Note on a Theorem connected with the area of a $2n$-sided polygon.

By THOMAS MUIR, M.A., F.R.S.E.

The theorem is :—*If* $a_1, a_2, a_3, \ldots, a_{2n}$ *be the middle points of the sides of any convex polygon* $A_1A_2A_3 \ldots A_{2n}$ *then as regards areas*

$$a_1 a_2 \ldots a_{2n} = \tfrac{1}{2}A_1A_2\ldots A_{2n} + \tfrac{1}{4}A_1A_3\ldots A_{2n-1} + \tfrac{1}{4}A_2A_4\ldots A_{2n}.$$

The following proof depends only on the theorem that the line joining the points of bisection of two sides of a triangle cuts off a triangle equal in area to a quarter of the original triangle. For convenience in writing, let us take the case where $n = 4$. Then

$$\left.\begin{aligned} &\tfrac{1}{4}A_1A_3A_5A_7 = \\ &\qquad \tfrac{1}{4}(A_1A_2A_3\ldots A_8 - A_1A_2A_3 - A_3A_4A_5 - A_5A_6A_7 - A_7A_8A_1) \\ \text{and }\ &\tfrac{1}{4}A_2A_4A_6A_8 = \\ &\qquad \tfrac{1}{4}(A_1A_2A_3\ldots A_8 - A_2A_3A_4 - A_4A_5A_6 - A_6A_7A_8 - A_8A_1A_2) \end{aligned}\right\}$$

$$\therefore \left.\begin{aligned} &\tfrac{1}{4}A_1A_3A_5A_7 \\ +\ &\tfrac{1}{4}A_2A_4A_6A_8 \end{aligned}\right\} = \tfrac{1}{2}A_1A_2A_3\ldots A_8 + a_1a_2a_3\ldots a_8 - A_1A_2A_3\ldots A_8$$

and $\therefore a_1 a_2 a_3 \ldots a_8 = \tfrac{1}{2}A_1A_2A_3\ldots A_8 + \tfrac{1}{4}A_1A_3A_5A_7 + \tfrac{1}{4}A_2A_4A_6A_8$

as was to be proved.

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Spherical Geometry.

By R. E. ALLARDICE, M.A.

The object of this paper is to bring together the principal properties of figures described on the surface of the sphere that can be established without the use of Solid Geometry or of Trigonometry.

The following properties of the spherical surface, which correspond to the definitions and axioms in Plane Geometry, are assumed. They may be considered as derived from one's general notion of the sphere.

a. On the surface of the sphere certain circles (great circles) can be drawn, which are closely analogous to straight lines in a plane. These great circles are all equal in circumference.

b. Through any two points one great circle can be drawn, and in general only one; if the distance between the two points be half a great circle, any number may be drawn.

c. Any two great circles intersect in two points (called antipodal points), the distance between which is half a great circle.

d. With any centre and any radius a circle may be described, called a small circle, unless the radius be a quadrant of a great circle, in which case the circle becomes a great circle.

e. Every circle, great or small, has two centres (or poles), these centres being antipodal points.

f. If two antipodal points move continuously on the sphere, they trace out what are called symmetric figures. These figures have corresponding elements equal, and are equal in area, but are not in general superposable. The one is, in fact, the perverse of the other.

An angle may be conceived as generated by the revolution of a great circular arc about a fixed point. Since the two characteristic properties of angles, which are that two equal angles are superposable and therefore identical, and that if a straight line trace out the whole (finite) angular space at a point it will return to its original position, are possessed also by circular arcs (a tracing point taking the place of a tracing line), arcs may evidently be treated as if they were angles, and arcs and angles may be spoken of as equal. The angle to which any arc corresponds is evidently the angle between the radii drawn to its extremities. From this it follows that the angle between two lines (great circular arcs) is equal to the angle between their middle points.

§ 1. The angle between two lines is equal or supplementary to the angle between their poles.

