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PAPER

On (co)products of partial combinatory algebras, with an application to pushouts of realizability toposes

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Abstract

We consider two preorder-enriched categories of ordered partial combinatory algebras: OPCA, where the arrows are functional (i.e., projective) morphisms, and $\mathsf{OPCA}^\dagger$, where the arrows are applicative morphisms. We show that OPCA has small products and finite biproducts, and that $\mathsf{OPCA}^\dagger$ has finite coproducts, all in a suitable 2-categorical sense. On the other hand, $\mathsf{OPCA}^\dagger$ lacks all nontrivial binary products. We deduce from this that the pushout, over Set, of two nontrivial realizability toposes is never a realizability topos. In contrast, we show that nontrivial subtoposes of realizability toposes *are* closed under pushouts over Set.

Keywords: Partial combinatory algebra; realizability; toposes

1. Introduction

Intuitionistic mathematics, or more precisely, the BHK interpretation of intuitionistic mathematics, employs a notion of *function* or *construction*. For example, a proof of $A \to B$ is a construction that transforms proofs of A into proofs of B. This notion of construction can be taken to be fundamental, but one can also give an interpretation of this notion which is more accessible to classical mathematics. The most prominent example of the latter approach is Kleene's realizability interpretation of (first-order) intuitionistic arithmetic, where the notion of construction is interpreted as recursive function application. Hyland later constructed the Effective Topos (Hyland 1982), which is a topos, hence a model for higher-order intuitionistic logic, whose internal logic is governed by recursive function application. In particular, the first-order theory of the natural numbers object in the Effective Topos coincides with Kleene realizability.

Similar toposes can be constructed on the basis of other models of computation (Hyland et al. 1980), giving rise to a class of toposes called *realizability toposes*. These realizability toposes are (almost) never Grothendieck toposes, and thus provide a whole new class of models of constructive mathematics. The relevant notion of a "model of computation" here is that of a *partial combinatory algebra* (PCA). The standard example of a PCA is given by the natural numbers equipped with ordinary Turing computability. This PCA is called Kleene's first model, and its associated realizability topos is the Effective Topos. One may also define PCA structures (i.e., notions of computability) on the Baire space (Kleene's second model) and on the powerset of $\mathbb{N}$ (Scott's graph model); see Example 2.7 below.

Morphisms between PCAs, which are called *applicative morphisms*, were introduced by Longley, thus making PCAs into the objects of a category (Longley 1994). This paper is concerned with two categories of *ordered* partial combinatory algebras (OPCAs). These OPCAs were first introduced in Hofstra and Van Oosten (2003) in order to give a criterion when an applicative morphism gives rise to a geometric morphism between the corresponding realizability toposes. First, we study the category OPCA, introduced in Hofstra and Van Oosten (2003), where the arrows are *functional* morphisms. Second, we consider the category $\mathsf{OPCA}^\dagger$, which arises as the Kleisli category for a monad on OPCA and whose arrows are called applicative morphisms. Restricting the latter category to discrete, i.e., unordered OPCAs yields the category of PCAs first introduced by Longley. Not much is known about the structure of these various categories of (ordered) PCAs. Indeed, the comprehensive monograph on the subject by van Oosten states: "It should be stressed that the category [of PCAs] is not very well understood at the moment of writing" (Van Oosten 2008, p. 28). That moment was more than a decade ago, and since then, progress has been made (see, e.g., Faber and Van Oosten 2014). However, there is one construction available in this category that, to my knowledge, has thus far escaped attention or at least publication in the literature. It turns out that the category of PCAs has finite coproducts. Their construction, in the slightly more general setting of ordered PCAs, is the first main result of the paper.

The second main result of the paper concerns realizability toposes. Given the class of realizability toposes, one can inquire into its closure properties under various constructions on toposes. In this respect, realizability toposes seem to be far less well behaved than Grothendieck toposes. For example, realizability toposes are not closed under slicing (see Zoethout 2020 for an investigation on this matter). Another standard construction in topos theory is the pushout of a pair of geometric inclusions (Johnstone 1977, Proposition 4.26). Each realizability topos contains Set as a subtopos. This leads to the following question: if we take the pushout, over Set, of two realizability toposes, then what kind of topos do we obtain? More specifically, do we obtain another realizability topos? The second main result of the paper is that such a pushout is (barring trivial cases) never itself a realizability topos. On the other hand, we also offer a positive result, namely that (nontrivial) *subtoposes* of realizability toposes are closed under pushouts over Set.

Another version of the construction of coproducts of PCAs already appeared in the paper Zoethout (2020), which discusses yet another category of PCAs. The differences between the PCAs studied there and the OPCAs studied here are:

(i) the PCAs in Zoethout (2020) are unordered, whereas the objects of OPCA and $\mathsf{OPCA}^\dagger$ are ordered PCAs.
(ii) Zoethout (2020) discusses PCAs internal to "base categories" other than Set, whereas in this paper, we will work exclusively over Set.

Coproducts in Longley's category of PCAs may be obtained as a special case of both Zoethout (2020) (by restricting to the base category Set) as well as the current paper (by restricting to the unordered case). One reason for presenting the construction here for the special case of Set is to enable one to understand the construction of coproducts of OPCAs without having to work through the PCAs over generalized base categories from Zoethout (2020). Another reason is that, as we shall see below, coproducts of OPCAs interact in an interesting way with *products* of OPCAs. In Zoethout (2020), the situation with products is quite different and requires the variation in base categories offered there; see also Remark 6.4 below.

The categories OPCA and $\mathsf{OPCA}^\dagger$ are enriched over preorders, so they carry a (simple) 2-categorical structure. Moreover, in the final section, we will briefly consider the 2-category of regular categories, and the 2-category of toposes, so some remarks on 2-categorical terminology are in order. In general, we will append the prefix "pseudo-" to a term to indicate that we define this term in a "fully weak" 2-categorical sense. For pseudo(co)limits, this entails two things. First, it means that (co)cones only need to commute up to isomorphism. For example, if $A_0 \xrightarrow{f_0} A_2 \xleftarrow{f_1} A_1$ is a cospan in a preorder-enriched category, then a cone for this cospan

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consists of three arrows $p_i \colon X \to A_i$ such that $f_0p_0 \simeq p_2 \simeq f_1p_1$. Second, the universal property of a pseudo(co)limit is expressed by an *equivalence* of categories, rather than an isomorphism. For example, if $\mathscr{C}$ is a category enriched over preorders, then a pseudoproduct of A_0 and A_1 is an object $A_0 \times A_1$ equipped with projections $A_0 \xleftarrow{\pi_0} A_0 \times A_1 \xrightarrow{\pi_1} A_1$ such that the map

$$(\pi_0 \circ -, \pi_1 \circ -) \colon \mathscr{C}(B, A_0 \times A_1) \to \mathscr{C}(B, A_0) \times \mathscr{C}(B, A_1)$$

is an equivalence of preorders for each object B. For the pseudoproducts considered below, it will turn out that the map above is actually an *isomorphism* of preorders. We will express this by saying that they are 2-products; note that we do not use the adjective "strict" here. We will use the adjective "strict" at another occasion, however: a strict pseudoinitial object will be a pseudoinitial object 0 with the additional property that every arrow $A \to 0$ is an equivalence. Similarly, we will use the term "strict pseudoterminal object" for the dual notion. Another important use of the prefix "pseudo-" concerns monos and epis. An arrow f of a preorder-enriched category is called a pseudomono if postcomposition with f reflects the order. For epis, a similar definition applies.

The paper is structured as follows. First of all, in Section 2, we define the category OPCA and state some of its elementary properties. In Section 3, we show that OPCA has small pseudoproducts (which are in fact 2-products) and finite pseudocoproducts, which also yield finite pseudobiproducts. Next, in Section 4, we construct the category $\mathsf{OPCA}^\dagger$ from OPCA. Section 5 shows that the finite pseudocoproducts in OPCA also yield finite pseudocoproducts in $\mathsf{OPCA}^\dagger$. On the other hand, nontrivial binary pseudoproducts (i.e., where both factors are not the pseudoterminal object) never exist in $\mathsf{OPCA}^\dagger$. Finally, in Section 6, we deduce from this that the pushout, over Set, of two nontrivial realizability toposes is never itself a realizability topos. In contrast, we show that nontrivial *subtoposes* of realizability toposes are closed under pushouts over Set.

2. Ordered PCAs

In this section, we introduce ordered partial combinatory algebras and morphisms between them. Since we will not state any new results here, we will describe the important constructions, but omit most proofs.

A partial combinatory algebra is a nonempty set A equipped with a *partial* binary application map $(a, b) \mapsto ab$. We think of the elements of A simultaneously as inputs and as (codes of) algorithms that act on these inputs. The element ab stands for the output, if any, when the algorithm (with code) a is applied to b. Of course, in order to capture the intuition that the application map is computation, this map will need to satisfy certain axioms, to be specified below.

A useful generalization of partial combinatory algebras was introduced in Hofstra and Van Oosten (2003). Here, a partial combinatory algebra A is also equipped with a partial order $\leq$. We can think of the statement $a' \leq a$ as expressing that a' gives more information than a, or that a' is a specialization of a. Of course, this order will need to be compatible with the application map. Let us make this explicit.

Definition 2.1. *An* ordered partial applicative structure *(OPAS) is a poset $A = (A, \leq)$ equipped with a partial binary map $A \times A \rightharpoonup A$, $(a, b) \mapsto ab$ satisfying the following axiom:*

(0) if $a' \leq a$, $b' \leq b$ and ab is defined, then $a'b'$ is also defined, and $a'b' \leq ab$.

In other words, if a' and b' contain at least as much information as a and b, and ab is already defined, then $a'b'$ must also be defined and give at least as much information as ab.

Before we proceed to define ordered partial combinatory algebras, some remarks on notation are in order. First of all, the application map will not be associative, meaning that expressions involving application need to be bracketed properly. In order to prevent illegible expressions, we

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adopt the convention that application associates to the left, writing abc as an abbreviation for $(ab)c$. Moreover, we will sometimes write $a \cdot b$ instead of ab if this is necessary to avoid confusion.

