

SECOND ORDER DEHN FUNCTIONS AND *HNN*-EXTENSIONS

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Dedicated to M. F. (Mike) Newman on the occasion of his 65th birthday

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Abstract

In previous work [2] calculations of subquadratic second order Dehn functions for various groups were carried out. In this paper we obtain estimates for upper and lower bounds of second order Dehn functions of *HNN*-extensions, and use these to exhibit an infinite number of different superquadratic second order Dehn functions. At the end of the paper several open questions concerning second order Dehn functions of groups are discussed.

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1. Introduction

Second (and higher) order Dehn functions have been studied in [2, 3, 10, 11, 13, 15, 17]. In particular, in [2] exact calculations of some (subquadratic) second order Dehn functions $\delta_G^{(2)}$ for various groups G were carried out. For any positive integer r examples were given of groups with second order Dehn function

$$n \mapsto n^{2-1/r}.$$

Also, examples were given of groups with second order Dehn function

$$n \mapsto \frac{n^2}{\log n}.$$

Recent work in [7], coupled with results in [2, Section 6] show that for any positive integers $p \geq q$ there is a group with second order Dehn function

$$n \mapsto n^{2-1/\rho}$$

where $\rho = 2 \log_2(2p/q)$; in particular the set

$$\{\alpha : \alpha \in [\frac{3}{2}, 2], n \mapsto n^\alpha \text{ is a second order Dehn function}\}$$

is dense in $[3/2, 2]$ ([7, Theorem C]).

In this paper we obtain estimates for upper and lower bounds of second order Dehn functions of *HNN*-extensions (Theorem 1 and Theorem 2). (We remark that similar estimates for general graphs of groups can be obtained.) We then obtain estimates for certain special *HNN*-extensions (Theorem 3), and use these to exhibit an infinite number of different superquadratic second order Dehn functions. For any positive integer r we give an example (Example 1) of a group G_r with $\delta_{G_r}^{(2)}$ lying between the functions

$$n \mapsto n^{r/2+1}, \quad n \mapsto n^{r+1}.$$

Also, we give an example (Example 2) of a group G_∞ with $\delta_{G_\infty}^{(2)}$ lying between the functions

$$n \mapsto e^{\sqrt{n}}, \quad n \mapsto e^n.$$

At the end of the paper we raise some open questions concerning second order Dehn functions.

We refer the reader to [2, Section 2] (see also [3, 15, 17]) for information concerning second order Dehn functions, and for all unexplained notations. However, as suggested by the referee, we include here some of the main ideas for the readers convenience.

Let $\mathcal{P} = \langle \mathbf{x}; \mathbf{r} \rangle$ be a finite group presentation. The group $G(\mathcal{P})$ defined by \mathcal{P} is the quotient of the free group $F(\mathbf{x})$ on \mathbf{x} by the normal closure of the elements of $F(\mathbf{x})$ represented by the words in \mathbf{r} . If W is a word on \mathbf{x} then the element of $G(\mathcal{P})$ represented by W will be denoted by \overline{W} , and the word length of W will be denoted by $L(W)$. Usually we will blur the distinction between a word and the element of $F(\mathbf{x})$ it represents.

Diagrams, or their duals, *pictures*, over presentations are a well-established concept. In this paper we will use pictures and adopt [6] as our main reference (see also [14]).

If W is a word representing the identity of $G(\mathcal{P})$ then the *area* $A(W)$ of W (with respect to \mathbf{r}) is the minimum area (that is, the number of discs) of any picture over \mathcal{P} with boundary label W . The *first order Dehn function* $\delta_{\mathcal{P}}^{(1)}$ is defined by

$$\delta_{\mathcal{P}}^{(1)}(n) = \max\{A(W); \overline{W} = 1, L(W) \leq n\}.$$

As is well-known (see [1] for example) if \mathcal{Q} is another finite presentation with $G(\mathcal{P}) \cong G(\mathcal{Q})$, then for a certain standard equivalence \sim of functions (see below) we have $\delta_{\mathcal{P}}^{(1)} \sim \delta_{\mathcal{Q}}^{(1)}$. Thus up to \sim -equivalence, for a finitely presented group G we have the (well-defined) first order Dehn function $\delta_G^{(1)}$ of G .

The equivalence \sim is defined as follows. For two increasing functions

$$f, g : \mathbb{N} \rightarrow \mathbb{R}^+$$

write $f \preceq g$ if there is a constant $a > 0$ such that $f(n) \leq ag(an) + an$ ($n \in \mathbb{N}$). Then $f \sim g$ if $f \preceq g$ and $g \preceq f$.

There are certain operations (bridge moves, insertions or deletions of folding pairs, insertions or deletions of floating circles) which can be performed on (spherical) pictures (see [6, Section 1.2]), and two spherical pictures are said to be equivalent if one can be obtained from the other by a finite number of these operations. We write $\langle \mathbb{P} \rangle_{\mathcal{P}}$ (or simply $\langle \mathbb{P} \rangle$) for the equivalence class of the spherical picture \mathbb{P} . These equivalence classes form a left $\mathbb{Z}G(\mathcal{P})$ -module (the *second homotopy module* of \mathcal{P} , denoted $\pi_2(\mathcal{P})$) under the addition

$$\langle \mathbb{P}_1 \rangle + \langle \mathbb{P}_2 \rangle = \langle \mathbb{P}_1 + \mathbb{P}_2 \rangle$$

(where $\mathbb{P}_1 + \mathbb{P}_2$ is the spherical picture obtained by putting \mathbb{P}_1 next to \mathbb{P}_2), with $G(\mathcal{P})$ -action

$$\overline{W} \cdot \langle \mathbb{P} \rangle = \langle W \cdot \mathbb{P} \rangle$$

(where $W \cdot \mathbb{P}$ is the picture obtained from \mathbb{P} by surrounding it with a collection of concentric closed arcs with total label W —see [6, Figure V.6.(b)]).

