x}) = (w_1Bw_1B)^{-1}(\mathbf{x}+w_1B\mathbf{v} + \mathbf{v})$. Its attractor is shown in Figure 2. The attractor of $\{f_1, f_3\}$ is shown in Figure 3, whose Hausdorff dimension is $\ln\big((3+\sqrt{5})/2\big)/\ln 2$.

FIGURE 1.

FIGURE 2.

We can also 'mix up' these dragons. Here are some of the examples. By (i) and (ii), we get

$$\begin{aligned}\mathbb{Z}^2 &= B\mathbb{Z}^2 \cup \big(B\big(B\mathbb{Z}^2 \cup \big(w_1B\mathbb{Z}^2 + \mathbf{v}\big)\big) + \mathbf{d}\big)\\ &= B\mathbb{Z}^2 \cup \big(B^2\mathbb{Z}^2 + \mathbf{d}\big) \cup \big(Bw_1B\mathbb{Z}^2 + B\mathbf{v} + \mathbf{d}\big).\end{aligned}$$

The attractor is shown in Figure 4. Using (i) and (iii), we get

$$\begin{aligned}\mathbb{Z}^2 &= B\big(B\mathbb{Z}^2 \cup \big(w_2B\mathbb{Z}^2 + \mathbf{h}\big)\big) \cup \big(B\mathbb{Z}^2 + \mathbf{d}\big)\\ &= B^2\mathbb{Z}^2 \cup \big(Bw_2B\mathbb{Z}^2 + B\mathbf{h}\big) \cup \big(B\mathbb{Z}^2 + \mathbf{d}\big).\end{aligned}$$

Its attractor is in Figure 5. Also we have

$$\begin{aligned}\mathbb{Z}^2 &= B\big(B\mathbb{Z}^2 \cup \big(B\mathbb{Z}^2 + \mathbf{d}\big)\big) \cup \big(w_2B\mathbb{Z}^2 + \mathbf{h}\big)\\ &= B^2\mathbb{Z}^2 \cup \big(B^2\mathbb{Z}^2 + B\mathbf{d}\big) \cup \big(w_2B\mathbb{Z}^2 + \mathbf{h}\big).\end{aligned}$$

FIGURE 3.

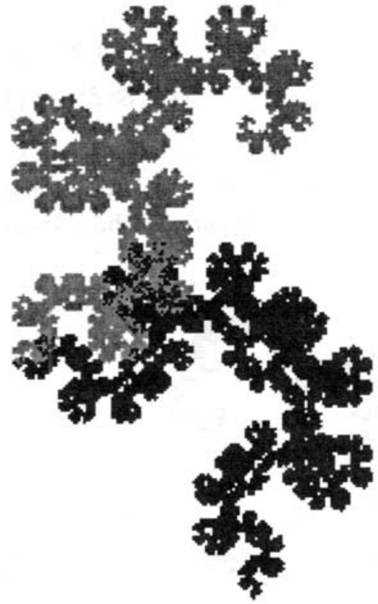

FIGURE 4.

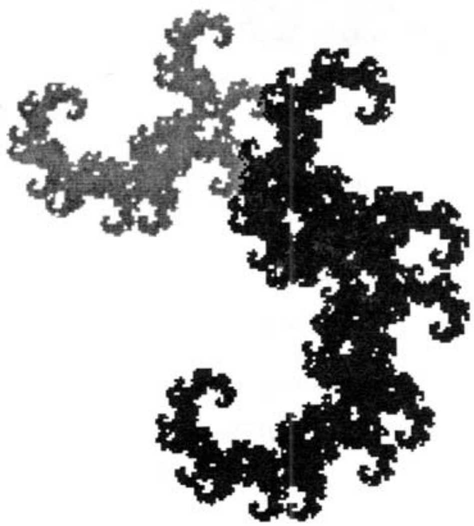

FIGURE 5.

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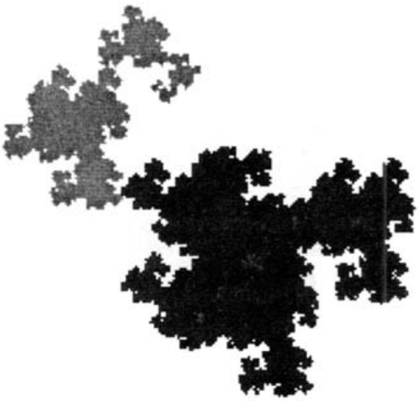

FIGURE 6.

FIGURE 7.

