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DECOMPOSITION OF JORDAN AUTOMORPHISMS OF TRIANGULAR MATRIX ALGEBRA OVER COMMUTATIVE RINGS

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Abstract. Let $T_{n+1}(R)$ be the algebra of all upper triangular $n+1$ by $n+1$ matrices over a 2-torsionfree commutative ring R with identity. In this paper, we give a complete description of the Jordan automorphisms of $T_{n+1}(R)$, proving that every Jordan automorphism of $T_{n+1}(R)$ can be written in a unique way as a product of a graph automorphism, an inner automorphism and a diagonal automorphism for $n \geq 1$.

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1. Introduction. Let $M_{n+1}(R)$ be the R-algebra of all square matrices of order $n+1$ over a commutative ring R with the identity 1. Jordan multiplication is defined by $x \circ y = xy + yx$ for any $x, y \in M_{n+1}(R)$. Obviously $x \circ y = y \circ x$. If an R-module automorphism φ of $M_{n+1}(R)$ satisfies $\varphi(x \circ y) = \varphi(x) \circ \varphi(y)$, then φ is called *Jordan automorphism* of $M_{n+1}(R)$. It is well known that an R-algebra automorphism, which is a ring automorphism and also an R-module automorphism of $M_{n+1}(R)$, must be a Jordan automorphism. However, there are Jordan automorphisms which are neither R-algebra automorphisms nor R-algebra anti-automorphisms [**3**]. Let A and B be subsets of $M_{n+1}(R)$. We denote Jordan multiplication of A and B by $A \circ B = \{x \circ y | x \in A, y \in B\}$. Let us consider the sub-algebra of $M_{n+1}(R)$ denoted by $T_{n+1}(R)$, which consists of all upper triangular matrices of $M_{n+1}(R)$. Jordan isomorphisms of associative algebras have been studied by many authors for several decades [**1–4**, **6**, **7**, **10**, **11**, **12**]. The algebra of all triangular matrices is an interesting topic for many researchers. Many papers are concerned with the study of automorphisms and Lie automorphisms [**5**, **8**, **9**, **13**]. On the basis of these papers, we consider the problem on decomposition of Jordan automorphism of upper triangular matrix algebra into some standard automorphisms.

Throughout this paper, R denotes a 2-torsionfree commutative ring with the identity 1. The main results are as follows:

THEOREM 1.1. *For any Jordan automorphism φ of $T_{n+1}(R)(n \geq 1)$, there exist unique graph, inner and diagonal automorphisms, respectively, ζ_ε, θ and λ_d of $T_{n+1}(R)$ such that*

$$\varphi = \zeta_\varepsilon \theta \lambda_d.$$

THEOREM 1.2. *Let $\mathcal{G}$, $\mathcal{I}$ and $\mathcal{D}$ be the graph, inner and diagonal automorphism group, respectively. When $n \geq 1$, then*

$$\mathrm{Aut}(\mathbf{n}_0) = \mathcal{G} \ltimes (\mathcal{I} \ltimes \mathcal{D}).$$

2. Preliminaries. Let e_{ij} denote the matrix unit of $M_{n+1}(R)$ and e the identity matrix of $M_{n+1}(R)$. The matrix set $\{e_{i,i+k}|i = 1, \ldots, n-k+1,\ \ k = 0, 1, \ldots, n\}$ is a basis of $T_{n+1}(R)$. For any $x \in T_{n+1}(R)$, it can be expressed $x = \sum_{k=0}^{n}\sum_{i=1}^{n-k+1} a_{i,i+k}e_{i,i+k}$ for some $a_{i,i+k} \in R$. Let $\mathbf{n}_1$ be the sub-algebra of all strictly upper matrices of $T_{n+1}(R)$. The matrix set $\{e_{i,i+k}|i = 1, \ldots, n-k+1,\ \ k = 1, \ldots, n\}$ is a basis of $\mathbf{n}_1$. Let $\mathbf{n}_0 = T_{n+1}(R)$ and $\mathrm{Aut}(\mathbf{n}_k)$, k=0,1 denote the Jordan automorphism group of $\mathbf{n}_k$, respectively. If R is 2-torsionfree, then a Jordan automorphism of $M_{n+1}(R)$ coincides with the semi-automorphism of $M_{n+1}(R)$ such that $\varphi(x^2) = [\varphi(x)]^2$ and $\varphi(xyx) = \varphi(x)\varphi(y)\varphi(x)$ for any $x, y \in M_{n+1}(R)$.