Since the application map is partial, we also introduce some notation dealing with partiality. If e is a possibly undefined expression, then we write $e\downarrow$ to indicate that e is in fact defined. We take this to imply that all subexpressions of e are defined as well. If e and e' are two possibly undefined expressions, then we write $e' \preceq e$ for the statement: if $e\downarrow$, then $e'\downarrow$ and $e' \leq e$. On the other hand, $e' \leq e$ always expresses the stronger statement that e' and e are defined and satisfy $e' \leq e$. Observe that axiom (0) can also be written as: if $a' \leq a$ and $b' \leq b$, then $a'b' \preceq ab$. Moreover, we write $e \simeq e'$ if both $e' \preceq e$ and $e \preceq e'$. In other words, $e \simeq e'$ expresses the Kleene equality of e and e', meaning that $e\downarrow$ iff $e'\downarrow$, and in this case, e and e' denote the same value. On the other hand, $e = e'$ will always mean that e and e' are defined and equal to each other.

Definition 2.2. *An OPAS A is an* ordered partial combinatory algebra *(OPCA) if there exist* $\mathsf{k}, \mathsf{s} \in A$ *satisfying:*

(1) $\mathsf{k}ab \leq a$;
(2) $\mathsf{s}ab\downarrow$;
(3) $\mathsf{s}abc \preceq ac(bc)$.

OPCAs satisfy an abstract version of the *Smn* Theorem for Turing computability on the natural numbers. In order to make this precise, we need the following definition.

Definition 2.3. *Let A be an OPCA. The set of* terms *over A is defined recursively as follows:*

(i) We assume given a countably infinite set of distinct variables, and these are all terms.
(ii) For every $a \in A$*, we assume that we have a* constant symbol *for a, and this is a term. The constant symbol for a is simply denoted by a.*
(iii) If t_0 *and* t_1 *are terms, then so is* $(t_0 t_1)$.

We omit brackets whenever possible, again subject to the convention that application associates to the left. Moreover, we may write $t_0 \cdot t_1$ if needed to avoid confusion.

Clearly, every *closed* term t can be assigned a (possibly undefined) interpretation in A, which will also be denoted by t. If $t(\vec{x})$ is a term in n free variables, then this term defines an obvious partial function $A^n \rightharpoonup A$, which sends a tuple $\vec{a} \in A^n$ to (the interpretation of) $t(\vec{a})$, if defined. The key fact about OPCAs is the all such functions are computable using an algorithm present in A.

Proposition 2.4 (Combinatory completeness)**.** *Let A be an OPCA. There exists a map that assigns, to each term* $t(\vec{x}, y)$ *in* $n+1$ *variables, an element* $\lambda^*\vec{x}y.t$ *of A, satisfying:*

- $(\lambda^*\vec{x}y.t)\vec{a}\downarrow$;
- $(\lambda^*\vec{x}y.t)\vec{a}b \preceq t(\vec{a}, b)$,

for all $\vec{a} \in A^n$, $b \in A$.

The proof is an easy adaptation of the proof of Theorem 1.1.3 in Van Oosten (2008) and is omitted. It is worth mentioning that the map $t(\vec{x}, y) \mapsto \lambda^*\vec{x}y.t$ can be constructed explicitly and only requires a choice for k and s as in Definition 2.2.

The elements k and s are usually called *combinators*. Using k, s and Proposition 2.4, we can construct additional useful combinators. For our purposes, the combinators $\mathsf{i} = \mathsf{skk}$, $\bar{\mathsf{k}} = \mathsf{ki}$, $\mathsf{p} = \lambda^* xyz.zxy$, $\mathsf{p}_0 = \lambda^* x.x\mathsf{k}$ and $\mathsf{p}_1 = \lambda^* x.x\bar{\mathsf{k}}$ will be relevant. These satisfy:

$$\mathsf{i}a \leq a, \quad \bar{\mathsf{k}}ab \leq b, \quad \mathsf{p}_0(\mathsf{p}ab) \leq a \quad \text{and} \quad \mathsf{p}_1(\mathsf{p}ab) \leq b.$$

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The combinators k and $\bar{\mathsf{k}}$ also serve as *booleans*, meaning that there exists a case combinator $\mathsf{C} \in A$ satisfying $\mathsf{C}\mathsf{k}ab \leq a$ and $\mathsf{C}\bar{\mathsf{k}}ab \leq b$. Observe that we may simply take $\mathsf{C} = \mathsf{i}$.

Remark 2.5. Even though k and s are not part of the structure of an OPCA, we will assume that, for each OPCA we discuss, we have made an explicit choice for k and s. Observe that this also yields a choice for the other combinators constructed above. If one has a lot of OPCAs, then this may require the Axiom of Choice; this situation will occur in the proof of Proposition 3.5.

Example 2.6. The prototypical example is the (discretely ordered) OPCA $\mathcal{K}_1$, known as Kleene's first model. Its underlying set is the set of natural numbers, and mn is the result, if any, when the m-th partial recursive function is applied to n.

Example 2.7. (i) Another discretely ordered OPCA is *Scott's graph model*, whose underlying set is the powerset of the natural numbers. In order to define the application map, fix a bijection $e_{(-)}$ between $\mathbb{N}$ and the set of *finite* subsets of $\mathbb{N}$, and a bijection $\langle\cdot,\cdot\rangle : \mathbb{N}^2 \to \mathbb{N}$. For $A, B \subseteq \mathbb{N}$, we define the application AB as

$$\{m \in \mathbb{N} \mid \exists n \in \mathbb{N}\,(e_n \subseteq B \text{ and } \langle n, m\rangle \in A)\}.$$

One can show that this yields a *total* OPCA, where "computable" means "continuous w.r.t. the Scott topology". For details, we refer to Van Oosten (2008).

(ii) The set of all functions $\mathbb{N} \to \mathbb{N}$ may also be equipped with a (again discretely ordered) OPCA structure, yielding *Kleene's second model* $\mathcal{K}_2$. Again, we refer to Van Oosten (2008) for details.

Example 2.8. Any poset with binary meets is an OPCA, where application is given by meet; for k and s, any choice of elements will work. These are examples of *pseudotrivial* OPCAs (Hofstra and Van Oosten 2003, Definition 2.3), i.e., OPCAs where any two elements have a common lower bound. This notion will not play a large role in this paper; we will need it only in Example 3.7 below.

We now proceed to define maps between OPCAs.

Definition 2.9. *Let A and B be OPCAs. A* morphism of OPCAs *is a function $f : A \to B$ satisfying the following requirements:*

- *there exists a $t \in B$ such that $t \cdot f(a) \cdot f(a') \preceq f(aa')$;*
- *there exists a $u \in B$ such that $u \cdot f(a') \leq f(a)$ whenever $a' \leq a$.*

We say that t tracks *f and that f preserves the order up to u.*

Definition 2.10. *Let A and B be OPCAs and consider two functions $f, f' : A \to B$. We say that $f \leq f'$ if there exists an $s \in B$ such that $s \cdot f(a) \leq f'(a)$ for all $a \in A$. Such an $s \in B$ is said to* realize *the inequality $f \leq f'$. Moreover, we write $f \simeq f'$ if both $f \leq f'$ and $f' \leq f$.*

Proposition 2.11. *OPCAs, morphisms of OPCAs and inequalities between them form a preorder-enriched category* OPCA.

We will be especially interested in morphisms with the following property, introduced in Hofstra and Van Oosten (2003).

Definition 2.12. *Let $f : A \to B$ be a morphism of OPCAs. We say that f is* computationally dense *(c.d.) if there exists an $n \in B$ satisfying:*

$$\forall s \in B\, \exists r \in A\, (n \cdot f(r) \leq s). \qquad \text{(cd)}$$

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The definition presented above is not the orginal definition from Hofstra and Van Oosten (2003), but a simplification due to Johnstone (2013). In the latter paper, computational density is also called "quasi-surjectivity". Indeed, Definition 2.12 informally says that, up to a realizer, every element of B is in the image of f. We warn the reader, however, that computationally dense morphisms are not necessarily (pseudo)epis in the category OPCA.

In Section 5, we will also need the following notion.

Definition 2.13. *A morphism of OPCAs $f\colon A \to B$ is called* discrete *if, for any subset $X \subseteq A$, we have: if $f(X) = \{f(a) \mid a \in X\}$ has a lower bound in B, then X has a lower bound in A.*

We list some elementary properties of computational density and discreteness, which we leave to the reader to prove.

Proposition 2.14. *Let $A \xrightarrow{f} B \xrightarrow{g} C$ be morphisms of OPCAs.*

- (*i*) *If f and g are c.d., then gf is c.d. as well.*
- (*ii*) *If gf is c.d., then g is c.d. as well.*
- (*iii*) *If gf is discrete, then f is discrete as well.*
- (*iv*) *Computational density and discreteness are downward closed. That is, if f is c.d. (resp. discrete) and $f' \le f$ is a morphism of OPCAs, then f' is also c.d. (resp. discrete).*

In particular, left adjoints are c.d., and right adjoints are discrete.

The following proposition provides the original definition of computational density from Hofstra and Van Oosten (2003), which we will need later on.

Proposition 2.15. (Johnstone 2013, Lemma 2.3)**.** *A morphism of OPCAs $f\colon A \to B$ is c.d. if and only if there exists an $m \in B$ satisfying:*

$$\forall s \in B \exists r \in A \forall a \in A \; (m \cdot f(ra) \preceq s \cdot f(a)). \tag{cdm}$$

In fact, any $m \in B$ satisfying (cdm) *also satisfies* (cd).

Proof. First of all, suppose that $m \in B$ satisfies (cdm). If $s \in B$, then we know that $\mathsf{k}s$ is defined, so by (cdm), there exists an $r \in A$ such that $m \cdot f(ra) \preceq \mathsf{k}s \cdot f(a) \le s$ for all $a \in A$. In particular, we have $m \cdot f(r\mathsf{i}) \le s$, so m satisfies (cd).

Conversely, suppose that $n \in B$ satisfies (cd). Let $t \in B$ we a tracker of f and let f preserve the order up to $u \in B$. We define

$$m = \lambda^* x.n(u(t \cdot f(\mathsf{p}_0) \cdot x))(u(t \cdot f(\mathsf{p}_1) \cdot x)).$$

Now let $s \in B$, and find an $r \in A$ such that $n \cdot f(r) \le s$. Now we compute

$$\begin{aligned} m \cdot f(\mathsf{p}ra) &\preceq n(u(t \cdot f(\mathsf{p}_0) \cdot f(\mathsf{p}ra)))(u(t \cdot f(\mathsf{p}_1) \cdot f(\mathsf{p}ra))) \\ &\preceq n(u \cdot f(\mathsf{p}_0(\mathsf{p}ra)))(u \cdot f(\mathsf{p}_1(\mathsf{p}ra))) \\ &\preceq n \cdot f(r) \cdot f(a) \\ &\preceq s \cdot f(a), \end{aligned}$$

as desired. □

3. Products and Coproducts in OPCA

In this section, we investigate the existence of pseudo(co)products in OPCA and their interaction with c.d. morphisms. We start with generalizing a result by Longley (1994, Proposition 2.1.7) to the ordered setting.