§ 2. The polar triangle. The triangle formed by joining the poles A′, B′, C′ of the sides BC, CA, AB of the triangle ABC, (A′ being the pole which lies on the same side of BC as A, and so on), is called the polar triangle of the triangle ABC. By § 1, B′C′, C′A′, A′B′ are either equal or supplementary to A, B, and C; and since motion from A′ to B′ corresponds to rotation from BC to the production of AC, the sides of the polar triangle must be the *supplements* of the angles of the original triangle. The polar property of the triangle is evidently reciprocal.

If both poles of each side of the original triangle be considered, eight triangles can be formed, the angles and sides of four of which are—(1) $\pi - a$, $\pi - b$, $\pi - c$, $\pi - A$, $\pi - B$, $\pi - C$; (2) $\pi - a$, b, c, $\pi - A$, B, C; (3) a, $\pi - b$, c, A, $\pi - B$, C; (4) a, b, $\pi - c$, A, B, $\pi - C$. The other four are the symmetric triangles. The triangles (2), (3), and (4) are the polars of the associated triangles of the original triangle, that is, the triangles formed by producing each pair of sides.

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§ 3. The Principle of Polar Transformation.

From any theorem that has been established another theorem may be deduced by consideration of the polar figure. Thus the polar figure corresponding to a line passing through a point is a point lying on a line; and hence, if it has been proved that three lines l, m, n pass through the same point P, the three points L, M, N, the poles of l, m, n, all lie on the same line p, of which P is the pole. It must be noticed that to the internal bisector of an angle corresponds the external bisector of the corresponding line, that is, the point which bisects the supplement of the line, and which is a quadrant distant from the internal point of bisection. This follows from the fact that it is the supplements of the sides of the polar triangle that are equal to the angles of the original triangle.

§ 4. The area of a spherical triangle $= \dfrac{A+B+C-\pi}{\pi} \cdot \tfrac{1}{4}$ surface of sphere.

§ 5. $A+B+C > \pi < 3\pi$; $b+c > a$, &c.; $a+b+c < 2\pi$.

Since $A+B+C-\pi$ varies as area of triangle,

$\therefore A+B+C > \pi$.

In the polar triangle $a'+b'+c' > 0$;

$\therefore \pi - A + \pi - B + \pi - C > 0$; $\therefore A+B+C < 3\pi$.

Transforming the inequality $A'+B'+C' > \pi$ by means of the second polar triangle, there results

$\pi - a + b + c > \pi$; $\therefore b+c > a$.

Again in the polar triangle

$A'+B'+C' > \pi$; $\therefore \pi - a + \pi - b + \pi - c > \pi$; $\therefore a+b+c < 2\pi$.

§ 6. Theorems analogous to Euclid I. 4, 5, 6, 8, 15, 24, 25, and 26 (the first case only) can be proved for the sphere in much the same way as they are proved in Plane Geometry; but where there are congruent triangles in Plane Geometry, there may be either congruent or symmetric triangles in Spherical Geometry. Of these theorems No. 6 is the polar of No. 5, and the first case of No. 26 the polar of No. 4. Theorem 16 is only true with limitations, which make it almost useless. The second case of No. 26 is an ambiguous proposition in the case of the sphere, being the polar theorem of the ordinary "ambiguous case" of Plane Geometry.

§ 7. The polar theorem of Euclid I. 8. If two triangles have the three angles of the one equal to the three angles of the other, the triangles are either congruent or symmetric.

The polar theorems of Euclid I. 24 and I. 25.

§ 8. From the proposition that any two sides of a triangle are together greater than the third, which is proved above, there is easily deduced Euclid I. 19, the polar of which gives Euclid I. 18.

Note.—Since through two given points there can be drawn only one small circle of given radius and concave in a given direction, an arc of such a circle may be substituted in some of the above propositions for a great circular arc.

§ 9. Euclid, III. 7 and 8, true both for great circles and for small circles, and proved for both in the same way.

§ 10. All the theorems of the Third Book of Euclid are true for the sphere, with the following exceptions :—

(1.) That angles in the same segment of a circle are equal.