Let \mathbf{X} be a set of generators of the module $\pi_2(\mathcal{P})$. Then if \mathbb{P} is a spherical picture we define the *volume* $V_{\mathbf{X}}(\langle \mathbb{P} \rangle)$ of $\langle \mathbb{P} \rangle$ with respect to \mathbf{X} to be the least value of m over all expressions

$$\sum_{i=1}^m \varepsilon_i g_i \zeta_i$$

($\varepsilon_i = \pm 1, g_i \in G(\mathcal{P}), \zeta_i \in \mathbf{X}, i = 1, 2, \dots, m$) equal to $\langle \mathbb{P} \rangle$. We then define the *second order Dehn function* $\delta_{\mathcal{P}, \mathbf{X}}^{(2)}$ by

$$\delta_{\mathcal{P}, \mathbf{X}}^{(2)}(n) = \max\{V_{\mathbf{X}}(\langle \mathbb{P} \rangle) : A(\mathbb{P}) \leq n\} \quad (n = 1, 2, \dots).$$

For a group G of type F_3 (that is, a group given by a finite presentation for which π_2 is finitely generated) this is a group invariant up to \sim -equivalence [3] (see also [15, 17]), which we write as $\delta_G^{(2)}$ —the second order Dehn function of G .

2. Preliminaries

Consider an HNN-extension

$$G = \langle H, t; t^{-1} K t \stackrel{\psi}{=} \hat{K} \rangle.$$

Here K, \hat{K} are subgroups of the group H , and

$$\psi : K \rightarrow \hat{K}$$

is an isomorphism. We will assume that H is of type F_3 and K, \hat{K} are finitely presented.

Choose a finite presentation $\mathcal{P}_H = \langle \mathbf{x}; \mathbf{r} \rangle$ for H . Let $\mathbf{a} = \{a_y : y \in \mathbf{y}\}$, $\hat{\mathbf{a}} = \{\hat{a}_y : y \in \mathbf{y}\}$ be finite sets of words on \mathbf{x} representing generators of K, \hat{K} respectively, where \hat{a}_y represents the image under ψ of the element of K represented by a_y . Let $F = F(\mathbf{y})$ be the free group on \mathbf{y} , and let B be the kernel of the epimorphism $F \rightarrow K$ induced by the mapping $y \mapsto a_y$ ($y \in \mathbf{y}$). Let \mathbf{s} be a finite set of words whose normal closure in F is B . Then

$$\mathcal{P}_K = \langle \mathbf{y}; \mathbf{s} \rangle$$

is a presentation of K .

For W a word on \mathbf{y} , say $W = y_1^{\varepsilon_1} y_2^{\varepsilon_2} \cdots y_n^{\varepsilon_n}$ ($y_i \in \mathbf{y}$, $\varepsilon_i = \pm 1$, $1 \leq i \leq n$), we let $W(\mathbf{a})$ (respectively, $W(\hat{\mathbf{a}})$) denote the word on \mathbf{x} obtained by replacing each y_i by a_{y_i} (respectively, \hat{a}_{y_i}). For $y \in \mathbf{y}$ let Q_y denote the word $a_y t \hat{a}_y^{-1} t^{-1}$ and let $\mathbf{q} = \{Q_y : y \in \mathbf{y}\}$. Then a presentation for G is given by

$$(1) \quad \mathcal{P} = \mathcal{P}_G = \langle \mathbf{x}, t; \mathbf{r}, \mathbf{q} \rangle.$$

For $S \in \mathbf{s}$ we let $\mathbb{S}, \hat{\mathbb{S}}$ be pictures over \mathcal{P}_H with (clockwise) boundary labels $S(\mathbf{a}), S(\hat{\mathbf{a}})$ respectively, and we set

$$c = \max\{A(\mathbb{S}), A(\hat{\mathbb{S}}) : S \in \mathbf{s}\}.$$

The abelianization $B^{ab} = B/B'$ (written additively) of B is a left $\mathbb{Z}K$ -module (called the *relation module* of \mathcal{P}_K) under the action

$$a_y \cdot WB' = yWy^{-1}B' \quad (y \in \mathbf{y}, W \in B).$$

This module is generated by the elements SB' ($S \in \mathbf{s}$). Given $W \in B$ we can thus write WB' as a sum of elements $\pm kSB'$ ($k \in K, S \in \mathbf{s}$). We define $A^{ab}(W)$ to be the minimum number of terms in any such sum equal to WB' . This is the ‘abelianized area’, considered in [5]. In general, $A^{ab}(W) \leq A(W)$.

Let \mathbb{D} be a picture over \mathcal{P}_K with discs $\Delta_1, \dots, \Delta_m$ labelled clockwise by $S_1^{\varepsilon_1}, \dots, S_m^{\varepsilon_m}$ respectively ($S_i \in \mathfrak{s}$, $\varepsilon_i = \pm 1$, $1 \leq i \leq m$). We let $\mathbb{D}(\mathbf{a})$ (respectively, $\mathbb{D}(\hat{\mathbf{a}})$) denote the picture over \mathcal{P}_H obtained from \mathbb{D} by replacing the disc Δ_i ($1 \leq i \leq m$) by the picture $\varepsilon_i S_i$ (respectively, $\varepsilon_i \hat{S}_i$), and for each $y \in \mathfrak{y}$ replacing all arcs labelled y by a sequence of parallel arcs with total label a_y (respectively \hat{a}_y).