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Proposition 3.1. *The category* OPCA *has a pseudozero object.*

Proof. The required pseudozero object is the OPCA $\mathbf{1} = \{*\}$, where $** = *$. For every OPCA A, there is only one function $!\colon A \to \mathbf{1}$, and this is clearly a morphism of OPCAs, so $\mathbf{1}$ is in fact a 2-terminal object. Conversely, every element $c \in A$ yields a morphism of OPCAs $¡\colon 1 \to A$ with $¡(*) = c$. Clearly, these are all isomorphic, so $\mathbf{1}$ is also a pseudoinitial object. □

The existence of a pseudozero object means that we also have *zero morphisms*.

Definition 3.2. *A morphism of OPCAs $A \to B$ is called a* zero morphism *if it factors, up to isomorphism, through* $\mathbf{1}$.

The following lemma provides two alternative characterizations of zero morphisms. We leave the proof to the reader.

Lemma 3.3. *For a morphism of OPCAs $f\colon A \to B$, the following are equivalent:*

(*i*) *f is a zero morphism;*
(*ii*) *$f(A) = \{f(a) \mid a \in A\}$ has a lower bound;*
(*iii*) *f is a top element of* $\mathsf{OPCA}(A, B)$.

It follows from (iii) that OPCA is even enriched over preorders with a top element. Before we continue, we characterize the OPCA $\mathbf{1}$ up to equivalence in a number of ways.

Lemma 3.4. *Let A be an OPCA. The following are equivalent:*

(*i*) *A is equivalent to* $\mathbf{1}$;
(*ii*) *A has a least element;*
(*iii*) id_A *is a zero morphism;*
(*iv*) $¡\colon \mathbf{1} \to A$ *is c.d.*

An OPCA A satisfying the equivalent conditions of Lemma 3.4 will be called *trivial*.

If A is an OPCA, then $! \circ ¡$ is isomorphic to the identity $\mathrm{id}_{\mathbf{1}}$. On the other hand, $¡ \circ !$ is, by definition, a zero morphism, so we also have $\mathrm{id}_A \leq ¡ \circ !$. This means that $! \dashv ¡$.

In Hofstra and Van Oosten (2003) (Remark (2) on p. 450), it is observed that OPCA has binary products. This construction generalizes to products of arbitrary (small) size, given choice on the index set.

Proposition 3.5. *The category* OPCA *has small pseudoproducts.*

Proof. Suppose we have an I-indexed sequence of OPCAs $(A_i)_{i \in I}$. We equip the product $A = \prod_{i \in I} A_i$ with an OPAS structure by defining the order and application coordinatewise. That is, if $a = (a_i)_{i \in I}$ and $b = (b_i)_{i \in I}$ are elements of A, then we set

- $a \leq b$ iff $a_i \leq b_i$ for all $i \in I$;
- $ab\downarrow$ iff $a_i b_i \downarrow$ for all $i \in I$, and in this case, $ab = (a_i b_i)_{i \in I}$.

Observe that A is nonempty by AC, and axiom (0) clearly holds for A, since it holds coordinatewise. For all $i \in I$, we may (using AC) pick suitable combinators k_i and s_i for A_i. Then it is not hard to check that $\mathsf{k} = (\mathsf{k}_i)_{i \in I}$ and $\mathsf{s} = (\mathsf{s}_i)_{i \in I}$ are suitable combinators for A, so A is an OPCA.

It is easy to show that the projections $\pi_i\colon A \to A_i$ are morphisms of OPCAs and to verify that this makes A even into the 2-product of the A_i. We leave this to the reader. □

If $f_i\colon A_i \to B$ are morphisms of OPCAs, then we denote their amalgamation by $\langle f_i \rangle_{i \in I}$. The projections π_i are clearly c.d., so if an amalgamation $\langle f_i \rangle_{i \in I}$ is c.d., then so are all the f_i. The converse only holds for *finite* products.

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Proposition 3.6. *If $(A_i)_{i\in I}$ is a finite sequence of OPCAs and the morphisms $f_i\colon B\to A_i$ are c.d., then $\langle f_i\rangle_{i\in I}\colon B\to\prod_{i\in I}A_i$ is also c.d.*

Proof. It suffices to treat the nullary and the binary case. The nullary case states that $!\colon B\to\mathbf{1}$ is always c.d., which follows from the adjunction $!\dashv$ ¡.

For the binary case, suppose we have c.d. morphisms $f_0\colon B\to A_0$ and $f_1\colon B\to A_1$. Let $t_i\in A_i$ track f_i, let f_i preserve the order up to $u_i\in A_i$, and let the computational density of f_i be witnessed by $n_i\in A_i$. We define $n_i'=\lambda^*x.n_i(u_i(t_i\cdot f_i(\mathsf{p}_i)\cdot x))\in A_i$. We claim that $n=(n_0',n_1')\in A_0\times A_1$ witnesses the computational density of $f=\langle f_0,f_1\rangle\colon B\to A_0\times A_1$.

In order to prove this, let $s=(s_0,s_1)\in A_0\times A_1$. Then we know that there exist $r_i\in B$ such that $n_i\cdot f_i(r_i)\le s_i$. Now define $r=\mathsf{pr}_0r_1\in B$. Then

$$n_i'\cdot f_i(r)\preceq n_i(u_i(t_i\cdot f_i(\mathsf{p}_i)\cdot f_i(r)))\preceq n_i(u_i\cdot f_i(\mathsf{p}_ir))\preceq n_i\cdot f(r_i)\le s_i,$$

so $n\cdot f(r)\le s$, as desired. □

Example 3.7. Let A be an OPCA that is not pseudotrivial. Then in particular, k and $\overline{\mathsf{k}}$ do not have a common lower bound, for if u were a lower bound of k and $\overline{\mathsf{k}}$, then uab would be a lower bound of a and b, for arbitrary $a,b\in A$. Let I be a set such that $2^{|I|}>|A|$. Then a morphism $f\colon A\to A^I$ is never c.d., where A^I denotes the I-fold product of A. Indeed, suppose for the sake of contradiction that f is c.d., witnessed by $n\in A^I$. Then every element of A^I is bounded from below by an element of $X=\{n\cdot f(r)\mid r\in A, n\cdot f(r)\downarrow\}$. This set X has cardinality at most $|A|$. However, the subset $\{a\in A^I\mid\forall i\in I(a_i\in\{\mathsf{k},\overline{\mathsf{k}}\})\}$ of A^I, which has cardinality $2^{|I|}>|A|\ge|X|$, has the property that every two distinct elements do not have a common lower bound in A^I: contradiction.

In particular, the diagonal $\delta\colon A\to A^I$ is not c.d., which means that Proposition 3.6 does not hold for infinite I.

Just as the 2-terminal object $\mathbf{1}$ is also pseudoinitial, *finite* 2-products in OPCA also serve as pseudocoproducts.

Theorem 3.8. *The category* OPCA *has finite pseudocoproducts.*

Proof. It suffices to treat the binary case. Let A_0 and A_1 be OPCAs. Then there is a morphism of OPCAs $\kappa_0\colon A_0\to A_0\times A_1$ given by $\kappa_0(a)=(a,\mathsf{i})$. Similarly, we have $\kappa_1\colon A_1\to A_0\times A_1$ given by $\kappa_1(a)=(\mathsf{i},a)$. We claim that this is a pseudocoproduct diagram. This means that we should show that the map

$$(-\circ\kappa_0,-\circ\kappa_1)\colon\mathsf{OPCA}(A_0\times A_1,B)\to\mathsf{OPCA}(A_0,B)\times\mathsf{OPCA}(A_1,B).$$

is an equivalence of preorders, for each OPCA C. If suffices to prove that this map is essentially surjective and full; it is automatically faithful.

For essential surjectivity, suppose that we have morphisms of OPCAs $f_0\colon A_0\to B$ and $f_1\colon A_1\to B$. Let $t_i\in B$ track f_i, and let f_i preserve the order up to $u_i\in B$. We define $f=[f_0,f_1]\colon A_0\times A_1\to B$ by $f(a_0,a_1)=\mathsf{p}\cdot f_0(a_0)\cdot f_1(a_1)$. Then f is tracked by

$$\lambda^*xy.\mathsf{p}(t_0(\mathsf{p}_0x)(\mathsf{p}_0y))(t_1(\mathsf{p}_1x)(\mathsf{p}_1y))\in B,$$

as a straightforward calculation will show. Similarly, one can show that f preserves the order up to $\lambda^*x.\mathsf{p}(u_0(\mathsf{p}_0x))(u_1(\mathsf{p}_1x))\in B$, so f is a morphism of OPCAs. We have $f(\kappa_0(a))=\mathsf{p}a\mathsf{i}$, so $\mathsf{p}_0\in B$ realizes $f\kappa_0\le f_0$ and $\lambda^*x.\mathsf{p}x\mathsf{i}$ realizes $f_0\le f\kappa_0$. Similarly, one shows that $f\kappa_1\simeq f_1$.

For fullness, suppose we have morphisms $g,g'\colon A_0\times A_1\to B$ such that $g\kappa_0\le g'\kappa_0$ and $g\kappa_1\le g'\kappa_1$. Let $s_i\in B$ realize $g\kappa_i\le g'\kappa_i$, let $t,t'\in B$ track g resp. g', and suppose that g and g' preserve the order up to $u,u'\in B$, respectively. We claim that $g\le g'$ is realized by

$$s=\lambda^*x.u'(t'(t'\cdot g'(\mathsf{k},\overline{\mathsf{k}})\cdot(s_0(u(t\cdot g(\mathsf{i},\mathsf{ki})\cdot x))))(s_1(u(t\cdot g(\mathsf{ki},\mathsf{i})\cdot x))))\in B.$$

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Let $(a_0, a_1) \in A_0 \times A_1$. Then we have:

$$\begin{aligned} s_0(u(t\cdot g(\mathsf{i},\mathsf{ki})\cdot g(a_0,a_1))) &\preceq s_0(u\cdot g(\mathsf{i}a_0,\mathsf{ki}a_1)) \\ &\preceq s_0\cdot g(a_0,\mathsf{i}) \\ &\simeq s_0\cdot g(\kappa_0(a_0)) \\ &\preceq g'(\kappa_0(a_0)) \\ &= g'(a_0,\mathsf{i}), \end{aligned}$$

and similarly, $s_1(u(t\cdot g(\mathsf{ki},\mathsf{i})\cdot g(a_0,a_1)))\preceq g'(\mathsf{i},a_1)$. This yields:

$$\begin{aligned} s\cdot g(a_0,a_1) &\preceq u'(t'(t'\cdot g'(\mathsf{k},\bar{\mathsf{k}})\cdot g'(a_0,\mathsf{i}))\cdot g'(\mathsf{i},a_1)) \\ &\preceq u'(t'\cdot g'(\mathsf{k}a_0,\bar{\mathsf{k}}\mathsf{i})\cdot g'(\mathsf{i},a_1)) \\ &\preceq u'\cdot g(\mathsf{k}a_0\mathsf{i},\bar{\mathsf{k}}\mathsf{i}a_1) \\ &\preceq g'(a_0,a_1), \end{aligned}$$

as desired. □

Corollary 3.9. *The category* OPCA *has finite pseudobiproducts*

Proof. The only thing left to check is that $A_0 \xrightarrow{\kappa_0} A_0 \times A_1 \xrightarrow{\pi_0} A_0$ is isomorphic to id_{A_0}, and that $A_0 \xrightarrow{\kappa_0} A_0 \times A_1 \xrightarrow{\pi_1} A_1$ is a zero morphism. Both are immediate. □

Moreover, Proposition 2.14(ii) immediately yields the following relation between coproducts and computational density.