The attractor is in Figure 6. Using (ii) and (iii), we get

$$\begin{aligned}\mathbb{Z}^2 &= B\mathbb{Z}^2 \cup \left(w_2 B(B\mathbb{Z}^2 \cup \left(w_1 B\mathbb{Z}^2 + \mathbf{v}\right)) + \mathbf{h}\right) \\ &= B\mathbb{Z}^2 \cup \left(w_2 B^2\mathbb{Z}^2 + \mathbf{h}\right) \cup \left(w_2 B w_1 B\mathbb{Z}^2 + w_2 B\mathbf{v} + \mathbf{h}\right).\end{aligned}$$

Its attractor is given in Figure 7. From this, we get that for any $\mathbf{a} \in \mathbb{Z}^2$,

$$\begin{aligned}\mathbb{Z}^2 &= \mathbb{Z}^2 + \mathbf{a} \\ &= \left(B\mathbb{Z}^2 + \mathbf{a}\right) \cup \left(w_2 B^2\mathbb{Z}^2 + \mathbf{h} + \mathbf{a}\right) \cup \left(w_2 B w_1 B\mathbb{Z}^2 + w_2 B\mathbf{v} + \mathbf{h} + \mathbf{a}\right).\end{aligned}$$

By using different values of $\mathbf{a}$, we may get different attractors. The attractor corresponding to $\mathbf{a} = \binom{-1}{-1}$ is shown in Figure 8.

Every attractor in these figures, except Figure 3, consists of three pieces of two different sizes. As in Example 5.1, we can do more iterations.

6. Tiling with more than one prototile

In general, the attractor of the IFS involving different matrices does not admit a monohedral tiling of $\mathbb{R}^d$. More than one prototile is needed. Consider the IFS $\left\{f_i : f_i(\mathbf{x}) = A_i\mathbf{x} + \mathbf{a}_i, A_i \in \mathbb{M}_d(\mathbb{R})\right\}_{i=1}^{N}$. Let K be the corresponding attractor.

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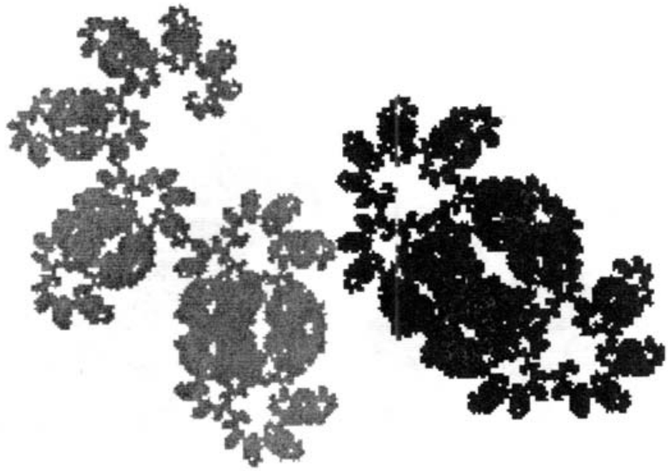

FIGURE 8.

For the non-overlapping attractors, we should be only concerned with the case that $\sum_{i=1}^{N} |\det A_i| = 1$, because if $\mu(K) > 0$, then

$$\mu\big(f_i(K) \cap f_j(K)\big) = 0 \text{ for } i \neq j \text{ if and only if } \sum_{i=1}^{N} |\det A_i| = 1.$$

We call a set E an affine copy of K if $E = A(K) + \mathbf{b}$ for some $A \in M_d(\mathbb{R})$ and $\mathbf{b} \in \mathbb{R}^d$. Let $\{K_i : i \in \mathbb{N}\}$ be a family of sets in $\mathbb{R}^d$, we say that K_i are *of approximately the same size* if there exist $0 < c_1 < c_2 < \infty$ such that $c_1 < \mu(K_i) < c_2$ for all $i \in \mathbb{N}$. The following general tiling theorem is easy to prove.

