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WHICH 3-MANIFOLDS EMBED IN TRIOD $\times I \times I$?

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ABSTRACT. We classify the compact 3-manifolds whose boundary is a union of 2-spheres, and which embed in $T \times I \times I$, where T is a triod and I the unit interval. This class is described explicitly as the set of punctured handlebodies. We also show that any 3-manifold in $T \times I \times I$ embeds in a punctured handlebody.

1**. Introduction and statement of results.** Working in the PL category, we let I denote the interval $[0, 1]$, and T the *triod* consisting of three intervals joined together at a common endpoint, that is, the cone on three points. One of the authors [4] showed that the contractible 3-complex $T \times T \times I$ is *universal* for 3-manifolds, meaning that every compact 3-manifold, which does not have a closed component, can be embedded in $T \times T \times I$. This sharpened an earlier result of [2]. Of course, a contractible 3-complex cannot contain *any* closed 3-manifolds.

Our goal is to investigate the question of which 3-manifolds with boundary embed in $T \times I \times I$, a space intermediate between the universal complex and the cube $I \times I \times I$. One reason this is of interest is that Gillman [1] showed that $T \times I \times I$ does not contain a fake cube, that is, a contractible compact 3-manifold topologically distinct from the cube. The Poincaré conjecture asserts fake cubes do not exist, and because of [4] one can restrict attention to submanifolds of $T \times T \times I$. An understanding of those which lie in $T \times I \times I$ can be considered a step toward understanding compact 3-manifolds (with boundary) in general.

Even in the cube $I \times I \times I$ (or ordinary 3-space), the classification of all 3-dimensional submanifolds is a complicated business. For example, if a torus boundary is considered, the problem is essentially classical knot theory. However, if all boundary components are 2-spheres, the situation is much simpler. Using standard results, it is easy to prove that any 3-manifold which embeds in $I \times I \times I$ and has boundary a union of spheres, must be a disjoint union of copies of Punctured S^3. By Punctured M, for any 3-manifold M we mean the result of deleting the interiors of finitely many disjoint 3-balls (at least one) from M.

DEFINITION. Let $\mathcal{C}$ denote the class of manifolds consisting of:

(1) Punctured S^3,
(2) Punctured $(S^1 \times S^2) \# (S^1 \times S^2) \# \cdots \# (S^1 \times S^2)$, ($k \geq 1$ summands),
(3) Punctured $(S^1 \tilde{\times} S^2) \# (S^1 \times S^2) \# \cdots \# (S^1 \times S^2)$, ($k \geq 1$ summands),
(4) Disjoint unions of finitely many of the above 3-manifolds.

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In the above, $S^1 \tilde{\times} S^2$ denotes the nonorientable 2-sphere bundle over S^1. We can now state our main results.

THEOREM 1. *Suppose M is a compact 3-manifold and ∂M is a union of 2-spheres. Then M embeds in $T \times I \times I$ if and only if M is in the class $\mathcal{C}$.*

COROLLARY. *If $M \in \mathcal{C}$ and $N \subset M$ is a sub-3-manifold with boundary a union of spheres, then $N \in \mathcal{C}$.*

THEOREM 2. *Suppose M is an arbitrary compact 3-manifold which embeds in $T \times I \times I$. Then M also embeds in a manifold of class $\mathcal{C}$.*

2**. Proof of Theorem 1.** First we'll check that any manifold in class $\mathcal{C}$ embeds in $T \times I \times I$. It clearly suffices to consider connected manifolds, and it is obvious that Punctured S^3 embeds. The connected sum and *boundary* connected sum (in which two manifolds with nonempty boundary are joined by a 1-handle attached to the two boundaries) are related: Punctured (A # B) is a boundary connected sum of Punctured A and Punctured B. Thus it suffices to embed Punctured ($S^1 \times S^2$) and Punctured ($S^1 \tilde{\times} S^2$) in $T \times I \times I$.