LEMMA 2.1. *Let φ be an R-module automorphism of $\mathbf{n}_1$. The following two statements are equivalent*:

(i) *φ is in* $\mathrm{Aut}(\mathbf{n}_1)$;

(ii) *For any $e_{i,i+k} \in \mathbf{n}_1$, $\varphi(e_{i,i+k}) = \varphi(e_{i,i+m}) \circ \varphi(e_{i+m,i+k})$ for 1$\leq m < k$ and $\varphi(e_{ij}) \circ \varphi(e_{mk}) = 0$ for $j \neq m$ and $i \neq k$.*

Proof. See [**12**, Lemma 2.1]. □

LEMMA 2.2. *Let φ be a Jordan automorphism of $\mathbf{n}_1$. The following two statements are equivalent*:

(i) *φ is in* $\mathrm{Aut}(\mathbf{n}_0)$;

(ii) *For any $e_{i,i+k} \in \mathbf{n}_1$, $[\varphi(e_{ii})]^2 = \varphi(e_{ii})$, $\varphi(e_{i,i+k}) = \varphi(e_{ii}) \circ \varphi(e_{i,i+k})$, $\varphi(e_{i,i+k}) = \varphi(e_{i,i+k}) \circ \varphi(e_{i+k,i+k})$, $\varphi(e_{jj}) \circ \varphi(e_{ii}) = 0 (j \neq i)$ and $\varphi(e_{jj}) \circ \varphi(e_{i,i+k}) = 0 (j \neq i, i+k)$.*

Proof. By Lemma 2.1 it is not difficult to prove Lemma 2.2. □

Lemma 2.2 implies that the set $\{\varphi(e_{11}), \varphi(e_{i+1,i+1}), \varphi(e_{i,i+1})|i = 1, \ldots, n\}$ generates $T_{n+1}(R)$. So we will investigate $\varphi(e_{11}), \varphi(e_{i+1,i+1}), \varphi(e_{i,i+1}),\ i = 1, \ldots, n$.

LEMMA 2.3. *Let φ be in* $\mathrm{Aut}(\mathbf{n}_0)$. *For any $x \in \mathbf{n}_0$ and $y, e_{ij} \in \mathbf{n}_1$, then $[\varphi(e_{ij})]^2 = 0$, $\varphi(e_{ij})x\varphi(e_{ij}) = 0$ and $e_{ii}ye_{ii} = 0$.*

Proof. For any $e_{ij} \in \mathbf{n}_1$, clearly $(e_{ij})^2 = 0$ so that $[\varphi(e_{ij})]^2 = 0$. It is easy to check that for $e_{mk} \in \mathbf{n}_0$, $e_{ij}e_{mk}e_{ij} = 0$ so that $e_{ij}xe_{ij} = 0$ for any $x \in \mathbf{n}_0$. Therefore $e_{ij}\varphi^{-1}(x)e_{ij} = 0$ then $\varphi(e_{ij})x\varphi(e_{ij}) = 0$. Similarly, for $e_{mk} \in \mathbf{n}_1$, $e_{ii}e_{mk}e_{ii} = 0$ leads to $e_{ii}ye_{ii} = 0$. □

LEMMA 2.4. *Let φ be in* $\mathrm{Aut}(\mathbf{n}_0)$. *Then $\varphi(\mathbf{n}_1) = \mathbf{n}_1$.*

Proof. We express $\varphi(e_{ii})$ and $\varphi(e_{i,i+1})$, respectively, as

$$\varphi(e_{ii}) = \sum_{k=1}^{n+1} a_{kk}^{(i)}e_{kk} \bmod \mathbf{n}_1,\ i = 1, 2, \ldots, n+1,$$

$$\varphi(e_{i,i+1}) = \sum_{k=1}^{n+1} b_{kk}^{(i)}e_{kk} \bmod \mathbf{n}_1,\ i = 1, \ldots, n.$$

Then, we have

$$\varphi(e_{ii}) = [\varphi(e_{ii})]^2 = \sum_{k=1}^{n+1} (a_{kk}^{(i)})^2 e_{kk} \bmod \mathbf{n}_1,\ i = 1, 2, \ldots, n+1.$$

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So $(a_{kk}^{(i)})^2 = a_{kk}^{(i)}$, $i = 1, 2, \ldots, n+1$, $k = 1, 2, \ldots, n+1$. Moreover,

$$\varphi(e_{i,i+1}) = \varphi(e_{ii}) \circ \varphi(e_{i,i+1}) = \sum_{k=1}^{n+1} 2a_{kk}^{(i)} b_{kk}^{(i)} e_{kk} \bmod \mathbf{n}_1, \ i = 1, \ldots, n.$$

Then $b_{kk}^{(i)} = 2a_{kk}^{(i)} b_{kk}^{(i)}$, $i = 1, \ldots, n$, $k = 1, 2, \ldots, n+1$. Therefore

$$a_{kk}^{(i)} b_{kk}^{(i)} = a_{kk}^{(i)} \big(2b_{kk}^{(i)} - b_{kk}^{(i)}\big) = a_{kk}^{(i)} \big(2b_{kk}^{(i)} - 2a_{kk}^{(i)} b_{kk}^{(i)}\big) = 2\big[a_{kk}^{(i)} - \big(a_{kk}^{(i)}\big)^2\big] b_{kk}^{(i)} = 0,$$

that is, $b_{kk}^{(i)} = 2a_{kk}^{(i)} b_{kk}^{(i)} = 0$, $i = 1, \ldots, n$, $k = 1, 2, \ldots, n+1$. That means $\varphi(\mathbf{n}_1) \subset \mathbf{n}_1$. So $\varphi^{-1}(\mathbf{n}_1) \subset \mathbf{n}_1$, that is, $\mathbf{n}_1 = \varphi\varphi^{-1}(\mathbf{n}_1) \subset \varphi(\mathbf{n}_1)$. □