We let $\mathbb{P}_{\mathbb{D}}$ denote the spherical picture over \mathcal{P} obtained from $\mathbb{D}(\mathbf{a})$, $\mathbb{D}(\hat{\mathbf{a}})$, as depicted in Figure 1, and we let $\xi_{\mathbb{D}}$ denote the element of $\pi_2(\mathcal{P})$ represented by $\mathbb{P}_{\mathbb{D}}$. When \mathbb{D} consists of a single disc labelled $S \in \mathfrak{s}$ then we write \mathbb{P}_S (respectively ξ_S) instead of $\mathbb{P}_{\mathbb{D}}$ (respectively $\xi_{\mathbb{D}}$).

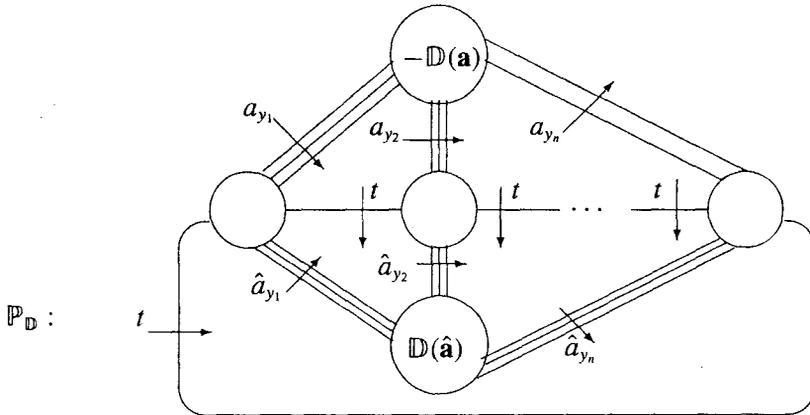


FIGURE 1.

Let W be the label of \mathbb{D} . Choose a spray (see [14] for the definition) $(\gamma_1, \dots, \gamma_m)$ for \mathbb{D} and let U_i be the label on γ_i ($i = 1, \dots, m$). Then W is freely equal to

$$\prod_{i=1}^m U_i S_i^{\varepsilon_i} U_i^{-1}.$$

It follows that in the relation module B^{ab} we have

$$WB' = \sum_{i=1}^m \varepsilon_i \bar{U}_i S_i B'.$$

LEMMA 1. *If \mathcal{P}_K is aspherical and if for each $S \in \mathfrak{s}$ there is no cancellation in the sum*

$$\sum_{S_i=S} \varepsilon_i \bar{U}_i,$$

then $A^{ab}(W) = m$.

PROOF. If \mathcal{P}_K is aspherical then B^{ab} is freely generated as a $\mathbb{Z}K$ -module by the elements SB' ($S \in \mathfrak{s}$). Thus, as an abelian group B^{ab} is generated freely by the elements kSB' ($k \in K, S \in \mathfrak{s}$), and the result is then immediate. \square

LEMMA 2. $\xi_{\mathbb{D}} = \sum_{i=1}^m \varepsilon_i \overline{U_i(\mathbf{a})} \xi_S.$

The proof is similar to the proof of Lemma 5.1 of [2] and is left to the reader.

3. Generators of the second homotopy module and upper bound for volume

Let $\mathbb{B}_1, \dots, \mathbb{B}_r$ be a collection of spherical pictures over \mathcal{P}_H representing a set \mathbf{X}_H of generators of $\pi_2(\mathcal{P}_H)$. Then each \mathbb{B}_i also represents an element of $\pi_2(\mathcal{P})$. We let \mathbf{X} denote the set of elements of $\pi_2(\mathcal{P})$ represented by $\mathbb{B}_1, \dots, \mathbb{B}_r$, together with the elements ξ_S ($S \in \mathfrak{s}$).

THEOREM 1. *The set \mathbf{X} generates $\pi_2(\mathcal{P})$, and if \mathbb{P} is a spherical picture over \mathcal{P} with n discs, of which l are \mathbf{q} -discs, then*

$$V_{\mathbf{X}}(\langle \mathbb{P} \rangle) \leq \overline{\delta}_{\mathcal{P}_H, \mathbf{X}_H}^{(2)} (2c\overline{\delta}_{\mathcal{P}_K}^{(1)}(l) + (n - l)) + \overline{\delta}_{\mathcal{P}_K}^{(1)}(l),$$

and hence

$$\delta_G^{(2)}(n) \leq \overline{\delta}_{\mathcal{P}_H, \mathbf{X}_H}^{(2)} (2c\overline{\delta}_{\mathcal{P}_K}^{(1)}(n) + n) + \overline{\delta}_{\mathcal{P}_K}^{(1)}(n).$$

(Here, as usual, for a function $\delta, \overline{\delta}$ denotes the subnegative closure, see [2, Section 2].)

PROOF. Let \mathbb{P} be a spherical picture over \mathcal{P} . A non-trivial t -circle (outward or inward directed) in \mathbb{P} consists of a collection of \mathbf{q} -discs $\Theta_1, \dots, \Theta_k$ ($k > 0$) and a collection of t -arcs $\alpha_1, \dots, \alpha_k$ where α_i joins Θ_{i-1} to Θ_i (subscripts mod k). We can assume that the numbering of the discs is such that they are encountered in the order $\Theta_1, \Theta_2, \dots, \Theta_k$ where we read clockwise around the t -circle starting at Θ_1 . Suppose Θ_i is labelled by $Q_{y_i}^{\varepsilon_i}$ ($y_i \in \mathbf{y}, \varepsilon_i = \pm 1$). Then the label on the t -circle is the word $y_1^{\varepsilon_1} y_2^{\varepsilon_2} \dots y_k^{\varepsilon_k}$. We also allow a trivial t -circle consisting of a single closed t -arc. The label on a trivial t -circle is the empty word. A t -circle C is minimal if there is no t -circle contained in the region enclosed by C .

The proof will be by induction on the number k of t -circles in \mathbb{P} .