For terminology, we will use subscripts for parameters in the three intervals $I_i = [0_i, 1_i]$ comprising the triod

$$T = [0_1, 1_1] \cup [0_2, 1_2] \cup [0_3, 1_3] / \{0_1 = 0_2 = 0_3 = 0\}$$

The space $T \times I \times I$ may be regarded as a "book" with three "pages" $E_i = I_i \times I \times I$, $i = 1, 2, 3$, joined along the "binding" $B = \{0\} \times I \times I$. It is useful to visualize $E_1 \cup E_2$ as (say) the cube $|x| \leq 1, |y| \leq 1, |z| \leq 1$ in 3-space with the binding B corresponding to the square $z = 0$, along which E_3 is attached. Now Punctured ($S^1 \times S^2$) is the result of attaching a 1-handle $H \cong D^2 \times D^1$ to $S^2 \times [0, 1]$ with one of the two disks $D^2 \times S^0$ identified with a disk in $S^2 \times \{0\}$ and the other with a disk in $S^2 \times \{1\}$. If the handle is attached to $S^2 \times [0, 1]$ as above, but in a nonorientable manner, then the result is Punctured ($S^1 \tilde{\times} S^2$).

$S^2 \times [0, 1]$ can be embedded in the cube $E_1 \cup E_2$ (as described above) so that a disk in $S^2 \times \{0\}$ and a disk in $S^2 \times \{1\}$ are subsets of B. Then the handle H can be attached to these disks, and embedded in E_3. In fact it is easy to see that there are different ways to embed $S^2 \times [0, 1]$ so that, when the handle is attached, the result is either Punctured ($S^1 \times S^2$) or Punctured ($S^1 \tilde{\times} S^2$), and we have achieved the desired embeddings.

For the converse to Theorem 1, we will need to consider certain "surgeries" on a manifold M. We will sort them into two types, according to whether the transition $M \longrightarrow M'$ increases or decreases the number of boundary components.

BOUNDARY-INCREASING SURGERIES. In the following, H denotes a 3-ball (handle) parametrized as $H = D^2 \times D^1$. Its boundary is the union of two parts: $S^1 \times D^1$ and $D^2 \times S^0$.

(1) Cutting along a disk: Let $H \subset M$ so that $H \cap \partial M = \partial H \cap \partial M \cong S^1 \times D^1$ separates ∂M. The new manifold M' is defined by $M' = M - \big(\operatorname{Int} H \cup \operatorname{Int}(\partial M \cap \partial H)\big)$. It will be useful to distinguish the two subcases:

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(1.a) non-separating disk: $M - H$ has the same number of connected components as M.

(1.b) separating disk: $M - H$ has more components than M.

(2) Adding a 2-handle: Attach H to M so that $M \cap \operatorname{Int} H = \emptyset$ and $\partial H \cap \partial M \cong S^1 \times D^1$ is a separating annulus in ∂M. The new manifold M' is defined by $M' = M \cup H$.

BOUNDARY-DECREASING SURGERIES. These are the inverses of the above, and may be described as follows:

(1′) Adding a 1-handle...: Let D_1, D_2 be two discs which lie in different boundary components of M. We add H to M to obtain a new 3-manifold $M' = M \cup H$ by attaching the two disks $D^2 \times S^0$ to D_1 and D_2 respectively, the interior of H being disjoint from M. Again we distinguish the subcases:

(1.a′) ... to the same component of M: $M \cup H$ and M have the same number of components.

(1.b′) Boundary connected sum: $M \cup H$ has fewer components than M.

(2′) Drilling a tunnel: Let $H \subset M$ such that $H \cap \partial M = \partial H \cap \partial M = D^2 \times S^0$, and assume the two discs of $D^2 \times S^0$ lie in different boundary components of M. The new 3-manifold M' is defined by $M' = M - \big(\operatorname{Int} H \cup \operatorname{Int}(H \cap \partial M)\big)$.

We note that for each of the above surgeries, ∂M is a union of 2-spheres if and only if the same is true of $\partial M'$. In the operations (1) and (2), $\partial M'$ has one more connected component than ∂M, whereas it has one fewer in the case of (1′) and (2′).

LEMMA 1. *The class $\mathcal{C}$ is closed under surgeries of type (1′) and (2′).*