Let $\mathbf{n}_2 = \mathbf{n}_1 \circ \mathbf{n}_1$, $\mathbf{n}_k = \mathbf{n}_1 \circ \mathbf{n}_{k-1}$, $k = 2, \ldots, n$. It is clear to know $\mathbf{n}_k = \sum_{m=k}^{n} \sum_{i=1}^{n-m+1} Re_{i,i+m}$, $k = 2, \ldots, n$. Notice that $\mathbf{n}_{n+1} = 0$. Without loss of generality, an element in $\mathbf{n}_k$ is often denoted by t_k. It is obvious that $t_m t_k, t_m \circ t_k \in \mathbf{n}_{m+k}$ for $m + k \le n$ or $t_m t_k = 0$ and $t_m \circ t_k = 0$ for $m + k > n$. For any $\varphi \in$Aut$(\mathbf{n}_0)$, we have that $\varphi(\mathbf{n}_1) = \mathbf{n}_1$, $\varphi(\mathbf{n}_2) = \varphi(\mathbf{n}_1) \circ \varphi(\mathbf{n}_1) = \mathbf{n}_1 \circ \mathbf{n}_1 = \mathbf{n}_2, \ldots, \varphi(\mathbf{n}_k) = \mathbf{n}_k$, $k = 2, \ldots, n$. Therefore $\varphi(\mathbf{n}_k \backslash \mathbf{n}_{k+1}) = \mathbf{n}_k \backslash \mathbf{n}_{k+1}$, $k = 0, 1, \ldots, n-1$. Let R^* be the multiplicative group of all the invertible elements of R. For any $\varphi \in$Aut$(\mathbf{n}_0)$, there exists $b \in R^*$ such that $\varphi(e_{1,n+1}) = be_{1,n+1}$.

Lemma 2.5. *Let* φ *in* Aut$(\mathbf{n_0})$. *Then*

$$\varphi(e_{11}) = a_{11}^{(1)} e_{11} + a_{n+1,n+1}^{(1)} e_{n+1,n+1} + t_1$$

where $a_{11}^{(1)} + a_{n+1,n+1}^{(1)} = 1$ *and* $a_{11}^{(1)}$ *is an idempotent of* R.

Proof. We express $\varphi(e_{11})$ as $\varphi(e_{11}) = \sum_{k=1}^{n+1} a_{kk}^{(1)} e_{kk} + t_1$. Let $e_{1m} \in \mathbf{n}_1$. By Lemma 2.4 $\varphi^{-1}(e_{1m}) \in \mathbf{n}_1$. By Lemma 2.3 $e_{11}\varphi^{-1}(e_{1m})e_{11} = 0$. Consequently,

$$\varphi(e_{11}) e_{1m} \varphi(e_{11}) = a_{11}^{(1)} a_{mm}^{(1)} e_{1m} + t_m = 0, \ m = 2, \ldots, n+1.$$

Let $e_{m,n+1} \in \mathbf{n}_1$. Similarly,

$$\varphi(e_{11}) e_{m,n+1} \varphi(e_{11}) = a_{mm}^{(1)} a_{n+1,n+1}^{(1)} e_{m,n+1} + t_{n-m+2} = 0, \ m = 1, \ldots, n.$$

So $a_{11}^{(1)} a_{mm}^{(1)} = 0$ and $a_{mm}^{(1)} a_{n+1,n+1}^{(1)} = 0$, $m = 2, \ldots, n$. From $\varphi(e_{1,n+1}) = be_{1,n+1}$, $b \in R^*$, we have

$$\varphi(e_{1,n+1}) = \varphi(e_{11}) \circ \varphi(e_{1,n+1}) = \big(a_{11}^{(1)} + a_{n+1,n+1}^{(1)}\big) be_{1,n+1},$$

then $a_{11}^{(1)} + a_{n+1,n+1}^{(1)} = 1$. So $a_{mm}^{(1)} = a_{mm}^{(1)}(a_{11}^{(1)} + a_{n+1,n+1}^{(1)}) = 0$, $m = 2, \ldots, n$. From the process of proving Lemma 2.4 we know $(a_{11}^{(1)})^2 = a_{11}^{(1)}$. □

Now let us introduce standard Jordan automorphisms of $T_{n+1}(R)$.