If $k = 0$ then \mathbb{P} is a picture over \mathcal{P}_H , so $\langle \mathbb{P} \rangle_{\mathcal{P}_H}$ can be expressed as a sum of at most $\overline{\delta}_{\mathcal{P}_H, \mathbf{X}_H}^{(2)}(n)$ elements of $\pm H\mathbf{X}_H$. Thus $\langle \mathbb{P} \rangle$ can be expressed as a sum of at most $\overline{\delta}_{\mathcal{P}_H, \mathbf{X}_H}^{(2)}(n)$ elements of $\pm G\mathbf{X}$.

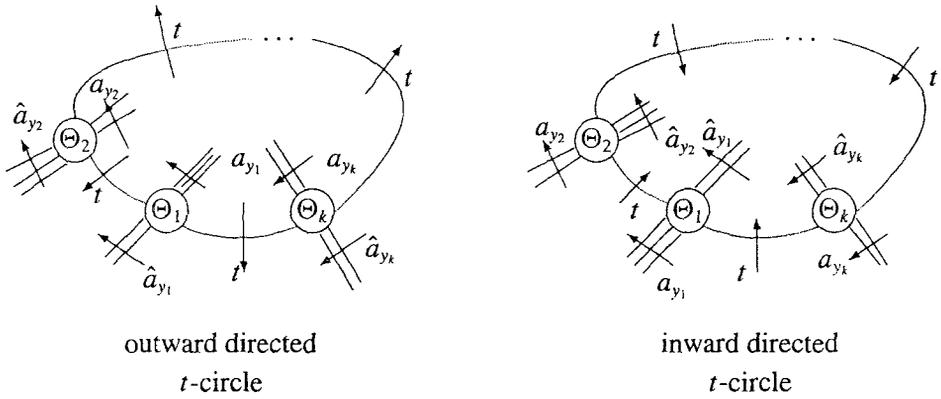


FIGURE 2.

Suppose $k > 0$ and let C be a minimal t -circle of \mathbb{P} with label W . Let us first assume that C is inward directed. Let A_0 be the subpicture of \mathbb{P} lying inside the area enclosed by C , and let A_1 be the geometric configuration obtained by deleting A_0 and C from \mathbb{P} . Let \mathbb{D}_W be a picture over \mathcal{S}_K with boundary label W and having $A(W)$ discs. Then A_0 and $-\mathbb{D}_W(\hat{\mathbf{a}})$ can be combined to form a spherical picture \mathbb{P}_H over \mathcal{S}_H , and A_1 and $\mathbb{D}_W(\mathbf{a})$ can be combined to form a spherical picture \mathbb{P}' over \mathcal{S} . We then have for some word U (see Figure 3)

$$(2) \quad \langle \mathbb{P} \rangle = \langle \mathbb{P}' \rangle + \overline{U}(\xi_{\mathbb{D}_W} + \bar{t}(\mathbb{P}_H)).$$

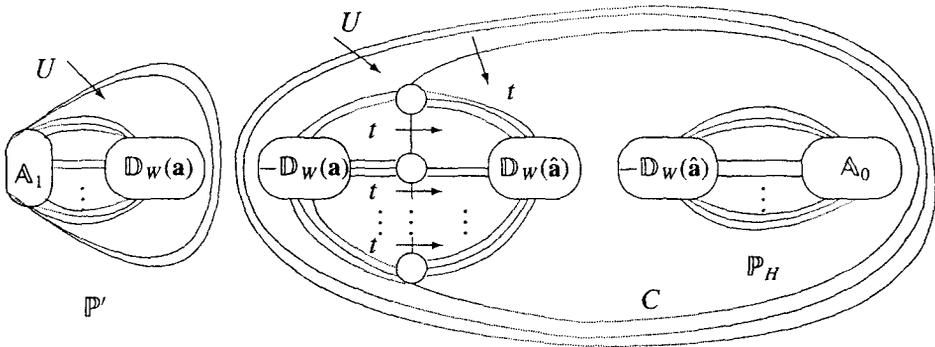


FIGURE 3.

Now \mathbb{P}' has fewer t -circles than \mathbb{P} . Moreover, the area of \mathbb{P}' is at most

$$n - L(W) - A(A_0) + cA(W)$$

and the number of \mathbf{q} -discs in \mathbb{P}' is $l - L(W)$. Thus by inductive hypothesis $\langle \mathbb{P}' \rangle$ can be expressed as a sum of at most

$$(3) \quad \bar{\delta}_{\mathcal{P}_H, \mathbf{X}_H}^{(2)} (2c\bar{\delta}_{\mathcal{P}_K}^{(1)} (l - L(W)) + n - l - A(\mathbb{A}_0) + cA(W)) + \bar{\delta}_{\mathcal{P}_K}^{(1)} (l - L(W))$$

elements of $\pm GX$.

By Lemma 2, $\xi_{\mathbb{D}_W}$ (and hence $\bar{U}\xi_{\mathbb{D}_W}$) can be expressed as a sum of $A(W)$ elements of $\pm GX$, so $V_X(\xi_{\mathbb{D}_W})$ is at most

$$(4) \quad \bar{\delta}_{\mathcal{P}_K}^{(1)} (L(W)).$$

Finally, since \mathbb{P}_H is a spherical picture over \mathcal{P}_H , and since $A(\mathbb{P}_H) \leq A(\mathbb{A}_0) + cA(W)$, we have that $\langle \mathbb{P}_H \rangle$ (and hence $\bar{U}\bar{t}(\mathbb{P}_H)$) can be expressed as a sum of at most

$$(5) \quad \bar{\delta}_{\mathcal{P}_H, \mathbf{X}_H}^{(2)} (A(\mathbb{A}_0) + cA(W))$$

elements of $\pm GX$.