PROOF. The operation (2′) can be interpreted, up to homeomorphism, as the same as the operation $M \longrightarrow \hat{M}$, where $\hat{M}$ is the union of M and a 3-cell attached to one of the boundary components involved. The homeomorphism $\hat{M} \cong M'$, can be visualized as excavating the tunnel, and then excavating the added 3-cell. And it is easy to see that $M \in \mathcal{C}$ implies $\hat{M} \in \mathcal{C}$ if no closed components have been created in $\hat{M}$. Closure under (1.b′), boundary-connected sum, is clear from the observation made earlier that a boundary connected sum of Punctured A with Punctured B is equivalent to Punctured $(A \# B)$. It may be necessary to use Lemma 2 (below) to replace $(S^1 \tilde{\times} S^2) \# (S^1 \tilde{\times} S^2)$ by $(S^1 \tilde{\times} S^2) \# (S^1 \times S^2)$. Finally, we need to check surgery of type (1.a′): attaching the 1-handle to distinct boundary components of a single component of M. Let $\tilde{M}$ denote M with two 3-cells attached to those boundary components. Then $M \cup H$ is homeomorphic with the connected sum of $\tilde{M}$ with either Punctured $(S^1 \times S^2)$ or Punctured $(S^1 \tilde{\times} S^2)$, and it follows that $M \in \mathcal{C}$ implies $M \cup H \in \mathcal{C}$.

REMARK. $\mathcal{C}$ is also closed under (1) and (2), although we will not need this for the proof of Theorem 1. Indeed, closure under (2) may be seen directly as above. Closure under (1) follows from the Corollary to Theorem 1, since $M' \subset M$. $\mathcal{C}$ can be considered to be the class generated by Punctured S^3 and addition of 1-handles, that is the class of punctured handlebodies.

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LEMMA 2. *Let M be a nonorientable* 3*-manifold. Then*

$$M \# (S^1 \tilde{\times} S^2) \cong M \# (S^1 \times S^2).$$

This is well-known. See [3], Lemma 3.17.

To resume the proof of Theorem 1, consider a 3-manifold $M \subset T \times I \times I$. We wish to show that $M \in \mathcal{C}$. The first step is to replace M by a submanifold which is "transverse" to the binding B. As noted in [1], interior points of M have neighborhoods (in M) which lie in just two pages of $T \times I \times I$, although this may not hold for boundary points of M. However, M contains a copy of itself, M', in its own interior, and we can suppose $\partial M'$ is in PL general position with respect to B. Then $B \cap \partial M'$ is a collection of simple closed curves, each of which has a neighborhood, in M', which lies in just two pages of $T \times I \times I$. Now we employ a standard argument: consider a component of $\partial M' \cap B$ which is innermost on B; it bounds a disk in B whose interior is either contained in or disjoint from M'. Then a neighborhood H of this disk can be removed from or added to M', a surgery of type (1) or (2), resulting in the elimination of that curve of intersection. Finitely many such surgeries gives us a 3-manifold $M'' \subset T \times I \times I$ such that $B \cup \partial M'' = \emptyset$. It follows that $B \cup M'' = \emptyset$, so each component of M'' lies in a single page of $T \times I \times I$ and hence embeds in R^3. It follows that M'' is a union of Punctured S^3, so $M'' \in \mathcal{C}$. Since the transition $M' \longrightarrow M''$ is effected by type (1) and (2) surgeries, the reverse transition $M'' \longrightarrow M'$ can be realized by surgeries of type (1') and (2'). Lemma 1 then implies $M \cong M' \in \mathcal{C}$.

3**. Proof of Theorem 2.** We consider a compact 3-manifold $M \subset T \times I \times I$, which may be assumed to be in general position with respect to the binding, as in the proof of Theorem 1. Then the set $B \cap \partial M$ is a union of simple closed curves interior to B, which divide the 2-cell B into regions, which we will visualize as black (a component F_i of $F = B \cap M$) and white (closures of components of the complement of F in B.) Each black region has a neighborhood in M which lies in just two pages of $T \times I \times I$, and we label that black region with the subscript f_i of the third page, whose interior is *not* intersected by this neighborhood. We may assume M is orthogonal to the binding $B = \{0\} \times I \times I$. That is, for some $\epsilon > 0$, the intersection of M with a neighborhood $N_\epsilon(B)$ is a union of sets

$$[0_j, \epsilon_j] \times F_i, \quad j \in \{1, 2, 3\} - \{f_i\}$$

in $T \times I \times I$, $F_i \subset I \times I \cong B$.

Our goal now is to engulf M by a 3-manifold whose boundary is a union of spheres. As a start, let

$$M' = M \cup ([\epsilon_1, 1_1] \times I \times I) \cup ([\epsilon_2, 1_2] \times I \times I) \cup ([\epsilon_3, 1_3] \times I \times I).$$

Also, if we let A be a thin annulus neighborhood of ∂B in B, we may adjoin $([0_2, \epsilon_2] \times A) \cup ([0_3, \epsilon_3] \times A)$ to M' to form M''. Then $M \subset M''$ and, using M'' in place of M in the above description of black and white regions in B, M'' is completely determined by the

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pattern of curves $B \cap \partial M''$ in B, together with the labels f_i of the black regions F_i. We have constructed M'' so that it has the property:

(1) The region of $B - (B \cap \partial M'')$ which is outermost on B is black (call it F_1), and has label $f_1 = 1$.