There is, however, a theorem analogous to this, which will be enunciated afterwards.

(2.) That the opposite angles of a quadrilateral inscribed in a circle are together equal to two right angles.

In the case of the sphere however, one pair of opposite angles of such a quadrilateral are together equal to the other pair

(3.) That the angle in a semicircle is a right angle, &c.

In the case of the sphere the triangle inscribed in a semicircle has its vertical angle equal to the sum of the other two, the triangle inscribed in a segment greater than a semicircle has its vertical angle less than the sum of the other two, and the triangle inscribed in a segment less than a semicircle its vertical angle greater than the sum of tne other two.

[The spherical triangle which has one angle equal to the sum of the other two, has many properties analogous to those of the right-angled plane triangle].

(4.) That the rectangle under the segments of secants passing through a fixed point is constant.

[In the sphere the product of the tangents of half the segments is constant].

Those of the above propositions that refer to angles reduce to the corresponding propositions of Plane Geometry if the condition be added that the three angles of a triangle are together equal to two right angles.

Definition. A spherical parallelogram is a quadrilateral which has its opposite sides equal.

§ 11. The opposite angles of a parallelogram are equal; the alternate angles made by the diagonals with the sides are equal; and the diagonals bisect one another.

§ 12. On a given base only one parallelogram can be described, having the side opposite this base in a given line. (Fig. 1.)

For if ABCD be a parallelogram, and AB and DC be produced to meet at E and F, then the triangles EAD and FCB are equiangular; $\therefore$ EA = CF, and ED = BF.

§ 13. Parallelograms on equal bases, and having a pair of opposite sides in the same lines, are equal. (Fig. 1.)

For $DD' = BB'$, $\angle D'DO = \angle B'BO'$, $\angle DD'O = \angle BB'O'$;
$\therefore \triangle DOD' = \triangle BO'B'$. Similarly, $OA'A = O'C'C$; $\therefore ABCD = A'B'C'D'$.

§ 14. If two parallel small circles be cut by a great circle in the points A, B, and C, D, then AC and BD are bisected by the great circle parallel to the two small circles, and the parts BA and CD intercepted by the small circles are equal. (Fig. 2.)

Draw OPO′ a great circle perpendicular to ABDC; and draw the great circle OQO′.

Then $OQ = O'Q$, $OA = O'C$, $\angle OQA = \angle O'QC$; $\therefore AQ = QC$.
Again, $OF = O'G$; $\therefore BA = CD$.

Cor.—If the great circle ABCD touch one of the small circles, it must touch the other.

§ 15. The quadrilateral formed by joining the extremities of two equal arcs of equal and parallel small circles is a parallelogram. (Fig. 3.)

Let AB and CD be the arcs.

Draw the great circles ODO′, OCO′.

Then $\angle CO'D = \angle COD = \angle AOB$.
$\therefore \angle AOD = \angle BOC$; and $AO = BO$, $OD = OC$.
$\therefore$ $AD = BC$, and chord AB = chord CD; $\therefore$ ABCD is a parallelogram.

§ 16. Parallelograms on the same (or equal) bases, and between the same equal and parallel small circles, are equal in area. (Fig. 2.)

Let ABCD, A′B′CD be the two parallelograms.

Then triangles A′DA, B′CB are equal, &c. (as in Euclid I. 35).

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Cor. 1.—Since the diagonal bisects a parallelogram, triangles on the same base and between the same parallel small circles are equal in area.

Cor. 2. From this it follows at once, that the locus of the vertex of a triangle of constant area on a fixed base is a small circle. (Lexell's Theorem.)

In order to find this small circle when one of the triangles ABC (fig. 4) is given, through B and C and through A two parallel and equal small circles must be drawn. Let O be the centre of the circle circumscribed to A′BC, O′ the point antipodal to O; then a circle with O′ as centre and O′A as radius is equal and parallel to the circle circumscribed to A′BC, and is the required locus.