Adding (3), (4), (5) and using the fact that $A(\mathbb{D}_W) \leq \bar{\delta}_{\mathcal{P}_K}^{(1)} (L(W))$, we find that $\langle \mathbb{P} \rangle$ can be expressed as a sum of at most

$$\bar{\delta}_{\mathcal{P}_H, \mathbf{X}_H}^{(2)} (2c\bar{\delta}_{\mathcal{P}_K}^{(1)} (l) + (n - l)) + \bar{\delta}_{\mathcal{P}_K}^{(1)} (l)$$

elements of $\pm GX$, as required.

If C is outward directed then a similar argument applies except now (2) becomes

$$\langle \mathbb{P} \rangle = \langle \mathbb{P}' \rangle - \bar{U}\bar{t}^{-1}\xi_{\mathbb{D}_W} + \bar{U}\bar{t}^{-1}\langle \mathbb{P}_H \rangle,$$

where \mathbb{P}_H is the result of combining \mathbb{A}_0 and $-\mathbb{D}_W(\mathbf{a})$, and \mathbb{P}' is the result of combining \mathbb{A}_1 and $\mathbb{D}_W(\hat{\mathbf{a}})$. □

REMARK. Theorem 1 can easily be extended to multiple extensions of H (where we allow several pairs of isomorphic subgroups of H). More generally, Theorem 1 can be extended to graphs of groups. Let Γ be a finite connected directed graph with vertex set \mathbf{v} and directed edge set \mathbf{e} . Suppose that for each vertex $v \in \mathbf{v}$ we have a group H_v of type F_3 , and for each directed edge e running from vertex u to vertex v say, we have isomorphic finitely presented subgroups $K_e \leq H_u, \hat{K}_e \leq H_v$. Let G be the corresponding fundamental group of this graph of groups. Then we get an upper bound for $\delta_G^{(2)}$ like that at the end of Theorem 1, with $\delta_{\mathcal{P}_H, \mathbf{X}_H}^{(2)}, \delta_{\mathcal{P}_K}^{(1)}$ replaced by

$$\max \left\{ \delta_{\mathcal{P}_{H_v}, \mathbf{X}_{H_v}}^{(2)} : v \in \mathbf{v} \right\}, \quad \max \left\{ \delta_{\mathcal{P}_{K_e}}^{(1)} : e \in \mathbf{e} \right\}$$

respectively.

4. Lower bound for volume

Let L be the submodule of $\pi_2(\mathcal{P})$ generated by the elements $\langle \mathbb{B}_1 \rangle, \dots, \langle \mathbb{B}_r \rangle$. Let μ be the standard embedding (see [2, Section 2]) of $\pi_2(\mathcal{P})$ into the free left $\mathbb{Z}G$ -module with basis e_r ($R \in \mathbf{r}$), e_y ($y \in \mathbf{y}$) in one-to-one correspondence with the defining relators of \mathcal{P} . Since $\mu(L) \subseteq \bigoplus_{R \in \mathbf{r}} \mathbb{Z}Ge_r$, we get an induced homomorphism

$$\mu^* : \frac{\pi_2(\mathcal{P})}{L} \longrightarrow \bigoplus_{y \in \mathbf{y}} \mathbb{Z}Ge_y.$$

We have the Magnus embedding κ (see [12, page 199]) of the relation module B^{ab} of \mathcal{P}_K into the free left $\mathbb{Z}K$ -module $\bigoplus_{y \in \mathbf{y}} \mathbb{Z}Ke_y$. By applying the exact functor $\mathbb{Z}G \otimes_{\mathbb{Z}K} \dashv$ we then obtain an embedding

$$\tilde{\kappa} : \mathbb{Z}G \otimes_{\mathbb{Z}K} B^{ab} \xrightarrow{1 \otimes \kappa} \mathbb{Z}G \otimes_{\mathbb{Z}K} \left(\bigoplus_{y \in \mathbf{y}} \mathbb{Z}Ke_y \right) \cong \bigoplus_{y \in \mathbf{y}} \mathbb{Z}Ge_y$$

and we have the commutative diagram

$$\begin{array}{ccc} B^{ab} & \xrightarrow{\kappa} & \bigoplus_{y \in \mathbf{y}} \mathbb{Z}Ke_y \\ \downarrow \scriptstyle WB' \mapsto 1 \otimes WB' & & \downarrow \scriptstyle \text{inclusion} \\ \mathbb{Z}G \otimes_{\mathbb{Z}K} B^{ab} & \xrightarrow{\tilde{\kappa}} & \bigoplus_{y \in \mathbf{y}} \mathbb{Z}Ge_y \end{array}$$

Now $\pi_2(\mathcal{P})/L$ is generated by the elements $\xi_S + L$ ($S \in \mathbf{s}$), and $\mathbb{Z}G \otimes_{\mathbb{Z}K} B^{ab}$ is generated by the elements $\beta_S = 1 \otimes SB'$ ($S \in \mathbf{s}$). Since

$$\mu^*(\xi_S + L) = \tilde{\kappa}(\beta_S) \quad (S \in \mathbf{s})$$

we have that $\text{Im } \mu^* = \text{Im } \tilde{\kappa}$, so we get a surjective homomorphism

$$\theta : \pi_2(\mathcal{P}) \xrightarrow{\text{nat}} \frac{\pi_2(\mathcal{P})}{L} \xrightarrow{\tilde{\kappa}^{-1} \mu^*} \mathbb{Z}G \otimes_{\mathbb{Z}K} B^{ab}.$$