Corollary 3.10. *If $f_0\colon A_0 \to B$ and $f_1\colon A_1 \to B$ are morphisms of OPCAs and f_0 is c.d., then $[f_0, f_1]\colon A_0 \times A_1 \to B$ is also c.d.*

In analogy with ordinary coproducts, we say that finite pseudocoproducts are *disjoint* if, for every pseudocoproduct diagram $A_0 \to A_0 \sqcup A_1 \leftarrow A_1$, the coprojections are pseudomonos, and

$$\begin{array}{ccc} 0 & \longrightarrow & A_1 \\ \downarrow & \searrow & \downarrow \\ A_0 & \longrightarrow & A_0 \sqcup A_1 \end{array}$$

is a pseudopullback, where 0 denotes the pseudoinitial object.

Proposition 3.11. *The finite pseudocoproducts in* OPCA *are disjoint.*

Proof. Since $\pi_i\kappa_i \simeq \mathrm{id}_{A_i}$, it is immediate that the κ_i are pseudomonos. In order to establish the required pseudopullback, we need to show the following: if we have morphisms $f_0\colon B \to A_0$ and $f_1\colon B \to A_1$ such that $\kappa_0 f_0 \simeq \kappa_1 f_1$, then f_0 and f_1 are both zero morphisms. Let $s = (s_0, s_1) \in A_0 \times A_1$ realize $\kappa_0 f_0 \leq \kappa_1 f_1$. Then for all $b \in B$, we have $(s_0 \cdot f_0(b), s_1\mathsf{i}) \simeq s\cdot\kappa_0(f_0(b)) \leq \kappa_1(f_1(b)) = (\mathsf{i}, f_1(b))$. In particular, we have $s_1\mathsf{i} \leq f_1(b)$ for all $b \in B$, so f_1 is a zero morphism. The proof that f_0 is a zero morphism proceeds analogously. □

The "dual" result to Proposition 3.11 also holds; this will be useful in Section 5.

Proposition 3.12. *If A_0 and A_1 are OPCAs, then $\pi_i\colon A_0 \times A_1 \to A_i$ is a pseudoepi and*

$$\begin{array}{ccc} A_0 \times A_1 & \longrightarrow & A_1 \\ \downarrow & \searrow & \downarrow \\ A_0 & \longrightarrow & \mathbf{1} \end{array}$$

is a pseudopushout diagram.

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Proof. Since $\pi_i\kappa_i \simeq \mathrm{id}_{A_i}$, we know that π_i is indeed pseudoepi.

For the pseudopushout, we need to show the following: if $f_0: A_0 \to B$ and $f_1: A_1 \to B$ are morphisms such that $f_0\pi_0 \simeq f_1\pi_1$, then f_0 and g_0 are both zero morphisms. If $s \in B$ realizes $f_0\pi_0 \leq f_1\pi_1$, then we have $s \cdot f_0(a_0) \leq f_1(a_1)$ for all $a_0 \in A_0$ and $a_1 \in A_1$. In particular, we have $s \cdot f_0(\mathsf{i}) \leq f_1(a_1)$ for all $a_1 \in A_1$, so f_1 is a zero morphism. The proof that f_0 is a zero morphism again proceeds analogously. □

We close this section by investigating coproducts in a category related to OPCA.

Definition 3.13. *The preorder-enriched category* $\mathsf{OPCA}_{\mathrm{adj}}$ *is defined as follows.*

- *Its objects are OPCAs.*
- *An arrow $f: A \to B$ is a pair of morphisms $f^*: B \to A$ and $f_*: A \to B$ with $f^* \dashv f_*$.*
- *If $f, g: A \to B$, then we say that $f \leq g$ iff $f^* \leq g^*$; equivalently, if $g_* \leq f_*$.*

Proposition 3.14. *The category* $\mathsf{OPCA}_{\mathrm{adj}}$ *has finite pseudocoproducts. Moreover, the pseudoinitial object is strict, and pseudocoproducts are disjoint.*

Proof. We have already seen that there are essentially unique morphisms $!: A \to \mathbf{1}$ and $¡: \mathbf{1} \to A$ satisfying $! \dashv ¡$, yielding the (essentially) unique arrow $\mathbf{1} \to A$ in $\mathsf{OPCA}_{\mathrm{adj}}$. Moreover, if we have an arrow $A \to \mathbf{1}$ in $\mathsf{OPCA}_{\mathrm{adj}}$, then also $¡ \dashv !$, so $!$ and $¡$ form an equivalence between A and $\mathbf{1}$, meaning that $\mathbf{1}$ is indeed strict.

Now consider two OPCAs A and B. We have the product diagram $A \xleftarrow{\pi_A} A \times B \xrightarrow{\pi_B} B$ and the coproduct diagram $A \xrightarrow{\kappa_A} A \times B \xleftarrow{\kappa_B} B$. We have already remarked that $\pi_A\kappa_A \simeq \mathrm{id}_A$. Moreover, it is easily computed that $\kappa_A\pi_A \geq \mathrm{id}_{A\times B}$, which means that $\pi_A \dashv \kappa_A$ is an arrow $A \to A \times B$ of $\mathsf{OPCA}_{\mathrm{adj}}$. Similarly, we have the arrow $\pi_B \dashv \kappa_B: B \to A \times B$. In order to show that this yields a pseudocoproduct diagram in $\mathsf{OPCA}_{\mathrm{adj}}$, we need to show the following: if $f: A \to C$ and $g: B \to C$ are arrows of $\mathsf{OPCA}_{\mathrm{adj}}$, then $h^* = \langle f^*, g^*\rangle$ is left adjoint to $h_* = [f_*, g_*]$. First of all, we may easily compute that $h_*(h^*(c)) = \mathsf{p} \cdot f_*(f^*(c)) \cdot g_*(g^*(c))$. So, if $r, s \in C$ realize $\mathrm{id}_C \leq f_*f^*$ and $\mathrm{id}_C \leq g_*g^*$, respectively, then $\lambda^* x.\mathsf{p}(rx)(sx)$ realizes $\mathrm{id}_C \leq h_*h^*$. The other inequality can be obtained completely from universal properties. We have:

$$\pi_A h^* h_* \kappa_A \simeq f^* f_* \leq \mathrm{id}_A \simeq \pi_A\kappa_A \quad \text{and} \quad \pi_A h^* h_* \kappa_B \simeq f^* g_* \leq \pi_A \kappa_B,$$

so from the universal property of the coproduct $A \times B$, it follows that $\pi_A h^* h_* \leq \pi_A$. Similarly, we obtain $\pi_B h^* h_* \leq \pi_B$, and the universal property of the product $A \times B$ yields $h^* h_* \leq \mathrm{id}_{A\times B}$, as desired.

For disjointness, we first note that $\pi_A \dashv \kappa_A$ is a pseudomono because $\pi_A\kappa_A \simeq \mathrm{id}_A$. Now suppose we have arrows $f: C \to A$ and $g: C \to B$ of $\mathsf{OPCA}_{\mathrm{adj}}$ such that $\kappa_A f_* \simeq \kappa_B g_*$. Then we know from Proposition 3.11 that f_* and g_* are both zero morphisms. From $\mathrm{id}_C \geq f^* f_*$, it follows that id_C is also a zero morphism, i.e., C is trivial. Now it is immediate that $\mathbf{1}$ is the pseudopullback of $A \to A \times B \leftarrow B$ in $\mathsf{OPCA}_{\mathrm{adj}}$. □

For an OPCA A, the codiagonal $\varepsilon: A \times A \to A$ can be defined as $\varepsilon(a, a') = \mathsf{p}aa'$. Proposition 3.14 tells us in particular that ε is right adjoint to the diagonal $\delta: A \to A \times A$. Together with the fact that $!: A \to 1$ has a right adjoint $¡$, we deduce that every OPCA is a cartesian object in the preorder-enriched category OPCA (compare with the internal finite meets of BCOs in Hofstra (2006), p. 246). Moreover, if $f, g: A \to B$ are morphisms of OPCAa, then the composition

$$A \xrightarrow{\langle f,g\rangle} B \times B \xrightarrow{\varepsilon_B} B$$

is readily seen to be the meet of f and g in $\mathsf{OPCA}(A, B)$. From this, it is easy to deduce that OPCA is even enriched over preorders with finite meets.

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Remark 3.15. We have seen that OPCA is enriched over preorders, preorders with a top element, and preorders with finite meets. For pseudo(co)limits in OPCA, it does not matter which of these enrichments we consider. The reason for this is that all these enrichments equip the homsets with the same *structure* (namely, a preorder), and differ only in which *properties* they ascribe to this structure.

4. Applicative Morphisms

In this section, we introduce the category of ordered PCAs and *applicative* morphisms between them. Applicative morphisms (between unordered PCAs) were the morphisms originally considered in Longley (1994). Applicative morphisms are no longer functions between the underlying sets, but total relations. In Hofstra and Van Oosten (2003), it is shown how to reconstruct the notion of applicative morphism by introducing a certain pseudomonad on OPCA. This is also the treatment we follow here.