(i) Let ε be an idempotent of R. Then ε, $1 - \varepsilon$ are orthogonal idempotents, that is, $\varepsilon(1 - \varepsilon) = 0$. Let $e_0 = \sum_{i=1}^{n+1} e_{i,n-i+2}$. We define a map ζ_ε: $x \mapsto \varepsilon x + (1 - \varepsilon)(e_0 x e_0)^\tau$, where τ denotes the transpose of a matrix. If both ε and $\bar{\varepsilon}$ are idempotents of R, then $1 - (\varepsilon - \bar{\varepsilon})^2$ is also an idempotent of R and $\zeta_\varepsilon \zeta_{\bar{\varepsilon}} = \zeta_{1-(\varepsilon-\bar{\varepsilon})^2}$. This implies that $\zeta_\varepsilon^{-1} = \zeta_\varepsilon$ and ζ_ε is an R-module automorphism of $T_{n+1}(R)$.

Obviously, ζ_1 is the identity automorphism of $T_{n+1}(R)$ and $\zeta_\varepsilon = \varepsilon\zeta_1 + (1-\varepsilon)\zeta_0$. From $\zeta_\varepsilon(x \circ y) = \zeta_\varepsilon(x) \circ \zeta_\varepsilon(y)$ for any $x, y \in T_{n+1}(R)$, we know that ζ_ε is a Jordan automorphism of $T_{n+1}(R)$. We call ζ_ε a *graph automorphism*. If ε is non-trivial, the graph automorphism ζ_ε is neither an R-algebra automorphism nor an R-algebra anti-automorphism of $T_{n+1}(R)$, unless one of the ideals $\varepsilon T_{n+1}(R)$ or $(1-\varepsilon)T_{n+1}(R)$ of $T_{n+1}(R)$ is commutative. The graph automorphism ζ_ε on the basis of $T_{n+1}(R)$ acts as $\zeta_\varepsilon(e_{kj}) = \varepsilon e_{kj} + (1-\varepsilon)e_{n-j+2,n-k+2}(1 \le k \le [\frac{n+1}{2}], k \le j \le n-k+1)$, $\zeta_\varepsilon(e_{k,n-k+2}) = e_{k,n-k+2}(1 \le k \le 1+[\frac{n}{2}])$ and $\zeta_\varepsilon(e_{n-j+2,n-k+2}) = (1-\varepsilon)e_{kj} + \varepsilon e_{n-j+2,n-k+2}(1 \le k \le [\frac{n+1}{2}], k \le j \le n-k+1)$, where $[\frac{n+1}{2}]$ is the integer part of $\frac{n+1}{2}$. The set of all graph automorphisms of $T_{n+1}(R)$ is a subgroup of $\mathrm{Aut}(\mathbf{n_0})$, which is denoted by $\mathcal{G}$.

(ii) For any $y \in \mathbf{n_1}$, let $h = e + y$. The map θ_h: $x \mapsto hxh^{-1}$ is called an *inner automorphism* which is an R-algebra automorphism of $T_{n+1}(R)$. If $h = h_{ij}(a) = e + ae_{ij}(i < j)$ with some $a \in R$, then $\theta_{h_{ij}(a)}$ is called the 'simple' form. Using $[h_{ij}(a)]^{-1} = h_{ij}(-a)$ we know that $\theta_{h_{ij}(a)}(e_{ii}) = e_{ii} - ae_{ij}$, $\theta_{h_{ij}(a)}(e_{jj}) = e_{jj} + ae_{ij}$ for $i < j$ and $\theta_{h_{ij}(a)}(e_{kk}) = e_{kk}$ for $k \ne i, j$ and that $\theta_{h_{mi}(a)}(e_{i,i+1}) = e_{i,i+1} + ae_{m,i+1}$ and $\theta_{h_{i+1,j}(a)}(e_{i,i+1}) = e_{i,i+1} - ae_{ij}$ also $\theta_{h_{mi}(a)}(e_{k,k+1}) = e_{k,k+1}$ and $\theta_{h_{i+1,j}(a)}(e_{k,k+1}) = e_{k,k+1}$ for $k \ne i, m, j$. It is easy to see that $\theta^{-1}_{h_{ij}(a)} = \theta_{h_{ij}(-a)}$. The set of all the 'simple' inner automorphisms of $T_{n+1}(R)$ generates a subgroup of $\mathrm{Aut}(\mathbf{n_0})$, which is denoted by $\mathcal{I}$.

(iii) Let $d = \sum_{i=1}^{n+1} d_i e_{ii}$ where $d_i \in R^*$, $i = 1, 2, \ldots, n+1$. The map λ_d: $x \mapsto dxd^{-1}$ is called a *diagonal automorphism* which is an R-algebra automorphism of $T_{n+1}(R)$. It is obvious that $\lambda_d^{-1} = \lambda_{d^{-1}}$. A diagonal automorphism on the basis of $T_{n+1}(R)$ yields that $\lambda_d(e_{ii}) = e_{ii}$ and $\lambda_d(e_{i,i+k}) = \prod_{m=1}^{k} c^{-1}_{i+m-1,i+m} e_{i,i+k}$ for $d_1 = 1$, $d_i = \prod_{m=2}^{i} c_{i-m+1,i-m+2} \in R^*$, $i = 2, \ldots, n+1$. The set of all diagonal automorphisms of $T_{n+1}(R)$ is a subgroup of $\mathrm{Aut}(\mathbf{n_0})$, which is denoted by $\mathcal{D}$.