Let W be any word on \mathbf{y} which defines an element of B . Consider a minimal expression

$$(6) \quad \sum_{j=1}^n \varepsilon_j g_j \beta_{S_j}$$

($\varepsilon_j = \pm 1$, $g_j \in G$, $S_j \in \mathbf{s}$ for $j = 1, \dots, n$) for $1 \otimes WB'$ as a sum of elements from the set $\{\pm g\beta_S : g \in G, S \in \mathbf{s}\}$. Clearly $n \leq A^{ab}(W)$. We want to show that in fact

equality holds. We can suppose that for some $0 \leq l \leq n$, $g_j \in K$ if and only if $j \leq l$. Now $\tilde{\kappa}(1 \otimes WB')$ and $\tilde{\kappa}(\beta_S)$ ($S \in \mathfrak{s}$) belong to $\bigoplus_{y \in Y} \mathbb{Z}Ke_y$ and so we deduce that

$$\tilde{\kappa} \left(\sum_{j>l} \varepsilon_j g_j \beta_{S_j} \right) = 0.$$

Hence, by the injectivity of $\tilde{\kappa}$ and the minimality of (6), $n = l$, that is, each $g_j \in K$. Thus we have

$$\kappa(WB') = \tilde{\kappa}(1 \otimes WB') = \tilde{\kappa} \left(\sum_{j=1}^n \varepsilon_j g_j \beta_{S_j} \right) = \kappa \left(\sum_{j=1}^n \varepsilon_j g_j S_j B' \right).$$

So (using the injectivity of κ), we deduce that $n \geq A^{ab}(W)$, as required.

Let $\mathbb{E}, \hat{\mathbb{E}}$ be any pictures over \mathcal{P}_H with boundary labels $W(\mathbf{a}), W(\hat{\mathbf{a}})$ respectively. Then we can form a spherical picture as in Figure 1 with $\mathbb{D}(\mathbf{a}), \mathbb{D}(\hat{\mathbf{a}})$ replaced by $\mathbb{E}, \hat{\mathbb{E}}$. For simplicity we denote the element of $\pi_2(\mathcal{P})$ represented by this picture by ζ_w (though note that ζ_w depends on the choice of $\mathbb{E}, \hat{\mathbb{E}}$).

LEMMA 3. $V_X(\zeta_w) \geq A^{ab}(W)$.

PROOF. Suppose we have an expression for $\zeta = \zeta_w$ as a sum of $V_X(\zeta)$ terms from $\pm GX$. Applying θ to this expression we will get an expression for $\theta(\zeta)$ as a sum of at most $V_X(\zeta)$ terms from $\{\pm g\beta_S : g \in G, S \in \mathfrak{s}\}$. But $\theta(\zeta) = 1 \otimes WB'$ and so from our discussion above

$$V_X(\zeta) \geq A^{ab}(W). \quad \square$$

THEOREM 2. Let $f : \mathbb{N} \rightarrow \mathbb{N}, h : [1, \infty) \rightarrow [1, \infty)$ be functions with h strictly increasing. Suppose there are words W_i ($i \in \mathbb{N}$) on y with:

- (a) $L(W_i) \leq h(i)$;
- (b) $A(W_i(\mathbf{a})), A(W_i(\hat{\mathbf{a}})) \leq h(i)$;
- (c) $A^{ab}(W_i) \geq f(i)$

for all i . Then

$$\delta_G^{(2)}(n) \geq f([h^{-1}(n)])$$

(where $[\]$ denotes the integral part).

In the special case when $K = \hat{K}$ and ψ is the identity, if we replace (a) by

- (a') $l(i)L(W_i) \leq h(i)$,

where $l(i)$ is a function of the natural number i , we obtain

$$\delta_G^{(2)}(n) \geq l([h^{-1}(n)])f([h^{-1}(n)]).$$

PROOF. For each i we may choose $\mathbb{E}_i, \hat{\mathbb{E}}_i$ each to have area at most $h(i)$ (by (b)), and then $A(\zeta_{w_i}) \leq 3h(i)$ (using (a)).

Now let $n \in \mathbb{N}$, and let $i = [h^{-1}(n)]$. Then $A(\zeta_{w_i}) \leq 3n$, and by Lemma 3 and (c),

$$V_{\mathbf{x}}(\zeta_{w_i}) \geq A^{ab}(W_i) \geq f([h^{-1}(n)]).$$

Hence,

$$\delta_{\varnothing, \mathbf{x}}^{(2)}(3n) \geq f([h^{-1}(n)]).$$

For the second part, first note that if

$$\zeta_w^{(l)} = (1 + \bar{t} + \dots + \bar{t}^{l-1})\zeta_w, \quad l = 1, 2, \dots,$$

then since powers of \bar{t} all lie in different cosets modulo K , we can modify the argument leading to Lemma 3 to show that

$$V_{\mathbf{x}}(\zeta_w^{(l)}) \geq lA^{ab}(W).$$

Now in the special case when $K = \hat{K}$ and ψ is the identity, $\zeta_w^{(l)}$ can be represented by a small picture, built up by using $-\mathbb{E}, \mathbb{E}$ and l concentric t -circles (similar to Figure 13 in [2]), so that

$$A(\zeta_w^{(l)}) \leq 2A(W(\mathbf{a})) + lL(W).$$

Thus, for this special case, if we replace condition (a) by (a') and modify the argument above we get

$$\delta_{\varnothing, \mathbf{x}}^{(2)}(3n) \geq l([h^{-1}(n)])f([h^{-1}(n)]).$$