Definition 4.1. *Let A be an OPCA.*

(i) We define a new OPCA TA as follows:
   - *TA is the set of all nonempty downsets of A, i.e.,*

$$TA = \{\emptyset \neq \alpha \subseteq A \mid \text{if } a \in \alpha \text{ and } a' \leq a, \text{ then } a' \in \alpha\}.$$

   - *TA is ordered by inclusion.*
   - *For $\alpha, \beta \in TA$, we say that $\alpha\beta\downarrow$ iff $ab\downarrow$ for all $a \in \alpha$ and $b \in \beta$; and in this case,*

$$\alpha\beta = \downarrow\{ab \mid a \in \alpha, b \in \beta\}.$$

(ii) For a morphism of OPCAs $f\colon A \to B$, we define $Tf\colon TA \to TB$ by $Tf(\alpha) = \downarrow f(\alpha) = \downarrow\{f(a) \mid a \in \alpha\}$.

(iii) We define $\delta_A\colon A \to TA$ and $\bigcup_A\colon TTA \to TA$ by $\delta_A(a) = \downarrow\{a\}$ and $\bigcup_A(\mathscr{A}) = \bigcup \mathscr{A}$.

Observe that for the combinators in TA, we may simply take $\downarrow\{\mathsf{k}\}$ and $\downarrow\{\mathsf{s}\}$.

Proposition 4.2. *The triple $(T, \delta, \bigcup)$ is a KZ-pseudomonad on* OPCA.

Recall that, in the preorder-enriched setting, a KZ-pseudomonad is a pseudomonad (T, η, μ) for which $\eta T \leq T\eta$. For a KZ-pseudomonad, algebra structures on an object X are left adjoint to the unit $\eta_X\colon X \to TX$. Consequently, algebras are really objects with a property, rather than objects with structure. A well-known example of a KZ-(pseudo)monad is the downset monad on the category of posets, whose algebras are suplattices. The proof of Proposition 4.2 is very similar to case of the *nonempty* downset monad on the category of posets, but one has to insert some realizers at appropriate positions. We leave this to the reader. For a description of the pseudoalgebras for $(T, \delta, \bigcup)$, we refer to Hofstra and Van Oosten (2003), Theorem 4.2.

Definition 4.3. *The preorder-enriched category* $\mathsf{OPCA}^\dagger$ *is defined as the Kleisli category for the pseudomonad T. An arrow of* $\mathsf{OPCA}^\dagger$ *will be called an* applicative morphism *and will be denoted by $f\colon A \multimap B$.*

Let us consider for a moment what this means. The objects of $\mathsf{OPCA}^\dagger$ are still OPCAs. An applicative morphism $f\colon A \multimap B$ is a morphism of OPCAs $f\colon A \to TB$. This means that f does not assign an *element* of B to $a \in A$, but rather a (nonempty and downward closed) set of elements. For this reason, we use the multimap sign $\multimap$ for applicative morphisms. The identity on A is δ_A, and the composition of $f\colon A \multimap B$ and $g\colon B \multimap C$ is $\bigcup_C \circ Tg \circ f$, i.e., $gf(c) = \bigcup_{b \in f(a)} g(b)$. The requirements for an applicative morphism can be reformulated completely in terms of elements

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of B (rather than TB). It is convenient to use the following notation: if $a \in A$ and $\alpha \in TA$, then we write

$$a \cdot \alpha := \downarrow\{a\} \cdot \alpha = \downarrow\{aa' \mid a' \in \alpha\}.$$

Now, a function $f\colon A \to TB$ is an applicative morphism iff the following hold:

- There exists an $r \in B$ such that $r \cdot f(a) \cdot f(a') \subseteq f(aa')$ whenever $aa'\downarrow$; such an r will also be called a tracker of f (even though the tracker is really $\downarrow\{r\} \in TB$).
- There exists a $u \in B$ such that $u \cdot f(a') \subseteq f(a)$ whenever $a' \leq a$. We will say that f preserves the order up to u

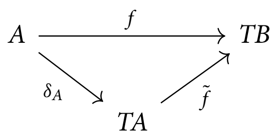
Similarly, if $f, f'\colon A \multimap B$, then we have that $f \leq f'$ iff there exists an $s \in B$ such that $s \cdot f(a) \subseteq f'(a)$ for all $a \in A$; and such an s will be called a realizer of $f \leq f'$.

It turns out for applicative morphisms, one can get rid of the realizer u above.

Lemma 4.4. *Every applicative morphism is isomorphic to an order-preserving applicative morphism.*

Proof. Given $f\colon A \multimap B$, define $f'\colon A \multimap B$ by $f'(a) = \bigcup_{a' \leq a} f(a')$. Clearly, $\mathsf{i} \in B$ realizes $f \leq f'$, and if f preserves the order up to $u \in B$, then u realizes $f' \leq f$. So we have $f \simeq f'$, which also implies that f' is, in fact, an applicative morphism. Clearly, f' preserves the order on the nose. □

If $f\colon A \multimap B$ is an applicative morphism, then there exists an essentially unique T*-algebra morphism* $\tilde{f}\colon TA \to TB$ such that the following diagram commutes:



Explicitly, we have $\tilde{f} \simeq \bigcup_B \circ Tf$. It is well known from the general theory of (pseudo)monads that this yields an equivalence between $\mathsf{OPCA}^\dagger$ and the full subcategory of T-Alg on the free T-algebras. Moreover, it is easy to show that δ_A is c.d., so Proposition 2.14 implies that f is c.d. iff $\tilde{f}$ is c.d. This means we have an unambiguous notion of computational density for applicative morphisms. Explicitly, there should be an $n \in B$ such that

$$\forall s \in B\, \exists r \in A\, (n \cdot f(r) \subseteq \downarrow\{s\}).$$

The results from Proposition 2.14 automatically hold for $\mathsf{OPCA}^\dagger$ as well. For example, suppose that $f\colon A \multimap B$ and $g\colon B \multimap C$ are c.d. Then $\tilde{f}$ and $\tilde{g}$ are c.d., so by Proposition 2.14(i), $\widetilde{gf} \simeq \tilde{g}\tilde{f}$ is c.d., hence gf is c.d.

Moreover, there exists a pseudofunctor $\mathsf{OPCA} \to \mathsf{OPCA}^\dagger$ sending a morphism $f\colon A \to B$ to $\delta_B f\colon A \multimap B$. This is not a 2-functor, since it does not preserve composition in the nose. Because δ_B is always a pseudomono, this pseudofunctor is fully faithful on 2-cells. Furthermore, one easily shows that this pseudofunctor preserves and reflects computational density.

Definition 4.5. *An applicative morphism $f\colon A \multimap B$ is called* projective *iff f belongs to the essential image of* $\mathsf{OPCA} \to \mathsf{OPCA}^\dagger$. *Equivalently, iff $\tilde{f}$ belongs to the essential image of T.*

In other words, f is projective iff there exists a morphism of OPCAs $f_0\colon A \to B$ such that $f \simeq \delta_B f_0$, and in this case, we have $\tilde{f} \simeq Tf_0$. In fact, it suffices that there be a *function* $f_0\colon A \to B$ such that $f \simeq \delta_B f_0$; such an f_0 will the automatically be a morphism of OPCAs, given that f is an applicative morphism. At various occasions in the remainder of the paper, we will view morphisms of OPCAs as projective applicative morphisms.

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The following result was obtained in Faber and Van Oosten (2014) (Corollary 1.15). For the 'if' direction of the theorem (which is the difficult part), this paper uses an analysis of the corresponding realizability toposes (to be defined in Section 6 below). Here we offer a more direct proof of this direction, using ideas from Faber and Van Oosten (2014).

Theorem 4.6. *An applicative morphism has a right adjoint in* $\mathsf{OPCA}^\dagger$ *if and only if it is both projective and c.d.*

Proof. First, suppose that $f\colon A \multimap B$ has a right adjoint $g\colon B \multimap A$. We already know from Proposition 2.14 that this implies that f is c.d. For projectivity, suppose that $r \in A$ realizes $\mathrm{id}_A \leq gf$ and $s \in B$ realizes $fg \leq \mathrm{id}_B$. Then for all $a \in A$, we have that $ra\downarrow$ and $ra \in gf(a) = \bigcup_{b \in f(a)} g(b)$. By the Axiom of Choice, there exists a function $f_0\colon A \to B$ such that $f_0(a) \in f(a)$ and $ra \in g(f_0(a))$ for all $a \in A$. We claim that $f \simeq \delta_B f_0$. First of all, we have that $\downarrow\{f_0(a)\} \subseteq f(a)$, so the identity combinator i realizes $\delta_B f_0 \leq f$. The converse inequality is realized by $s' := \lambda^* x.s(tr'x) \in B$, where r' is an element from $f(r)$ and $t \in B$ tracks f. Indeed, if $b \in f(a)$, then $tr'b \in f(ra) \subseteq \bigcup_{a' \in g(f_0(a))} f(a') = fg(f_0(a))$. So we see that $s'b \preceq s(tr'b)$, which is defined and an element of $\mathrm{id}_B\,(f_0(a)) = \downarrow\{f_0(a)\}$, as desired.

For the converse, let $f\colon A \to B$ be a c.d. morphism of OPCAs; we need to show that $f' = \delta_B f\colon A \multimap B$ has a right adjoint $g\colon B \multimap A$. Let $m \in B$ satisfy (cdm) from Proposition 2.15 for f. We define $g\colon B \multimap A$ by:

$$g(b) = \downarrow\{a \in A \mid m \cdot f(a) \leq b\}.$$

First, let us show that g is indeed an applicative morphism. Because m also satisfies (cd) from Definition 2.12 for f, we know that $g(b)$ is nonempty for every $b \in B$. Moreover, g clearly preserves the order on the nose. In order to construct a tracker, let

$$s = \lambda^* x.m(u(t \cdot f(\mathsf{p}_0) \cdot x))(m(u(t \cdot f(\mathsf{p}_1) \cdot x))) \in B,$$

where t tracks f and f preserves the order up to u. Find $r \in A$ such that $m \cdot f(ra) \preceq s \cdot f(a)$, and define $q = \lambda^* xy.r(\mathsf{p}xy) \in A$. We claim that q tracks g. We need to show that, if $bb'\downarrow$, then

$$q \cdot g(b) \cdot g(b') = \downarrow\{qaa' \mid m \cdot f(a) \leq b \text{ and } m \cdot f(a') \leq b\}$$

is a subset of $g(bb')$. So suppose that $m \cdot f(a) \leq b$ and $m \cdot f(a') \leq b$. Then $qaa' \preceq r(\mathsf{p}aa')$ and:

$$\begin{aligned}
m \cdot f(r(\mathsf{p}aa')) &\preceq s \cdot f(\mathsf{p}aa') \\
&\preceq m(u(t \cdot f(\mathsf{p}_0) \cdot f(\mathsf{p}aa')))(m(u(t \cdot f(\mathsf{p}_1) \cdot f(\mathsf{p}aa')))) \\
&\preceq m(u \cdot f(\mathsf{p}_0(\mathsf{p}aa')))(m(u \cdot f(\mathsf{p}_1(\mathsf{p}aa')))) \\
&\preceq m \cdot f(a)(m \cdot f(a')) \\
&\preceq bb',
\end{aligned}$$

so $qaa' \in g(bb')$, as desired.