3. Lemmas for main results. In order to achieve our goal, we also need other lemmas.

LEMMA 3.1. *Let φ be in* $\mathrm{Aut}(\mathbf{n}_0)$. *There exists a graph automorphism ζ_ε such that $\zeta_\varepsilon\varphi(e_{11}) = e_{11} + t_1$.*

Proof. By Lemma 2.5, $\varphi(e_{11}) = a^{(1)}_{11}e_{11} + a^{(1)}_{n+1,n+1}e_{n+1,n+1} + t_1$. Take $\varepsilon = a^{(1)}_{11}$, then

$$\begin{aligned}
\zeta_{a^{(1)}_{11}}(\varphi(e_{11})) &= a^{(1)}_{11}\zeta_{a^{(1)}_{11}}(e_{11}) + a^{(1)}_{n+1,n+1}\zeta_{a^{(1)}_{11}}(e_{n+1,n+1}) + \zeta_{a^{(1)}_{11}}(t_1) \\
&= a^{(1)}_{11}\big[a^{(1)}_{11}e_{11} + \big(1 - a^{(1)}_{11}\big)e_{n+1,n+1}\big] + \big(1 - a^{(1)}_{11}\big)\big[a^{(1)}_{11}e_{n+1,n+1} + \big(1 - a^{(1)}_{11}\big)e_{11}\big] + t_1 \\
&= \big(a^{(1)}_{11}\big)^2 e_{11} + \big(1 - a^{(1)}_{11}\big)^2 e_{11} + t_1 = e_{11} + t_1.
\end{aligned}$$

This completes the proof. □

LEMMA 3.2. *Let φ be in* $\mathrm{Aut}(\mathbf{n}_0)$. *If $\varphi(e_{11}) = e_{11} + t_1$, then $\varphi(e_{ii}) = e_{ii} + t_1$, $i = 1, 2, \ldots, n+1$ and $\varphi(e_{i,i+1}) = b^{(i)}_{i,i+1}e_{i,i+1} + t_2$, $i = 1, \ldots, n$ where $b^{(i)}_{i,i+1} \in R^*$.*

Proof. If $e_{jl} \in \mathbf{n}_1$, then $\varphi^{-1}(e_{jl}) \in \mathbf{n}_1$. By Lemma 2.3 we have $e_{ii}\varphi^{-1}(e_{jl})e_{ii} = 0$ then $\varphi(e_{ii})e_{jl}\varphi(e_{ii}) = 0$. Therefore,

$$\varphi(e_{ii})e_{im}\varphi(e_{ii}) = a^{(i)}_{ii}a^{(i)}_{mm}e_{im} + t_{m-i+1} = 0,\ i = 1, \ldots, m-1 (m \ge 2),$$
$$\varphi(e_{ii})e_{mi}\varphi(e_{ii}) = a^{(i)}_{ii}a^{(i)}_{mm}e_{ki} + t_{i-m+1} = 0,\ i = m+1, \ldots, n+1 (m \le n),$$

so $a_{ii}^{(i)}a_{mm}^{(i)} = 0$, $i \neq m$. When $i \neq j$, $\varphi(e_{ii}) \circ \varphi(e_{jj}) = \sum_{k=1}^{n+1} a_{kk}^{(i)}a_{kk}^{(j)}e_{kk} + t_1 = 0$, so $a_{kk}^{(i)}a_{kk}^{(j)} = 0$, $i \neq j$. Let us express $\varphi(e_{i,i+1})$ as $\varphi(e_{i,i+1}) = \sum_{k=1}^{n} b_{k,k+1}^{(i)}e_{k,k+1} + t_2$. Therefore $\varphi(e_{12}) = \varphi(e_{11}) \circ \varphi(e_{12}) = b_{12}^{(1)}e_{12} + t_2$. From $\varphi^{-1}\varphi(e_{11}) = \varphi^{-1}(e_{11}) + t_1$, we have $\varphi^{-1}(e_{11}) = e_{11} + t_1$. Then $\varphi^{-1}(e_{12}) = \hat{b}_{12}^{(1)}e_{12} + t_2$. Furthermore, $e_{12} = \varphi^{-1}\varphi(e_{12}) = b_{12}^{(1)}\hat{b}_{12}^{(1)}e_{12} + t_2$, then $b_{12}^{(1)}\hat{b}_{12}^{(1)} = 1$, that is, $b_{12}^{(1)} \in R^*$. Also we have $\varphi(e_{12}) = \varphi(e_{12}) \circ \varphi(e_{22}) = (a_{11}^{(2)} + a_{22}^{(2)})b_{12}^{(1)}e_{12} + t_2$. Then $a_{11}^{(2)} + a_{22}^{(2)} = 1$. From $a_{11}^{(1)}a_{11}^{(2)} = 0$, we know $a_{11}^{(2)} = 0$, that is, $a_{22}^{(2)} = 1$. Using induction we assume that $\varphi(e_{m-1,m-1}) = e_{m-1,m-1} + t_1$, $\varphi(e_{m-1,m}) = b_{m-1,m}^{(m-1)}e_{m-1,m} + t_2$, $b_{m-1,m}^{(m-1)} \in R^*$ and $a_{mm}^{(m)} = 1$ hold. Then $a_{kk}^{(m)} = 0$, $k \neq m$, that is, $\varphi(e_{mm}) = e_{mm} + t_2$. From