□

5. Application

Let F be a finitely generated free group with basis \mathbf{x} , and let ϕ be a positive automorphism of F (that is $\phi(x)$ is a positive word on \mathbf{x} for all $x \in \mathbf{x}$). Form the semi-direct products $K = F \rtimes \langle s, \bar{s} \rangle, A = F \rtimes \langle u \rangle$ with s, \bar{s}, u all acting via ϕ . Let $H = A \times \langle v, \bar{v} \rangle$. There is an embedding of K into H given by

$$F \xrightarrow{id} F, \quad s \mapsto uv, \quad \bar{s} \mapsto u\bar{v},$$

and we can form the *HNN*-extension

$$G = \langle H, t; t^{-1}Kt = K \rangle,$$

which has presentation

$$(7) \quad \mathcal{P} = \langle \mathbf{x}, u, v, \bar{v}, t; u^{-1}xu = \phi(x), [v, x], [\bar{v}, x], [x, t](x \in \mathbf{x}), [v, u], [\bar{v}, u], [uv, t], [u\bar{v}, t] \rangle.$$

Let Z_n ($n = 1, 2, \dots$) be a sequence of positive words on \mathbf{x} such that $L(Z_n) \leq \alpha n$ for some constant α (independent of n), and let

$$f(n) = \sum_{i=0}^{n-1} L(\phi^i(Z_n)).$$

THEOREM 3. $\sqrt{nf}([\sqrt{n}]) \leq \delta_G^{(2)}(n) \leq \bar{\delta}_K^{(1)}(n) \quad (n = 1, 2, 3, \dots)$.

PROOF. By making the substitution $s = uv, \bar{s} = u\bar{v}$ in the presentation \mathcal{P} above, and performing Tietze transformations we find that

$$G \cong K \rtimes \langle u, t \rangle,$$

where u acts via the automorphism $\hat{\phi}$ of K defined by

$$F \xrightarrow{\hat{\phi}} F, \quad s \mapsto s, \quad \bar{s} \mapsto \bar{s},$$

and t acts via the identity automorphism. We can thus regard G as a (multiple) HNN-extension where the base-group and associated subgroups are all equal to K , and can then apply Theorem 1 (for multiple HNN-extensions). Since K has the aspherical presentation

$$(8) \quad \mathcal{P}_K = \langle \mathbf{x}, s, \bar{s}; s^{-1}xs = \phi(x), \bar{s}^{-1}x\bar{s} = \phi(x) \quad (x \in \mathbf{x}) \rangle$$

we have that $\bar{\delta}_{\mathcal{P}_K, \emptyset}^{(2)}$ is the zero function. So we deduce from Theorem 1 that

$$\delta_G^{(2)}(n) \leq \bar{\delta}_K^{(1)}(n).$$

To establish the lower bound we revert to the original presentation (7) of G , and use the second part of Theorem 2 with f as above, $h(\lambda) = (4 + 2\alpha)\lambda^2$ and $l(n) = n$. Note that

$$\mathcal{P}_H = \langle \mathbf{x}, u, v, \bar{v}; u^{-1}xu = \phi(x), [v, x], [\bar{v}, x](x \in \mathbf{x}), [v, u], [\bar{v}, u] \rangle,$$

\mathcal{P}_K is as in (8), and

$$a_x = \hat{a}_x = x \quad (x \in \mathbf{x}), \quad a_s = \hat{a}_s = uv, \quad a_{\bar{s}} = \hat{a}_{\bar{s}} = u\bar{v}.$$

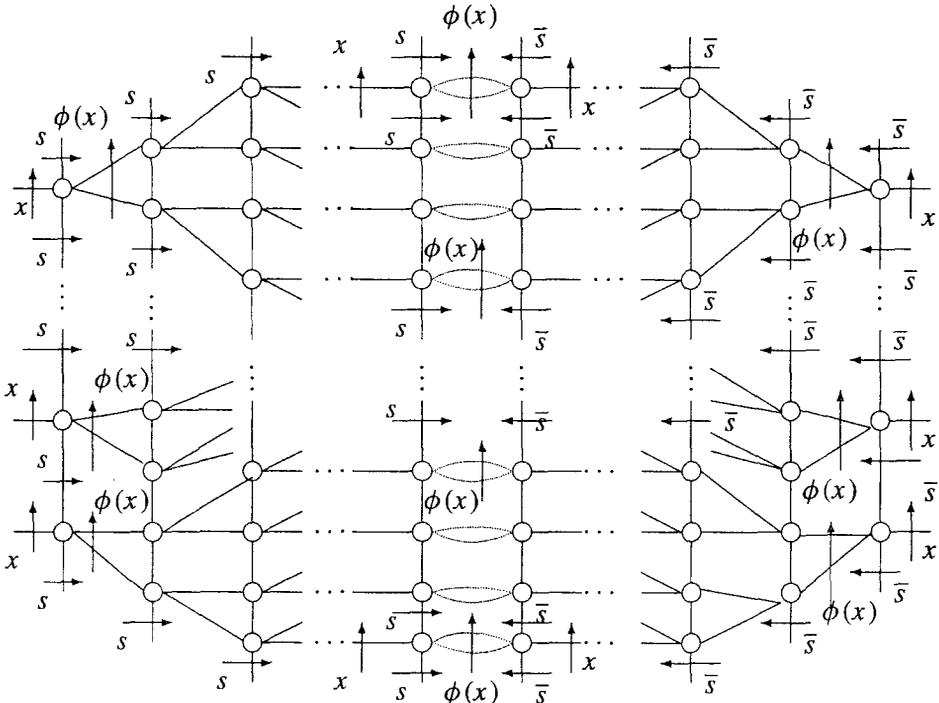


FIGURE 4.