In order to establish the adjunction $f' \dashv g$, we first note that

$$gf'(a) = \bigcup_{b \leq f(a)} g(b) = \downarrow\{a' \in A \mid m \cdot f(a') \leq f(a)\}.$$

According to (cdm), there exists an $r \in A$ such that $m \cdot f(ra) \preceq \mathsf{i} \cdot f(a) \leq f(a)$ for all $a \in A$. This immediately implies that $ra \in gf'(a)$ for all $a \in A$, so r realizes $\mathrm{id}_A \leq gf'$. Conversely, we have

$$f'(g(b)) = \bigcup_{a \in g(b)} \downarrow\{f(a)\} = \downarrow\{f(a) \mid m \cdot f(a) \leq b\},$$

so it is immediate that $m \in B$ realizes $f'g \leq \mathrm{id}_B$. □

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We observe that, as an immediate corollary of the 'only if' direction, any two OPCAs that are equivalent in $\mathsf{OPCA}^\dagger$ are already equivalent in OPCA. This means that we can speak unambiguously about the equivalence of OPCAs.

5. Products and Coproducts in PCA

In this section, we investigate to what extent the results from Section 3 carry over to the category $\mathsf{OPCA}^\dagger$. For pseudocoproducts, this is quite easy.

Corollary 5.1. *The pseudofunctor $\mathsf{OPCA} \to \mathsf{OPCA}^\dagger$ preserves finite pseudocoproducts. In particular, $\mathsf{OPCA}^\dagger$ has all finite pseudocoproducts.*

Proof. For every OPCA A, we have $\mathsf{OPCA}^\dagger(\mathbf{1}, A) \simeq \mathsf{OPCA}(\mathbf{1}, TA)$, which we know to be equivalent to the one-element preorder. Similarly, if A_0, A_1, and B are OPCAs, then

$$\begin{aligned}\mathsf{OPCA}^\dagger(A_0 \times A_1, B) &\simeq \mathsf{OPCA}(A_0 \times A_1, TB)\\ &\simeq \mathsf{OPCA}(A_0, TB) \times \mathsf{OPCA}(A_1, TB)\\ &\simeq \mathsf{OPCA}^\dagger(A_0, B) \times \mathsf{OPCA}^\dagger(A_1, B),\end{aligned}$$

finishing the proof. □

Explicitly, if $f_0 \colon A_0 \multimap B$ and $f_1 \colon A_1 \multimap B$ are applicative morphisms, then their amalgamation $[f_0, f_1] \colon A_0 \times A_1 \multimap B$ is given by

$$[f_0, f_1](a_0, a_1) = \downarrow\{\mathsf{p} b_0 b_1 \mid b_0 \in f_0(a_0) \text{ and } b_1 \in f_1(a_1)\}.$$

By Proposition 2.14(ii) (or rather, its counterpart for $\mathsf{OPCA}^\dagger$), we immediately have the following corollary.

Corollary 5.2. *If $f_0 \colon A_0 \multimap B$ and $f_1 \colon A_1 \multimap B$ are applicative morphisms and f_0 is c.d., then $[f_0, f_1] \colon A_0 \times A_1 \multimap B$ is also c.d.*

Since $T\mathbf{1} \simeq \mathbf{1}$, we have that $\mathbf{1}$ is not only pseudoinitial in $\mathsf{OPCA}^\dagger$ but also pseudoterminal. Therefore, we also define zero morphisms in $\mathsf{OPCA}^\dagger$, by saying that $f \colon A \multimap B$ is a zero morphism iff it factors (in $\mathsf{OPCA}^\dagger$) through $\mathbf{1}$. This is in fact equivalent to $f \colon A \to TB$ being a zero morphism in OPCA, which is equivalent to $\bigcap_{a \in A} f(a) \neq \emptyset$. The proof of the following proposition is now completely analogous to the proof Proposition 3.11 and is therefore omitted.

Proposition 5.3. *Pseudocoproducts in $\mathsf{OPCA}^\dagger$ are disjoint.*

If we want to show that $A_0 \times A_1$ is also the pseudoproduct of A_0 and A_1 in $\mathsf{OPCA}^\dagger$, then we should show that $T(A_0 \times A_1) \simeq TA_0 \times TA_1$. However, it turns out that this is *not* true in general and that $\mathsf{OPCA}^\dagger$ does not have finite pseudoproducts. On the other hand, $A_0 \times A_1$ is still a product of A_0 and A_1 in $\mathsf{OPCA}^\dagger$ in a weak sense. Explicitly, if $f_0 \colon B \multimap A_0$ and $f_1 \colon B \multimap A_1$, then there exists a *maximal* mediating arrow $f \colon B \multimap A_0 \times A_1$. Using the theory developed in Section 3, we can tie things together quite nicely.

Because T is a pseudofunctor, we have arrows $T\pi_0 \dashv T\kappa_0 \colon TA_0 \to T(A_0 \times A_1)$ and $T\pi_1 \dashv T\kappa_1 \colon TA_1 \to T(A_0 \times A_1)$ of $\mathsf{OPCA}_{\mathrm{adj}}$. By Proposition 3.14, there exists a mediating arrow $h^* \dashv h_* \colon TA_0 \times TA_1 \to T(A_0 \times A_1)$. Explicitly, we have $h_*(\alpha_0, \alpha_1) = \alpha_0 \times \alpha_1$ for $\alpha_i \in TA_i$, whereas

$$\begin{aligned}h^*(\alpha) &= (T\pi_0(\alpha), T\pi_1(\alpha))\\ &= (\{a_0 \in A_0 \mid \exists a_1 \in A_1\, ((a_0, a_1) \in \alpha)\}, \{a_1 \in A_1 \mid \exists a_0 \in A_0\, ((a_0, a_1) \in \alpha)\})\end{aligned}$$

for $\alpha \in T(A_0 \times A_1)$. One easily computes that $h^* h_*$ is in fact isomorphic to $\mathrm{id}_{TA_0 \times TA_1}$. (This also follows from the fact that $T\pi_i \circ T\kappa_i \simeq \mathrm{id}_{TA_i}$, whereas $T\pi_j \circ T\kappa_i$ is a zero morphism for $i \neq j$.) Now we see that

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$$\begin{aligned}\mathsf{OPCA}^{\dagger}(B, A_0) \times \mathsf{OPCA}^{\dagger}(B, A_1) &\simeq \mathsf{OPCA}(B, TA_0) \times \mathsf{OPCA}(B, TA_1)\\ &\simeq \mathsf{OPCA}(B, TA_0 \times TA_1)\\ &\leftrightarrows \mathsf{OPCA}(B, T(A_0 \times A_1))\\ &\simeq \mathsf{OPCA}^{\dagger}(B, A_0 \times A_1),\end{aligned}$$

where

$$\mathsf{OPCA}(B, TA_0 \times TA_1) \underset{h_* \circ -}{\overset{h^* \circ -}{\leftrightarrows}}_{\perp} \mathsf{OPCA}(B, T(A_0 \times TA_1))$$

is an adjunction whose counit is an isomorphism. In particular, if $f_0 \colon B \multimap A_0$ and $f_1 \colon B \multimap A_1$ are applicative morphisms, then

$$B \xrightarrow{\langle f_0, f_1 \rangle} TA_0 \times TA_1 \xrightarrow{h_*} T(A_0 \times A_1)$$

is the maximal mediating applicative morphism $B \multimap A_0 \times A_1$. Conversely, $g \colon B \multimap A_0 \times A_1$ is such a maximal mediating morphism iff $g \colon B \to T(A_0 \times A_1)$ factors through h_*; or equivalently, $h_* h^* g \simeq g$. Observe that this includes all *projective* $g \colon B \multimap A_0 \times A_1$. Indeed if $g \simeq \delta_{A_0 \times A_1} \circ g_0$ with $g_0 \colon B \to A_0 \times A_1$, then we also have $g \simeq \delta_{A_0 \times A_1} \circ g_0 \simeq h_* \circ (\delta_{A_0} \times \delta_{A_1}) \circ g_0$.

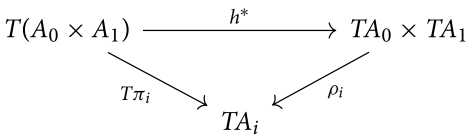
The above shows that pseudoproducts exist in in $\mathsf{OPCA}^{\dagger}$ in a weak sense. Now let us turn to the existence of actual pseudoproducts in $\mathsf{OPCA}^{\dagger}$. Obviously, if A_0 (resp. A_1) is trivial, then the pseudoproduct of A_0 and A_1 exists in $\mathsf{OPCA}^{\dagger}$, and it is equivalent to A_1 (resp. A_0). Using the morphism h^* above, we can show that this is the *only* situation in which A_0 and A_1 have a product in $\mathsf{OPCA}^{\dagger}$.

Theorem 5.4. *If A_0 and A_1 are OPCAs that have a pseudoproduct in $\mathsf{OPCA}^{\dagger}$, then at least one of A_0 and A_1 is trivial.*

Proof. The proof is divided into two parts.

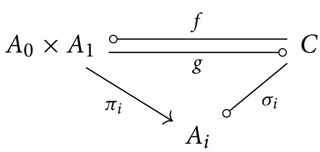
(1) First, we show that $h^* \colon T(A_0 \times A_1) \to TA_0 \times TA_1$ has a left adjoint and is therefore discrete. Here $\times$ stands for the product in OPCA, and h^* is the amalgamation $\langle T\pi_0, T\pi_1 \rangle \colon T(A_0 \times A_1) \to TA_0 \times TA_1$ defined above.
(2) Second, we show that h^* cannot be discrete if A_0 and A_1 are both nontrivial.

For the first part, denote the pseudoproduct projections $TA_0 \times TA_1 \to TA_i$ by ρ_i, then h^* is the essentially unique morphism such that



commutes up to isomorphism, for $i = 0, 1$.