$$\varphi(e_{m,m+1}) = \varphi(e_{mm}) \circ \varphi(e_{m,m+1}) = b_{m,m+1}^{(m)}e_{m,m+1} + b_{m-1,m}^{(m)}e_{m-1,m} + t_2,$$

we have $b_{k,k+1}^{(m)} = 0$, $k \neq m-1, m$. From

$$\varphi(e_{m-1,m-1}) \circ \varphi(e_{m,m+1}) = b_{m-1,m}^{(m)}e_{m-1,m} + t_2 = 0,$$

we have $b_{m-1,m}^{(m)} = 0$, that is, $\varphi(e_{m,m+1}) = b_{m,m+1}^{(m)}e_{m,m+1} + t_2$. In the same way, we know $b_{m,m+1}^{(m)} \in R^*$. Furthermore,

$$\varphi(e_{m,m+1}) = \varphi(e_{m,m+1}) \circ \varphi(e_{m+1,m+1}) = \left(a_{mm}^{(m+1)} + a_{m+1,m+1}^{(m+1)}\right)b_{m,m+1}^{(m)}e_{m,m+1} + t_2.$$

Then $a_{mm}^{(m+1)} + a_{m+1,m+1}^{(m+1)} = 1$. So $a_{m+1,m+1}^{(m+1)} = 1$. When $m = n$, the proof is completed. □

LEMMA 3.3. *Let φ be in* $\mathrm{Aut}(\mathbf{n}_0)$. *If $\varphi(e_{ii}) = e_{ii} + t_1$, $i = 1, 2, \ldots, n+1$, then*

$$\begin{aligned}
\varphi(e_{11}) &= e_{11} + a_{12}^{(1)}e_{12} + t_2,\\
\varphi(e_{ii}) &= e_{ii} + a_{i,i+1}^{(i)}e_{i,i+1} - a_{i-1,i}^{(i-1)}e_{i-1,i} + t_2,\ i = 2, \ldots, n (n \geq 2),\\
\varphi(e_{n+1,n+1}) &= e_{n+1,n+1} - a_{n,n+1}^{(n)}e_{n,n+1} + t_2.
\end{aligned}$$

Proof. We write $\varphi(e_{ii})$ as

$$\varphi(e_{ii}) = e_{ii} + \sum_{k=1}^{n} a_{k,k+1}^{(i)}e_{k,k+1} + t_2,\ i = 1, 2, \ldots, n+1.$$

From $\varphi(e_{ii}) = [\varphi(e_{ii})]^2$ we have

$$\begin{aligned}
\varphi(e_{11}) &= e_{11} + a_{12}^{(1)}e_{12} + t_2,\\
\varphi(e_{ii}) &= e_{ii} + a_{i,i+1}^{(i)}e_{i,i+1} + a_{i-1,i}^{(i)}e_{i-1,i} + t_2,\ i = 2, \ldots, n,\\
\varphi(e_{n+1,n+1}) &= e_{n+1,n+1} + a_{n,n+1}^{(n+1)}e_{n,n+1} + t_2.
\end{aligned}$$

Then

$$\varphi(e_{ii}) \circ \varphi(e_{i+1,i+1}) = (a_{i,i+1}^{(i)} + a_{i,i+1}^{(i+1)})e_{i,i+1} + t_2 = 0,\ i = 1, \ldots, n.$$

So $a_{i,i+1}^{(i)} = -a_{i,i+1}^{(i+1)}$, $i = 1, \ldots, n$. □

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LEMMA 3.4. *Let φ be in* $\mathrm{Aut}(\mathbf{n}_0)$. *If $\varphi(e_{ii}) = e_{ii} + t_1$, $i = 1, 2, \ldots, n+1$, we take that*

$$\theta = \prod_{i=1}^{n} \theta_{h_{i,i+1}(a^{(i)}_{i,i+1})}.$$

Then

$$\theta\varphi(e_{ii}) = e_{ii} + t_2, \; i = 1, 2, \ldots, n+1.$$

Proof. From $\theta_{h_{i,i+1}(a^{(i)}_{i,i+1})}(e_{ii}) = e_{ii} - a^{(i)}_{i,i+1}e_{i,i+1}$ and $\theta_{h_{i,i+1}(a^{(i)}_{i,i+1})}(e_{i+1,i+1}) = e_{i+1,i+1} + a^{(i)}_{i,i+1}e_{i,i+1}$ and then by Lemma 3.3 it is not difficult to complete the proof. □

