Appendix C

Solution to a common differential equation

We have often encountered a differential equation of the type

$$-\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + \left[\epsilon - v\cosh 2\mu - v\sinh 2\mu\ \tanh x + v\cosh^2\mu\ \mathrm{sech}^2\mathrm{x}\right]\psi = 0 \tag{C.1}$$

where v, μ are parameters and ϵ is the eigenvalue. This differential equation has been solved in Section 12.3 of [113] where the Schrödinger problem has also been extensively studied. Here we reproduce the solution.

The solution is given in terms of new parameters a and b

$$a = \frac{1}{2}\sqrt{v\mathrm{e}^{2\mu} - \epsilon} - \frac{1}{2}\sqrt{v\mathrm{e}^{-2\mu} - \epsilon} \equiv \frac{1}{2}\kappa_+ - \frac{1}{2}\kappa_- \tag{C.2}$$

$$b = \frac{1}{2}\sqrt{v\mathrm{e}^{2\mu} - \epsilon} + \frac{1}{2}\sqrt{v\mathrm{e}^{-2\mu} - \epsilon} \equiv \frac{1}{2}\kappa_+ + \frac{1}{2}\kappa_- \tag{C.3}$$

Then, with

$$\psi = \mathrm{e}^{-ax}\mathrm{sech}^b x\ F(x) \tag{C.4}$$

the equation for F becomes

$$F'' - 2[a + b\tanh x]F' + [v\cosh^2\mu - b(b+1)]\mathrm{sech}^2 x\ F = 0 \tag{C.5}$$

where primes denote derivatives with respect to x. Defining

$$u = \frac{1}{2}[1 - \tanh x] \tag{C.6}$$

we get the hypergeometric equation

$$u(1-u)\frac{\mathrm{d}^2F}{\mathrm{d}u^2} + [a + b + 1 - 2(b+1)u]\frac{\mathrm{d}F}{\mathrm{d}u} + [v\cosh^2\mu - b(b+1)]F = 0 \tag{C.7}$$

The general solution may be found in [71]

$$F = AF_1 + BF_2 \tag{C.8}$$

where A and B are constants of integration and

$$F_1 = F(\alpha, \beta; \gamma; u) \tag{C.9}$$

$$F_2 = u^{1-\gamma}F(\alpha - \gamma + 1, \beta - \gamma + 1; 2 - \gamma; u) \tag{C.10}$$

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where

$$\alpha = b + \frac{1}{2} - \sqrt{v\cosh^2\mu + \frac{1}{4}}$$

$$\beta = b + \frac{1}{2} + \sqrt{v\cosh^2\mu + \frac{1}{4}} \tag{C.11}$$

$$\gamma = a + b + 1 \tag{C.12}$$

and γ is assumed to not be an integer.

The general analysis can be taken further by considering the solution at $x = \pm\infty$. A solution that is regular at $x \to \infty$ (i.e. $u = 0$) is obtained by setting $B = 0$ in Eq. (C.8). Regularity at $x = -\infty$ ($u = 1$) is only obtained for certain values of ϵ, and thus the energy levels are quantized. The details of the general analysis may be found in Section 12.3 of [113].

In this book, we have often encountered the special case with $\mu = 0$. Then, bound states are obtained for the following discrete values of $b > 0$

$$b_n = \sqrt{v + \frac{1}{4}} - \left(n + \frac{1}{2}\right) \tag{C.13}$$

where $n = 0, 1, 2, \ldots, N$ with N determined by $b_{N+1} \leq 0$. The discrete eigenvalues of ϵ follow from the definition in Eq. (C.3)

$$\epsilon_n = (2n + 1)\sqrt{v + \frac{1}{4}} - \left(n^2 + n + \frac{1}{2}\right) \tag{C.14}$$

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