Next, we enlarge M'' to ensure that its boundary is spherical. Suppose there are any white regions which are not an annulus or disk. We can split such a region up by adding strips to the black region which surrounds it, separating the black islands within the white. The enlarged black region is labelled the same as the original. Each strip S adds a 1-handle to the black set $B \cap M''$, which in turn adds a 2-handle $([0_i, \epsilon_i] \cup [0_j, \epsilon_j]) \times S$ to M''.

In the same way, if a white annulus adjoins black regions having the same label, we can connect those black regions by a strip and convert the white annulus into a white disk. Finally, any white region which is a disk can be eliminated simply by painting it black and annexing it to the surrounding black region (this adds a 3-handle to M''). After these operations we are able to assume the further property of M'':

(2) Each white region (component of $B - (B \cap M'')$) is an annulus and it lies between black regions having different labels.

LEMMA. *Assuming the properties (1) and (2) above, $\partial M''$ is a union of 2-spheres.*

PROOF. We will use induction on the number of white regions. If there are no white regions, then $M'' = ([\epsilon_1, 1_1] \times I \times I) \cup ([0_2, 1_2] \times I \times I) \cup ([0_3, 1_3] \times I \times I)$ and its boundary consists of two 2-spheres.

Next consider the case of exactly one white region; call it W. Then there are exactly two black regions: F_1 is an annulus between ∂B and a curve C_1, and the other black region F_2 is a disk with boundary curve C_2; $\partial W = C_1 \cup C_2$. The labels are: $f_1 = 1, f_2 \neq 1$ (say $f_2 = 2$, without loss of generality). Note that $\{\epsilon_1\} \times F_1$ is a subset of $\partial M''$; denote the curve $\{\epsilon_1\} \times C_1$ by γ. We will argue that both sides of γ in $\partial M''$ are disks. On one side we have the set

$$(\{\epsilon_1\} \times F_1) \cup \big([\epsilon_1, 1_1] \times \partial(I \times I)\big) \cup (\{1_1\} \times I \times I),$$

which is clearly a disk. The other side of γ is the union of seven annuli and a disk, which are (working inward from γ):

$$\begin{aligned}(\{\epsilon_1\} \times W) \cup ([0_1, \epsilon_1] \times C_2) \cup ([0_3, \epsilon_3] \times C_2) \cup (\{\epsilon_3\} \times W) \cup \\ \cdots \cup ([0_3, \epsilon_3] \times C_1) \cup ([0_2, \epsilon_2] \times C_1) \cup (\{\epsilon_2\} \times W) \cup (\{\epsilon_2\} \times F_2).\end{aligned}$$

They clearly fit together to form a disk and so the boundary component of M'' which contains γ is a 2-sphere. There is one other boundary component of M'', namely

$$(\{1_2\} \times I \times I) \cup \big([0_2, 1_2] \times \partial(I \times I)\big) \cup \big([0_3, 1_3] \times \partial(I \times I)\big) \cup (\{1_3\} \times I \times I).$$

This is also a 2-sphere and we've established the lemma in this case.

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Now consider the general case, with more than one white region. Being all annuli, there must be (at least) one which is innermost in B; call it W. Let j be the label of the black region surrounding W, let C denote the outer boundary of W and define $\gamma = \{\epsilon_j\} \times C$. We can argue exactly as in the preceding case that γ bounds a disk in $\partial M''$. It follows that the *boundary* of M'' is the same as if W were "removed" from the diagram—that is W and the black disk it encloses all become part of the surrounding black region (and get the label j). Note that this last operation could destroy the property $M \subset M''$. Nevertheless, it allows us to conclude inductively that $\partial M''$ is a union of 2-spheres. The proof of the lemma, and of Theorem 2, are now complete.

We close this paper with the remark that the triod could be replaced by an arbitrary graph in the following sense. The class of 3-manifolds with boundary a union of 2-spheres, and which embed in a 3-complex of the form $G \times I \times I$, for some graph G, is exactly the class $\mathcal{C}$. Verification is left to the interested reader.

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