LEMMA 3.5. *Let φ be in* $\mathrm{Aut}(\mathbf{n}_0)$. *If $\varphi(e_{ii}) = e_{ii} + t_{m-1}$, $i = 1, 2, \ldots, n+1$, then*

$$\varphi(e_{ii}) = e_{ii} + a^{(i)}_{i,i+m-1}e_{i,i+m-1} + t_m, \; 1 \le i \le min\{m-1, n-m+2\},$$

$$\varphi(e_{ii}) = e_{ii} + a^{(i)}_{i,i+m-1}e_{i,i+m-1} - a^{(i-m+1)}_{i-m+1,i}e_{i-m+1,i} + t_m,$$

$$m \le i \le n-m+2 \left(m \le \left[\frac{n+1}{2}\right] \right),$$

$$\varphi(e_{ii}) = e_{ii} + t_m,$$

$$n-m+3 \le i \le m \left(m \ge \left[\frac{n+1}{2}\right] + 1 \text{ or when } n \text{ is odd, } m > \left[\frac{n+1}{2}\right] + 1 \right),$$

$$\varphi(e_{ii}) = e_{ii} - a^{(i-m+1)}_{i-m+1,i}e_{i-m+1,i} + t_m, \; max\{n-m+3, m\} \le i \le n+1.$$

Proof. It is the case in Lemma 3.3 if $m = 2$. Using the method of proving Lemma 3.3 we may verify the consequence. □

LEMMA 3.6. *Let φ be in* $\mathrm{Aut}(\mathbf{n}_0)$. *If $\varphi(e_{ii}) = e_{ii} + t_{m-1}$, $i = 1, 2, \ldots, n+1$, we take that*

$$\theta = \prod_{i=1}^{n-m+2} \theta_{h_{i,i+m-1}(a^{(i)}_{i,i+m-1})}.$$

Then

$$\theta\varphi(e_{ii}) = e_{ii} + t_m, \; i = 1, 2, \ldots, n+1.$$

When $m = n+1$, $\theta\varphi(e_{ii}) = e_{ii}$, $i = 1, 2, \ldots, n+1$.

Proof. The process for verifying the result is similar to that of Lemma 3.4. □

LEMMA 3.7. *When $n \ge 1$, let φ be in* $\mathrm{Aut}(\mathbf{n_0})$. *If $\varphi(e_{ii}) = e_{ii}$, there exists a diagonal automorphism λ_d such that $\lambda_d\varphi(e_{i,i+1}) = e_{i,i+1} + t_2$, $i = 1, \ldots, n$.*

Proof. By Lemma 3.2 we know that $\varphi(e_{i,i+1}) = b^{(i)}_{i,i+1}e_{i,i+1} + t_2$, $i = 1, \ldots, n$, where $b^{(i)}_{i,i+1} \in R^*$. Let λ_d satisfy $e_{i,i+1} \mapsto (b^{(i)}_{i,i+1})^{-1}e_{i,i+1}$, where $d_1 = 1$, $d_i = \prod_{m=2}^{i} b^{(i-m+1)}_{i-m+1,i-m+2}$, $i = 2, \ldots, n+1$. Applying $\lambda_d\varphi$ to $e_{i,i+1}$ we get the asserted property. □

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LEMMA 3.8. *When* $n \geq 1$, *let* φ *be in* $\mathrm{Aut}(\mathbf{n_0})$. *If* $\varphi(e_{ii}) = e_{ii}$, $i = 1, 2, \ldots, n+1$ *and* $\varphi(e_{i,i+1}) = e_{i,i+1} + t_2$, $i = 1, \ldots, n$, *then* $\varphi(e_{i,i+1}) = e_{i,i+1}$, $i = 1, \ldots, n$.

Proof. We express $\varphi(e_{i,i+1})$ as

$$\varphi(e_{i,i+1}) = e_{i,i+1} + \sum_{k=2}^{n} \sum_{m=1}^{n-k+1} b^{(i)}_{m,m+k} e_{m,m+k}, \ i = 1, \ldots, n.$$

Therefore,

$$\begin{aligned}
\varphi(e_{12}) &= \varphi(e_{11}) \circ \varphi(e_{12}) = e_{12} + \sum_{k=2}^{n} b^{(1)}_{1,1+k} e_{1,1+k} (n \geq 2),\\
\varphi(e_{23}) &= \varphi(e_{22}) \circ \varphi(e_{23}) = e_{23} (n = 2),\\
\varphi(e_{23}) &= \varphi(e_{22}) \circ \varphi(e_{23}) = e_{23} + \sum_{k=2}^{n-1} b^{(2)}_{2,2+k} e_{2,2+k} (n \geq 3),\\
\varphi(e_{i,i+1}) &= \varphi(e_{ii}) \circ \varphi(e_{i,i+1}) = e_{i,i+1} + \sum_{k=2}^{n-i+1} b^{(i)}_{i,i+k} e_{i,i+k} + \sum_{k=2}^{n-1} b^{(i)}_{i-k,i} e^{(i)}_{i-k,i}\\
&\quad \times (3 \leq i \leq n-1, n \geq 4),\\
\varphi(e_{n,n+1}) &= \varphi(e_{nn}) \circ \varphi(e_{n,n+1}) = e_{n,n+1} + \sum_{k=2}^{n-1} b^{(n)}_{n-k,n} e^{(n)}_{n-k,n} (n \geq 3).
\end{aligned}$$