Suppose that C is a pseudoproduct of A_0 and A_1 in $\mathsf{OPCA}^{\dagger}$, with projections $\sigma_i \colon C \multimap A_i$. Then σ_0 and σ_1 induce a maximal mediating arrow $f \colon C \multimap A_0 \times A_1$. On the other hand, π_0 and π_1, seen as projective applicative morphisms, induce a unique mediating map $g \colon A_0 \times A_1 \multimap C$. So for $i = 0, 1$ we get a diagram in $\mathsf{OPCA}^{\dagger}$:



(1)

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where the triangles commute up to isomorphism. Since C is a pseudoproduct, we have $gf \simeq \mathrm{id}_C$. Moreover, we have $\pi_i fg \simeq \sigma_i g \simeq \pi_i \simeq \pi_i \circ \mathrm{id}_{A_0\times A_1}$ for $i = 0, 1$, and since $\mathrm{id}_{A_0\times A_1}$ is certainly projective, this yields $fg \leq \mathrm{id}_{A_0\times A_1}$. We can conclude that $f \dashv g$.

For every OPCA B, we have natural equivalences

$$\begin{aligned}\mathsf{OPCA}(B, TC) &\simeq \mathsf{OPCA}^\dagger(B, C)\\ &\simeq \mathsf{OPCA}^\dagger(B, A_0) \times \mathsf{OPCA}^\dagger(B, A_1)\\ &\simeq \mathsf{OPCA}(B, TA_0) \times \mathsf{OPCA}^\dagger(B, TA_1),\end{aligned}$$

so $TA_0 \xleftarrow{\tilde{\sigma}_0} TC \xrightarrow{\tilde{\sigma}_1} TA_1$ is a product diagram in OPCA. This means there exists an equivalence $\iota\colon TC \to TA_0 \times TA_1$ such that the diagram

$$\begin{array}{ccc} TC & \xrightarrow{\iota} & TA_0 \times TA_i \\ & \tilde{\sigma}_i \searrow \quad \swarrow \rho_i & \\ & TA_i & \end{array}$$

commutes up to isomorphism for $i = 0, 1$. Taking the image of the diagram (1) under the equivalence between $\mathsf{OPCA}^\dagger$ and free T-algebras, we get the diagram

$$\begin{array}{ccccc} T(A_0 \times A_1) & \underset{\tilde{g}}{\overset{\tilde{f}}{\rightleftarrows}} & TC & \xrightarrow{\iota} & TA_0 \times TA_1 \\ & T\pi_i \searrow & \downarrow \tilde{\sigma}_i & \swarrow \rho_i & \\ & & TA_i & & \end{array}$$

in OPCA for $i = 0, 1$, where all triangles commute up to isomorpism. In particular, $\rho_i \iota \tilde{g} \simeq \tilde{\sigma}_i \tilde{g} \simeq T\pi_i$, so $\iota\tilde{g}$ must be isomorphic to h^*. Since $f \dashv g$, we also have $\tilde{f} \dashv \tilde{g}$, hence also $\tilde{f}\iota^{-1} \dashv \iota\tilde{g} \simeq h^*$. We conclude that h^* has a left adjoint, so by Proposition 2.14, h^* is discrete.

For the second part, suppose that A_0 and A_1 are both nontrivial and that h^* is discrete. Consider the set

$$X \subseteq \{\alpha \in T(A_0 \times A_1) \mid h^*(\alpha) = (A_0, A_1)\}.$$

We claim that $\bigcap X$ is empty. Let $(a_0, a_1) \in A_0 \times A_1$ be arbitrary and consider the downset

$$\alpha = \{(b_0, b_1) \in A_0 \times A_1 \mid a_0 \not\leq b_0 \text{ or } a_1 \not\leq b_1\}$$

of $A_0 \times A_1$. Since a_0 is, by assumption, not the least element of A_0, there exists a $b_0 \in A_0$ such that $a_0 \not\leq b_0$. This implies that $\{b_0\} \times A_1 \subseteq \alpha$, so α is nonempty and satisfies $T\pi_1(\alpha) = A_1$. Similarly, we show that $T\pi_0(\alpha) = A_0$, so $\alpha \in X$. On the other hand, we clearly do *not* have $(a_0, a_1) \in \alpha$, so $(a_0, a_1) \notin \bigcap X$. Since this holds for all $(a_0, a_1) \in A_0 \times A_1$, we can conclude that $\bigcap X = \emptyset$.

But $h^*(X) = \{(A_0, A_1)\}$ obviously has a lower bound in $TA_0 \times TA_1$, so since h^* is discrete, X should have a lower bound in $T(A_0 \times A_1)$. However, this is impossible given that $\bigcap X$ is empty, so we have reached a contradiction. □

We close this section by investigating, in analogy with $\mathsf{OPCA}_{\mathrm{adj}}$, the category $\mathsf{OPCA}^\dagger_{\mathrm{adj}}$.

Definition 5.5. *The preorder-enriched category* $\mathsf{OPCA}^\dagger_{\mathrm{adj}}$ *is defined as follows.*

- *Its objects are OPCAs.*
- *An arrow* $f\colon A \to B$ *is a pair of applicative morphisms* $f^*\colon B \multimap A$ *and* $f_*\colon A \multimap B$ *with* $f^* \dashv f_*$.
- *If* $f, g\colon A \to B$, *then we say that* $f \leq g$ *iff* $f^* \leq g^*$; *equivalently, if* $g_* \leq f_*$.

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From Theorem 4.6, we know that $\mathsf{OPCA}^{\dagger}_{\mathrm{adj}}$ is actually equivalent to $\mathsf{OPCA}^{\mathrm{op}}_{\mathrm{cd}}$, where $\mathsf{OPCA}_{\mathrm{cd}}$ denotes the wide subcategory of OPCA on the c.d. morphisms, and $(\,\cdot\,)^{\mathrm{op}}$ indicates a reversal of the *1-cells*. The following result is now immediate.

Corollary 5.6. *The category* $\mathsf{OPCA}^{\dagger}_{\mathrm{adj}}$ *has finite pseudocoproducts. Moreover, the pseudoinitial object is strict, and pseudocoproducts are disjoint.*

Proof. It suffices to prove the dual statements in $\mathsf{OPCA}_{\mathrm{cd}}$. By Proposition 3.6, $\mathsf{OPCA}_{\mathrm{cd}}$ has *finite* pseudoproducts. Moreover, by Lemma 3.4, the terminal object is strict in $\mathsf{OPCA}_{\mathrm{cd}}$. The final statement is Proposition 3.12. □

6. The Realizability Topos

In this final section, we briefly investigate what we can say about coproducts of the *realizability toposes* associated to OPCAs; in particular, to which extent realizability toposes are closed under coproducts. First, let us give the appropriate definitions.

Definition 6.1. *Let A be an OPCA.*

(i) An assembly *over A is a pair* $X = (|X|, E_X)$*, where* $|X|$ *is a set, and* E_X *is a function* $|X| \to TA$.
(ii) A morphism of assemblies $X \to Y$ *is a function* $f\colon X \to Y$ *for which there exists an* $r \in A$ *(called a* tracker *of f) such that* $r \cdot E_X(x) \subseteq E_Y(f(x))$ *for all* $x \in |X|$.

Assemblies and morphisms between them form a quasitopos $\mathsf{Asm}(A)$. Moreover, there is an obvious forgetful funtor $\Gamma_A\colon \mathsf{Asm}(A) \to \mathsf{Set}$ sending X to $|X|$, and there is a functor $\nabla_A\colon \mathsf{Set} \to \mathsf{Asm}(A)$, sending a set Y to the assembly $(Y, y \mapsto A)$. These functors are both regular, and they satisfy $\Gamma_A \dashv \nabla_A$ with $\Gamma_A \nabla_A \cong \mathrm{id}_{\mathsf{Set}}$.

The ex/reg completion of $\mathsf{Asm}(A)$ turns out to be a topos, which is called the *realizability topos* of A and denoted by $\mathsf{RT}(A)$. Since there is an inclusion $\mathsf{Asm}(A) \hookrightarrow \mathsf{RT}(A)$, we can also view ∇_A as a functor $\mathsf{Set} \to \mathsf{RT}(A)$. Moreover, since Γ_A is regular and Set is exact, Γ_A may be lifted to a functor $\mathsf{RT}(A) \to \mathsf{Set}$, which we denote by $\hat{\Gamma}_A$. This yields an adjunction

$$\mathsf{Set} \underset{\nabla_A}{\overset{\hat{\Gamma}_A}{\leftrightarrows}} \mathsf{RT}(A)$$

where $\hat{\Gamma}_A \nabla_A \cong \mathrm{id}_{\mathsf{Set}}$ and $\hat{\Gamma}_A$ preserves finite limits. This means that Set is a subtopos of $\mathsf{RT}(A)$, and in fact, this is precisely the inclusion of double negation sheaves. The $\neg\neg$-separated objects are precisely those objects that are isomorphic to an assembly.

The following result was first obtained by Longley for the unordered case (Longley 1994, Theorem 2.3.4) and generalized to OPCAs in Hofstra and Van Oosten (2003). We denote by REG the 2-category of regular categories, regular functors, and natural transformations. Moreover, $\mathsf{REG}/\mathsf{Set}$ will denote the pseudoslice of REG over Set, i.e., its objects are regular functors with codomain Set, its 1-cells are triangles that commute up to specified isomorphism, and its 2-cells are natural transformations that are compatible with these specified isomorphisms.

Theorem 6.2. *The assignment* $A \mapsto (\Gamma_A\colon \mathsf{Asm}(A) \to \mathsf{Set})$ *may be extended to a local equivalence* $\mathsf{OPCA}^{\dagger} \to \mathsf{REG}/\mathsf{Set}$.

Remark 6.3. Theorem 6.2 tells us applicative morphisms $A \multimap B$, or equivalently, arrows $A \to TB$ of OPCA, correspond to regular functors $\mathsf{Asm}(A) \to \mathsf{Asm}(B)$ that commute with Γ. The category

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OPCA can be used to capture two other classes of functors between categories of assemblies, the first being smaller and the second being larger.

(i) Arrows $A \to B$ of OPCA correspond to regular functors $\mathsf{Asm}(A) \to \mathsf{Asm}(B)$ that commute with Γ and preserve projective objects. This also explains the terminology in Definition 4.5.
(ii) Arrows $TA \to TB$ correspond to *left exact* functors $\mathsf{Asm}(A) \to \mathsf{Asm}(B)$ that commute with Γ (Faber and Van Oosten 2014, Theorem 2.2).