Lexell's Theorem may also be proved as follows:—

If ABC be a triangle on a fixed base BC, and inscribed in a fixed small circle, then $B+C-A$ is constant. (Fig. 5.)

[This is the analogue to Euclid III. 21, to which reference was made in § 10.]

Let BAC, BA′C be two of the triangles.

Then $ABC+ACB-BAC=A'BC+A'CB-BA'C$;

if $\quad BAC-ABA'=BA'C-ACA'$;

if $\quad CAO+A'BO=BA'O+ACO$; which is true.

Now, let BAC (fig. 4) be one of the triangles of given area of Lexell's Theorem. Circumscribe a circle to BA′C, and let A′ move along this circle. Then $A'BC+A'CB-A'=$constant.

$\therefore\ \pi-ABC+\pi-ACB-A=$constant.

$\therefore\ ABC+ACB+BAC=$constant.

$\therefore$ the area of ABC is constant; and A moves along the figure antipodal to the circle circumscribed to BA′C, that is, an equal and parallel circle.

Part of the circle is not included in the locus; for if tangents be drawn from C and D (fig. 3) to the small circle A′B′AB meeting the circle in C′ and D′, one at least of the lines joining any point between C′ and D′ to C and D must cut the circle in another point. Hence C′D′ is excluded from the locus. Since C′ and D′ are points antipodal to C and D, $C'D'=CD$.

This is also evident from the second method of proving Lexell's theorem, since the loci for a number of triangles of different areason the same base are a number of circles all passing through the two points antipodal to the extremities of the common base.

§ 17. All triangles formed with CD as base and vertex in C′D′ are equal; and one of these triangles, together with one of the other set of equal triangles, forms half of the surface of the sphere.

Let BAC and BDC (fig. 5) be two triangles on the same base BC, but on opposite sides of it, and inscribed in the same circle.

Then $BAC + BDC = ABC + ACB + DBC + DCB$.

Now let AB and AC be produced to form a triangle, and also DB and DC; and let the angles of these triangles be A, B, C and A′, B′, C′ respectively.

Then from the above equality—

$$A + A' = \pi - B + \pi - C + \pi - B' + \pi - C';$$

$\therefore A + B + C - \pi + A' + B' + C' - \pi = 2\pi = \frac{1}{2}$ surface of sphere;

and these two triangles have their vertices on the circle antipodal to the circle BACD. (The two triangles are on opposite sides of the sphere).

§ 18. The polar of Lexell's Theorem. If one angle of a triangle be fixed in position, and the sum of the sides containing this angle be constant, the side opposite the fixed angle will envelope a circle.

§ 19. If two sides of a triangle be given, the area is a maximum when the angle contained by the two given sides is equal to the sum of the other two. (Fig. 6.)

Let the side AB be supposed fixed, and the triangle to vary by change of the position of AC, the other given side.

Then the locus of the vertices of triangles of given area is a circle whose centre lies on OO′, the perpendicular bisector of AB; and the area will be greater the further the circle is from AB. Hence for a given length of AC the area is greatest when AC produced passes through the centre of the circle, as in the figure.

Let OO′ meet the circle in D; CA meet OO′ in E; produce DA and DB to meet at D′; and let O be the centre of the circle circumscribed to AD′B. Hence O′A passes through O.

Again $OA = O'D$; $\therefore \angle ODA = \angle O'AD$.

But since $\triangle ADB = \triangle ACB$,

$\therefore ADB + DAB + ABD = ACB + CAB + ABC$;

$\therefore \quad 2(DAE + EDA) = ACB + DAC + DAE$;

$\therefore \quad 2(DAE + DAC) = ACB + DAC + DAE$;

$\therefore EAC = ECA$. $\therefore BAC = ABC + BCA$.

§ 20. The perpendicular bisectors of the sides and the bisectors of the angles of a triangle are concurrent.

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§ 21. The internal points of bisection of any two sides of a triangle, and the external point of bisection of the remaining side (and also the three external points of bisection), are collinear.