THEOREM 6.1. *Let* $\{f_i : f_i(\mathbf{x}) = A_i\mathbf{x} + \mathbf{a}_i,\ A_i \in \mathbb{M}_d(\mathbb{R}),\ \det A_i \neq 0\}_{i=1}^{N}$ *be a family of functions which satisfy the weak contractivity condition. If the attractor K has non-empty interior and* $\sum_{i=1}^{N} |\det A_i| = 1$, *then* $\mathbb{R}^d$ *can be tiled by affine copies of K of approximately the same size.*

PROOF. Since $K^\circ \neq \emptyset$, we can find some $g = f_{i_1} \cdots f_{i_m}$ such that its fixed point is in K°. Then $g^{-n}(K)$ can be broken down into affine copies of K of approximately the same size. As n goes to infinity, $g^{-n}(K)$ will tile the whole space $\mathbb{R}^d$. □

In this theorem, if $A_1 = \cdots = A_N = A$, then every tile has the same shape and size. The attractor K tiles $\mathbb{R}^d$ by translation. If the base of the lattice is orthonormal and $A_i = w_i A$, where $w_i \in \mathbb{W}$ and $\mathbb{W}$ is a symmetry group of A consisting of Euclidean isometries, then K also admits a monohedral tiling of $\mathbb{R}^d$. But in this case rotations and reflections of K are needed in the tiling. For the attractor K in Theorem 1.1 with $\sum_{i=1}^{N} |\det B_i|^{-1} = 1$, every affine copy of K has the form $wB^{-k}(K) + \mathbf{b}$ for some $k \in \mathbb{N}$, where $\mathbf{b} \in \mathbb{R}^d$ and w belongs to a finite group. Since they are of approximately the same size, the number of such integers k must be finite. Hence we have proved the following:

COROLLARY 6.2. *Let K be the attractor in Theorem 1.1 and $\sum_{i=1}^{N} |\det B_i|^{-1} = 1$. Then there exists a tiling of $\mathbb{R}^d$ with each tile congruent to some $w_i B^{-k_i}(K)$, where $w_i \in \mathbb{W}$, $k_i \in \mathbb{N}$, and there are only finitely many such k_i.*

Furthermore, we have

COROLLARY 6.3. *Suppose that in Theorem 1.1, the base $\mathscr{B}$ of the lattice L is orthonormal, $\sum_{i=1}^{N} |\det B_i|^{-1} = 1$, and $\mathbb{W}$ consists of Euclidean isometries. Let $m_1 \le m_2 \le \cdots \le m_N$. Then there exists an n-hedral tiling of $\mathbb{R}^d$ by similar copies of K with $n \le m_N$. In particular, if $\{1, 2, \ldots, m_N\} = \{m : m = m_i$ for some $1 \le i \le N\}$, then $\{B^{-j}(K) : j = 1, 2, \cdots, m_N\}$ admits an m_N-hedral tiling.*

PROOF. We have

$$K = \bigcup_{i=1}^{N} B_i^{-1}(K + \mathbf{d}_i) = \bigcup_{i=1}^{N} B^{-m_i} w_i^{-1}(K + \mathbf{d}_i).$$

From the given conditions $w(E)$ is congruent to E for all $w \in \mathbb{W}$ and $E \subset \mathbb{R}^d$. By Corollary 6.2, there exists a tiling $\mathscr{T}$ of $\mathbb{R}^d$ using finitely many similar copies of K as the prototiles. Each tile T in $\mathscr{T}$ has the form $w_T B^{-k_T}(K) + \mathbf{b}_T$. Let $k_{\min} = \min_{T \in \mathscr{T}}\{k_T\}$ and $k_{\max} = \max_{T \in \mathscr{T}}\{k_T\}$. If $k_T < k_{\max} - m_N + 1$, then

$$\begin{aligned} T &= w_T B^{-k_T}\left(\bigcup_{i=1}^{N} B^{-m_i} w_i^{-1}(K + \mathbf{d}_i)\right) + \mathbf{b}_T \\ &= \left(\bigcup_{i=1}^{N} w_T B^{-k_T - m_i} w_i^{-1}(K + \mathbf{d}_i)\right) + \mathbf{b}_T. \end{aligned}$$

In the above expression, $k_T < k_T + m_1 \le \cdots \le k_T + m_N \le k_{\max}$. Hence T is broken down into smaller tiles. If still $k_T + m_i < k_{\max} - m_N + 1$ for some i, then we can continue the process for that tile. Eventually every tile will have the form $wB^{-k}(K) + \mathbf{b}$, where $k_{\max} - m_N + 1 \le k \le k_{\max}$. So there are at most m_N different sizes. If $\{1, 2, \ldots, m_N\} = \{m : m = m_i$ for some $1 \le i \le N\}$, then the m_N different sizes do exist. $\square$

From this corollary, it is easily seen that all the attractors in the examples in Section 5, except the one in Figure 3, give rise to a dihedral tiling. Also for any positive integer n, it is easy to generate a fractal tiling which is n-hedral.

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