So for $i = 1, 2, \ldots, n$

$$\varphi(e_{i,i+1}) = \varphi(e_{i,i+1}) \circ \varphi(e_{i+1,i+1}) = \varphi(e_{i,i+1}) \circ e_{i+1,i+1} = e_{i,i+1}.$$

In the case $n = 1$, $\varphi(e_{12}) = e_{12}$. □

4. Proofs of main results. *Proof of Theorem 1.1.* By Lemma 3.1, Lemma 3.4 and Lemmas 3.6–3.8 there are $\lambda_d^{-1}, \theta^{-1}$ and ζ_ε such that

$$\lambda_d^{-1}\theta^{-1}\zeta_\varepsilon\varphi(e_{ii}) = e_{ii}, \ i = 1, 2, \ldots, n+1.$$
$$\lambda_d^{-1}\theta^{-1}\zeta_\varepsilon\varphi(e_{i,i+1}) = e_{i,i+1}, \ i = 1, \ldots, n.$$

Since $e_{11}, e_{i+1,i+1}, e_{i,i+1}, i = 1, \ldots, n$, generate $T_{n+1}(R)$, then $\varphi = \zeta_\varepsilon\theta\lambda_d$. The uniqueness of the decomposition follows from Theorem 1.2. □

Proof of Theorem 1.2. By Theorem 1.1 we have $\mathrm{Aut}(\mathbf{n}_0) = \mathcal{GID}$. For any $x \in \mathbf{n}_0$ we have $\theta_h\lambda_d(x) = h(dxd^{-1})h^{-1} = \lambda_d\theta_{d^{-1}hd}(x)$, thus $\theta_h\lambda_d = \lambda_d\theta_{d^{-1}hd}$. So $\mathcal{I} \triangleleft \mathcal{ID}$. Obviously, $\mathcal{I} \cap \mathcal{D} = 1$, then $\mathcal{ID} = \mathcal{I} \ltimes \mathcal{D}$. Also we have $\zeta_0\theta_h(x) = [e_0(hxh^{-1})e_0]^\tau = \theta_{\zeta_0(h^{-1})}\zeta_0(x)$, that is, $\zeta_0\theta_h = \theta_{\zeta_0(h^{-1})}\zeta_0$. From

$$\begin{aligned}
\theta_{\varepsilon h+(1-\varepsilon)\zeta_0(h^{-1})}(x) &= [\varepsilon h + (1-\varepsilon)\zeta_0(h^{-1})]x[\varepsilon h + (1-\varepsilon)\zeta_0(h^{-1})]^{-1}\\
&= [\varepsilon h + (1-\varepsilon)\zeta_0(h^{-1})]x[\varepsilon h^{-1} + (1-\varepsilon)(\zeta_0(h^{-1}))^{-1}]\\
&= \varepsilon^2 hxh^{-1} + (1-\varepsilon)^2\zeta_0(h^{-1})x(\zeta_0(h^{-1}))^{-1}\\
&= \varepsilon\theta_h(x) + (1-\varepsilon)\theta_{\zeta_0(h^{-1})}(x)\\
&= [\varepsilon\theta_h + (1-\varepsilon)\zeta_0\theta_h\zeta_0](x),
\end{aligned}$$

we have $\theta_{\varepsilon h+(1-\varepsilon)\zeta_0(h^{-1})} = \varepsilon\theta_h + (1-\varepsilon)\zeta_0\theta_h\zeta_0$. Furthermore,

$$\begin{aligned}\zeta_\varepsilon\theta_h\zeta_\varepsilon &= [\varepsilon\zeta_1 + (1-\varepsilon)\zeta_0]\theta_h[\varepsilon\zeta_1 + (1-\varepsilon)\zeta_0]\\ &= \varepsilon^2\theta_h + (1-\varepsilon)^2\zeta_0\theta_h\zeta_0\\ &= \theta_{\varepsilon h+(1-\varepsilon)\zeta_0(h^{-1})}.\end{aligned}$$

Similarly, $\zeta_\varepsilon\lambda_d\zeta_\varepsilon = \lambda_{\varepsilon d+(1-\varepsilon)\zeta_0(d^{-1})}$. Thus $\mathcal{ID} \triangleleft \mathcal{GID}$. Clearly, $\mathcal{G} \cap \mathcal{ID} = 1$, then $\mathcal{GID} = \mathcal{G} \ltimes (\mathcal{I} \ltimes \mathcal{D})$, that is, $\mathrm{Aut}(\mathbf{n}_0) = \mathcal{G} \ltimes (\mathcal{I} \ltimes \mathcal{D})$. □

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