Let A_0 and A_1 be OPCAs. The pseudocoproduct of $\mathsf{RT}(A_0)$ and $\mathsf{RT}(A_1)$, in the 2-category of toposes and geometric morphisms, is the product category $\mathsf{RT}(A_0) \times \mathsf{RT}(A_1)$. In this topos, the logic may be computed componentwise, which implies that its subtopos of double negation sheaves is equivalent to Set^2, rather than Set. This immediately tells us that $\mathsf{RT}(A_0) \times \mathsf{RT}(A_1)$ is never equivalent to a realizability topos.

Remark 6.4. It should be mentioned, however, that (A_0, A_1) is an OPCA internal to the topos Set^2, and that constructing $\mathsf{RT}(A_0, A_1)$ over the base Set^2 rather than Set does yield $\mathsf{RT}(A_0) \times \mathsf{RT}(A_1)$. Moreover, the pair (A_0, A_1), viewed as an OPCA internal to Set^2, is the product of A_0 and A_1 in an appropriate category of OPCAs with variable base categories. See also the treatment (for the unordered case) in Zoethout (2020).

If we want to keep working over the base Set, on the other hand, then it makes more sense to take the pseudocoproduct over Set. That is, we consider the pseudopushout square

$$\begin{array}{ccc} \mathsf{Set} & \hookrightarrow & \mathsf{RT}(A_0) \\ \downarrow & \searrow & \downarrow \\ \mathsf{RT}(A_1) & \hookrightarrow & \mathscr{E} \end{array}$$

which always exists according to Proposition 4.26 from Johnstone (1977). This proposition also tells us that the inverse image part of this diagram:

$$\begin{array}{ccc} \mathscr{E} & \longrightarrow & \mathsf{RT}(A_0) \\ \downarrow & \searrow & \downarrow {\scriptstyle \hat{\Gamma}_{A_0}} \\ \mathsf{RT}(A_1) & \xrightarrow[\hat{\Gamma}_{A_1}]{} & \mathsf{Set} \end{array}$$

is a pseudo*pullback* of categories. Because all displayed functors are regular, this is also a pseudopullback in REG, as is not difficult to show. This means that the inverse image part $\mathscr{E} \to \mathsf{Set}$ is the pseudoproduct of $\hat{\Gamma}_{A_0}$ and $\hat{\Gamma}_{A_1}$ in $\mathsf{REG}/\mathsf{Set}$.

We finish the paper by determining when $\mathscr{E}$ above is itself a realizability topos. If A_0 is trivial, then the inclusion $\mathsf{Set} \to \mathsf{RT}(A_0)$ is an equivalence, so in that case, we will have $\mathscr{E} \simeq \mathsf{RT}(A_1)$. Similarly, if A_1 is trivial, then $\mathscr{E}$ will be equivalent to the realizability topos over A_0. It turns out that these are the only cases in which $\mathscr{E}$ is a realizability topos.

Proposition 6.5. *Let A_0 and A_1 be OPCAs such that the pseudocoproduct of* $\mathsf{RT}(A_0)$ *and* $\mathsf{RT}(A_1)$ *over* Set *is again a realizability topos. Then at least one of A_0 and A_1 is trivial.*

Proof. Suppose that the $\mathscr{E}$ constructed above is equivalent to $\mathsf{RT}(C)$ for some OPCA C. By Corollary 1.4 from Johnstone (2013), there exists (up to isomorphism) at most one geometric morphism $\mathsf{Set} \to \mathsf{RT}(C)$. In particular, $\mathsf{Set} \hookrightarrow \mathscr{E} \simeq \mathsf{RT}(C)$ is isomorphic to the inclusion of double

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negation sheaves. This means that the inverse image part $\mathsf{RT}(C) \to \mathsf{Set}$ is isomorphic to $\hat{\Gamma}_C$, so we have a pseudopullback

$$\begin{array}{ccc} \mathsf{RT}(C) & \xrightarrow{p_0} & \mathsf{RT}(A_0) \\ {\scriptstyle p_1}\downarrow & \searrow^{\hat{\Gamma}_C} & \downarrow{\scriptstyle \hat{\Gamma}_{A_0}} \\ \mathsf{RT}(A_1) & \xrightarrow[\hat{\Gamma}_{A_1}]{} & \mathsf{Set} \end{array}$$

of categories, where p_i denotes the inverse image of $\mathsf{RT}(A_i) \hookrightarrow \mathscr{E} \simeq \mathsf{RT}(C)$. By Johnstone (2013), Lemma 2.4, such an inverse image functor always commutes with the constant object functors, i.e., we have $p_i \nabla_C \simeq \nabla_{A_i}$ for $i = 0, 1$.

An object X of $\mathsf{RT}(C)$ is isomorphic to an assembly if and only if $X \to \nabla_C \hat{\Gamma}_C X$ is a monomorphism. By the pseudopullback diagram above, this is the case iff and $p_i X \to p_i \nabla_C \hat{\Gamma}_C X$ is mono for $i = 0, 1$. Since $p_i \nabla_C \hat{\Gamma}_C X \cong \nabla_{A_i} \hat{\Gamma}_{A_i} p_i X$, this is equivalent to saying that $p_i X$ is isomorphic an assembly, for $i = 0, 1$. So we also have a pseudopullback

$$\begin{array}{ccc} \mathsf{Asm}(C) & \longrightarrow & \mathsf{Asm}(A_0) \\ \downarrow & \searrow^{\Gamma_C} & \downarrow{\scriptstyle \Gamma_{A_0}} \\ \mathsf{Asm}(A_1) & \xrightarrow[\Gamma_{A_1}]{} & \mathsf{Set} \end{array}$$

of categories. But again, all the displayed functors are regular, so this is also a pseudopullback in REG, meaning that Γ_C is a pseudoproduct of Γ_{A_0} and Γ_{A_1} in $\mathsf{REG}/\mathsf{Set}$.

This, together with Theorem 6.2, implies that for any OPCA B, we have natural equivalences:

$$\begin{aligned} \mathsf{OPCA}^{\dagger}(B, C) &\simeq (\mathsf{REG}/\mathsf{Set})(\Gamma_B, \Gamma_C) \\ &\simeq (\mathsf{REG}/\mathsf{Set})(\Gamma_B, \Gamma_{A_0}) \times (\mathsf{REG}/\mathsf{Set})(\Gamma_B, \Gamma_{A_1}) \\ &\simeq \mathsf{OPCA}^{\dagger}(B, A_0) \times \mathsf{OPCA}^{\dagger}(B, A_1), \end{aligned}$$

so C is a pseudoproduct of A_0 and A_1 in $\mathsf{OPCA}^{\dagger}$. Applying Theorem 5.4 finishes the proof. □

Even though the pushout $\mathscr{E}$ constructed above is not a realizability topos, we can ask how far it is from being a realizability topos. The adjunctions $\pi_i \dashv \kappa_i$ between A_i and $A_0 \times A_1$ give rise to geometric inclusions $\mathsf{RT}(A_i) \hookrightarrow \mathsf{RT}(A_0 \times A_1)$. The pushout diagram above then also yields a geometric inclusion $\mathscr{E} \hookrightarrow \mathsf{RT}(A_0 \times A_1)$, so $\mathscr{E}$ is a subtopos of a realizability topos. We can wonder from which local operator on $\mathsf{RT}(A_0 \times A_1)$ this subtopos $\mathscr{E}$ arises. Local operators on a realizability topos $\mathsf{RT}(B)$ arise from *functions* $J\colon DB \to DB$ where DB stands for the set of *all* downsets of B (including $\emptyset$), and J should satisfy certain requirements analogous to the axioms for a local operator. For details, we refer to Lee and Van Oosten (2013). In this particular case, the subtopos $\mathscr{E}$ arises from $J\colon D(A_0 \times A_1) \to D(A_0 \times A_1)$ defined by

$$J(\alpha) = \{a_0 \in A_0 \mid \exists a_1 \in A_1\, ((a_0, a_1) \in \alpha)\} \times \{a_1 \in A_1 \mid \exists a_0 \in A_0\, ((a_0, a_1) \in \alpha)\},$$

i.e., $J(\alpha)$ is the smallest 'rectangular' subset of $A_0 \times A_1$ containing α. We can also describe this map by saying that $J(\alpha) = h_*(h^*(\alpha))$ for $\alpha \in T(A_0 \times A_1)$ (with $h^* \dashv h_*$ as in the previous section), and $J(\emptyset) = \emptyset$.

This discussion suggests a natural class of toposes that includes realizability toposes and is closed under pushouts over Set, namely the class of nontrivial *subtoposes* of realizability toposes. A standard fact on subtoposes of realizability toposes is that $\neg\neg$ is the largest topology on $\mathsf{RT}(A)$ below the maximal topology $\top$. Accordingly, Set is the smallest nontrivial subtopos of any realizability topos $\mathsf{RT}(A)$. So, if $\mathscr{S}_i$ is a nontrivial subtopos of $\mathsf{RT}(A_i)$, then we have inclusions:

$$\mathsf{Set} \hookrightarrow \mathscr{S}_i \hookrightarrow \mathsf{RT}(A_i).$$

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This means that we can take the pushout of $\mathscr{I}_0$ and $\mathscr{I}_1$ over Set and that this pushout is a subtopos of $\mathscr{E}$. Since $\mathscr{E}$ itself is a subtopos of $\mathsf{RT}(A_0 \times A_1)$, we get the following result.

Corollary 6.6. *The class of nontrivial subtoposes of realizability toposes is closed under pushouts over* Set.

7. Conclusion

In this paper, we have studied two categories of ordered partial combinatory algebras. The first category OPCA, where the arrows are functional morphisms, has both small 2-products and finite pseudocoproducts. The second category $\mathsf{OPCA}^{\dagger}$, where the arrows are applicative morphisms, does have finite pseudocoproducts, but it lacks all nontrivial binary pseudoproducts. In this regard, OPCA has a better categorical structure than $\mathsf{OPCA}^{\dagger}$. However, it also seems that $\mathsf{OPCA}^{\dagger}$ is more important for the study of functors between categories of assemblies and realizability toposes. For example, Johnstone (2013) shows, continuing work by Longley, Hofstra and Van Oosten, that in the unordered case, every geometric morphism $\mathsf{RT}(A) \to \mathsf{RT}(B)$ arises from a computationally dense applicative morphism $B \to A$. As we have seen in this paper, pushouts of realizability toposes over Set require pesudoproducts in $\mathsf{OPCA}^{\dagger}$ (rather than OPCA), which are not available. In order to take coproducts of realizability toposes, one must be prepared to consider a wider class of toposes than realizability toposes. One can allow the base category to vary, as in Zoethout (2020). If we want to keep working over Set, however, then we must be prepared to consider *subtoposes* of realizability toposes as well.

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