This is the polar of the theorem that two external and one internal bisector of the angles of a triangle are concurrent.

A direct proof may also be given.

§ 22. The perpendiculars from the vertices of a triangle on the opposite sides are concurrent. (Fig. 7).

Let ABC be the triangle; AD, CF perpendicular to BC, AB. Draw AB′, CB′ perpendicular to AD, CF.
Make AC′ = AB′; CA′ = CB′; and join C′A′; bisect C′A′ in E.

Then, since A, C, E, are the middle points of the sides of A′B′C′, CE, if produced, will meet B′C′ in its external point of bisection, that is, in the pole of the line ADD′. ∴ CD′ is perpendicular to AD; ∴ C is the pole of ADD′; ∴ CA is a quadrant.

Hence, if CA be not a quadrant, E must coincide with B; and as CA may be any one of the sides, it is always possible to form a triangle such that A, B, C shall be the middle points of its sides, unless the sides of the triangle ABC be all quadrants. Now, the perpendiculars of the triangle ABC are the perpendicular bisectors of the sides of A′B′C′, and are therefore concurrent. If two sides, BA and BC say, are quadrants, B is the pole of AC; and since any line from B is, in that case, perpendicular to AC, it is not necessarily perpendicular to A′C′, and the theorem does not hold.

Cor.—The points of intersection of CA and C′A′, AB and A′B′, BC and B′C′, are collinear.

The theorem of § 22 may also be stated as follows:—In a complete quadrangle, if two diagonal angles be right angles, the third must also be a right angle.

§ 23. If two diagonals of a complete quadrilateral be quadrants, the third must also be a quadrant.

This is the polar of the theorem of § 22, according to the second statement of that theorem.

The following direct proof may also be given.

Let ACKH (fig. 8) be the quadrilateral, AK and CH being quadrants. Draw the perpendiculars of the triangle ABC.

Then O is the pole of HKL, and BO is perpendicular to AL;
∴ L is the pole of BO; ∴ BL is a quadrant.

§ 24. The perpendiculars from the vertices on the opposite sides of a triangle bisect the angles of the triangle formed by joining the feet of the perpendiculars. (Proof by means of the polar figure).

Let ABC be the polar triangle, L, M, N, the poles of the perpendiculars in the original triangle, *i.e.*, L is a point in BC such that LA is a quadrant, &c. Then DEF is the polar of the triangle formed by joining the feet of the perpendiculars in the original triangle; and it is required to show that L, M, N bisect the sides of DEF externally.

L, M, N are the poles of the perpendiculars of ABC;
∴ A, B, C are the middle points of the sides of DEF (§ 22).
∴ L, M, N bisect the sides externally.

Although not strictly within the scope of this paper, the following proof of the theorem of § 23 may be interesting.

Let ABCD (fig. 10) be the quadrilateral, AC and BD being quadrants. Then $(AGCK) = -1$, and AC is a quadrant; ∴ $GC = CK$. Similarly, $GB = BL$.

Now, in the triangle LGK, B bisects LG internally, and A bisects GK externally; ∴ E bisects LK. And from triangle GLK F bisects LK externally; ∴ EF is a quadrant.

Note on the Condensation of a Special Continuant.

By Thomas Muir, M.A., F.R.S.E.

[*Held over from Third Meeting.*]

§ 1. The continuant referred to is that in which the elements of the main diagonal are all equal (to x, say), the elements of the one minor diagonal all equal (to b, say), and the elements of the other minor diagonal all equal (to c, say). It may be denoted by $F(b, x, c, n)$ when it is of the nth order. Professor Wolstenholme has recently given two elegant theorems regarding the condensation of $F(1, x, 1, n)$. I wish to establish the analogous theorems for $F(b, x, c, n)$.

§ 2. It may be necessary to premise that a determinant whose elements are all zeros, except those in the main diagonal and in the two diagonals drawn through the places (1, 3), (3, 1) parallel to the main diagonal, is expressible as the product of two continuants. Thus