Let $W_n = s^{-n} Z_n s^n \bar{s}^{-n} Z_n^{-1} \bar{s}^n$ ($n = 1, 2, \dots$). Then

$$L(W_n) \leq (4 + 2\alpha)n.$$

We have the picture \mathbb{D}_n with boundary label W_n over \mathcal{P}_K as in Figure 4. Since \mathcal{P}_K is aspherical it follows from Lemma 1 that

$$A^{ab}(W_n) = 2f(n).$$

We also have the picture \mathbb{E}_n with boundary label $W_n(\mathbf{a})$ over \mathcal{P}_H as in Figure 5. This picture has area $2n^2 + 2n(L(Z_n) + 1)$, so

$$A(W_n(\mathbf{a})) \leq 2(2 + \alpha)n^2. \quad \square$$

EXAMPLE 1. Let F be free on x_1, x_2, \dots, x_r and let ϕ be defined by

$$x_i \mapsto x_i x_{i+1} \quad (1 \leq i < r), \quad x_r \mapsto x_r.$$

Let $Z_n = x_1^n$. Then

$$f(n) = n \left[\binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{r} \right],$$

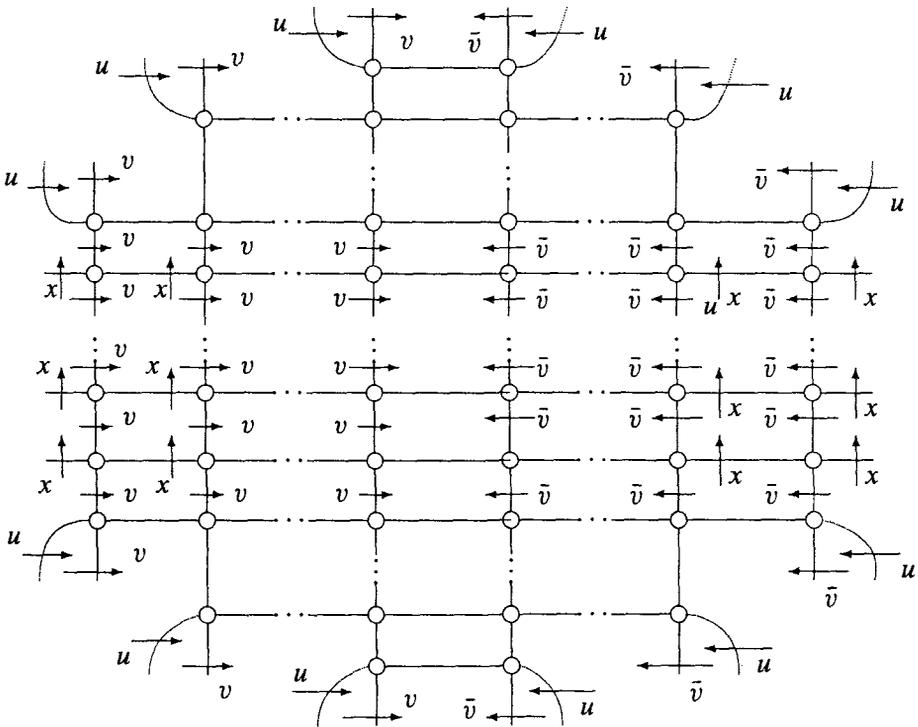


FIGURE 5.

and so it follows that

$$[\sqrt{n}]f([\sqrt{n}]) \sim n^{(r+2)/2}.$$

On the other hand, it has been shown by Bridson [8] that for this situation

$$\delta_K^{(1)}(n) \sim n^{r+1} (\sim \bar{\delta}_K^{(1)}(n)).$$

Thus we deduce that

$$n^{(r+2)/2} \preceq \delta_G^{(2)}(n) \preceq n^{r+1}.$$

EXAMPLE 2. Let F be free of rank 3 on a, b, c and let ϕ be defined by

$$a \rightarrow c, b \rightarrow ac, c \rightarrow bc.$$

Let $Z_n = c$ for all n . Then as shown in [2] (see page 26)

$$f(n) \sim e^n.$$

Also, it is shown in [4, Example 2] that

$$\delta_K^{(1)}(n) \sim e^n (\sim \bar{\delta}_K^{(1)}(n)).$$

Thus we have

$$e^{\sqrt{n}} \leq \delta_G^{(2)}(n) \leq e^n.$$

6. Open questions

The results of this paper combined with those in [2, 7] mentioned in the introduction give some idea of the behaviour of second order Dehn functions of groups, but the overall picture is still far from clear. We mention the following open questions.

(i) Obtain some exact calculations of superquadratic second order Dehn functions.

In particular:

(ii) Is there a group with exactly quadratic second order Dehn function? (Added in proof: Yes—Martin Bridson, to appear.) For the group G_2 mentioned in the introduction, is $\delta^{(2)}$ quadratic?

(iii) Describe the spectrum of second order Dehn functions, that is the set of α for which

$$n \mapsto n^\alpha$$

is a second order Dehn function.

(iv) Characterize groups with linear second order Dehn function. (Clearly groups with aspherical presentations are in this class. It is shown in [2] that hyperbolic groups are also in this class.)

(v) How small can $\delta^{(2)}$ be? The split extension of \mathbb{Z}^2 by \mathbb{Z} with the generator of \mathbb{Z} acting via the automorphism

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

was shown in [16] to have second order Dehn function bounded below by

$$n \mapsto n \log n.$$

It follows from [9] that this lower bound is in fact exact (we thank C. Pittet for pointing this out to us). This is the smallest non-linear second order Dehn function we know of.

(vi) How large can $\delta^{(2)}$ be? For the group G_∞ mentioned in the introduction, $\delta^{(2)}$ is more or less exponential. However, no examples above exponential are known. The techniques of this paper could probably be used to obtain such examples if the following question could be addressed.

(vii) Can one embed a finitely presented group with large (eg. $n \mapsto 2^{2^n}$) first order Dehn function into a group of type F_3 with small (for example polynomial) first order Dehn function?

In connection with (vi) and (vii) see also [13, Question 1 and Question 2].

In [13] it is shown that the second order Dehn function of a group is subrecursive (see also [17]). There then arises the following question.

(viii) Is every second order Dehn function equivalent to a recursive function?

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