

LOWER-ORDER TERMS OF THE ONE-LEVEL DENSITY OF A FAMILY OF QUADRATIC HECKE *L*-FUNCTIONS

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Abstract

In this paper, we study lower-order terms of the one-level density of low-lying zeros of quadratic Hecke *L*-functions in the Gaussian field. Assuming the generalized Riemann hypothesis, our result is valid for even test functions whose Fourier transforms are supported in $(-2, 2)$. Moreover, we apply the ratios conjecture of *L*-functions to derive these lower-order terms as well. Up to the first lower-order term, we show that our results are consistent with each other when the Fourier transforms of the test functions are supported in $(-2, 2)$.

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1. Introduction

The work of Montgomery on the pair-correlation of zeros of $\zeta(s)$ in [32] revealed, for the first time, the ties between zeros of *L*-functions and eigenvalues of random matrices. In more recent years, there has been growing interest in the study of low-lying zeros of *L*-functions due to the important roles they play in problems such as determining the rank of the Mordell–Weil groups of elliptic curves and the size of class numbers of imaginary quadratic number fields. The relation between these low-lying zeros and the random matrices is predicted by the density conjecture of Katz and Sarnak [24, 25], which asserts that the distribution of zeros near the central point of a

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family of L -functions is the same as that of the eigenvalues near 1 of a corresponding classical compact group.

A rich literature exists on the density conjecture for various families of L -functions. Some examples include results for Dirichlet [21, 34], Hecke [12, 31], automorphic [14, 20, 23], elliptic curve [1, 19, 44], Dedekind [39, 43], Artin [3] and symmetric power L -functions [7, 18]. Among these various families, the investigation of the families of quadratic Dirichlet L -functions has a relatively long history. They were first examined by Özlük and Snyder in [34] on the one-level density of low-lying zeros of the family, under the assumption of the generalized Riemann hypothesis (GRH). Further work in this direction can be found in [13, 29, 38].

The density conjecture predicts the main term behavior of the n -level density of low-lying zeros of families of L -functions for all n . One can actually do more on the number theory side by computing the lower-order terms of these n -level densities. These lower-order terms serve to provide a better understanding of the n -level densities. Examples of such computations can be found in [30, 35, 45].

For the family of quadratic Dirichlet L -functions, the lower-order terms of one-level density were first analyzed by Miller in [29] for test functions whose Fourier transforms are supported in $(-1, 1)$. On the other hand, we note that the above-mentioned result of Özlük and Snyder [34] on the one-level density is valid with the assumption of the GRH as long as the Fourier transforms of test functions are supported in $(-2, 2)$. Thus, one expects that the computation of the corresponding lower-order terms for all such functions should be possible. This was indeed achieved by a recent result of Fiorilli *et al.* in [10] which assumes the GRH.

In [15], we studied the one-level density of low-lying zeros of quadratic Hecke L -functions in the Gaussian field. Assuming the GRH, we showed that our result confirms the density conjecture when the Fourier transforms of test functions are supported in $(-2, 2)$, a result analogous to that of the family of quadratic Dirichlet L -functions. In view of this, it is natural to ask whether one can compute the lower-order terms as well, as in [10]. We also point out here that in [42], Waxman computed low-order terms of the one-level density for a symplectic family of L -functions attached to Hecke characters of infinite order in the Gaussian field. It is our goal in this paper to continue our work in this direction.

We write $K = \mathbb{Q}(i)$ for the Gaussian field and $\mathcal{O}_K = \mathbb{Z}[i]$ for the ring of integers in K . We write $N(n)$ for the norm of an element $n \in \mathcal{O}_K$ and we reserve the symbol χ_n for the quadratic Hecke character (n/\cdot) defined in Section 2.1. We denote by $\zeta_K(s)$ the Dedekind zeta function of K . We assume the GRH throughout this paper and we are concerned with the following family of L -functions:

$$\mathcal{F} = \{L(s, \chi_{i(1+i)^s c}) : c \text{ square-free}, (c, 1+i) = 1\}.$$

Let $L(s, \chi)$ be one of the L -functions in \mathcal{F} , and write χ here for the corresponding Hecke character. We denote the nontrivial zeros of $L(s, \chi)$ by $1/2 + i\gamma_{\chi, j}$ so that $\gamma_{\chi, j} \in \mathbb{R}$

under the GRH. We order them as

$$\cdots \leq \gamma_{\chi,-2} \leq \gamma_{\chi,-1} < 0 \leq \gamma_{\chi,1} \leq \gamma_{\chi,2} \leq \cdots .$$

Let X be a large real number. We set $\mathcal{L} = \log X$ throughout the paper, and we normalize the zeros by defining

$$\tilde{\gamma}_{\chi j} = \frac{\gamma_{\chi j}}{2\pi} \mathcal{L}.$$

Fix two even Schwartz class functions ϕ, w such that w is nonzero and nonnegative. We regard ϕ as a test function and w as a weight function. We define the one-level density for the single L -function $L(s, \chi)$ with respect to ϕ by the sum

$$S(\chi, \phi) = \sum_j \phi(\tilde{\gamma}_{\chi j}).$$

The one-level density of the family \mathcal{F} with respect to w is then defined as the weighted sum

$$D(\phi; w, X) = \frac{1}{W(X)} \sum_c^* w\left(\frac{N(c)}{X}\right) S(\chi_{i(1+i)^s c}, \phi), \tag{1-1}$$

where we use \sum^* to denote a sum over square-free elements in \mathcal{O}_K throughout the paper, and $W(X)$ here is the total weight given by

$$W(X) = \sum_c^* w\left(\frac{N(c)}{X}\right).$$

Our first result in this paper is an asymptotic expansion of $D(\phi; w, X)$ in descending powers of $\log X$.

THEOREM 1.1. *Suppose that the GRH holds for the family of L -functions in \mathcal{F} as well as for $\zeta_K(s)$. Let $\phi(x)$ be an even Schwartz function whose Fourier transform $\hat{\phi}(u)$ has compact support in $(-2, 2)$, and let w be an even nonzero and nonnegative Schwartz function. Let $D(\phi; w, X)$ be defined as in (1-1). Then, for any integer $M \geq 1$,*

$$D(\phi; w, X) = \widehat{\phi}(0) - \frac{1}{2} \int_{-1}^1 \widehat{\phi}(u) du + \sum_{m=1}^M \frac{R_{w,m}(\phi)}{\mathcal{L}^m} + O\left(\frac{1}{\mathcal{L}^{M+1}}\right), \tag{1-2}$$

where the coefficients $R_{w,m}(\phi)$ are linear functionals in ϕ that can be given explicitly in terms of w , and the derivatives of $\widehat{\phi}$ at the points 0 and 1 (see (4-4)).

We note that Theorem 1.1 gives a refinement of [15, Theorem 1.1], which can be regarded as computing only the main term of the expansion for ϕ given in (1-2). Our result is similar to [10, Theorem 1.1], and we follow closely many of the steps in [10] in the proof of Theorem 1.1. Additionally, our proof of Theorem 1.1 proceeds along the same lines as that of [15, Theorem 1.1] with extra efforts to keep track of all the lower-order terms.

When deriving these lower-order terms, a powerful tool to deploy is the L -functions ratios conjecture of Conrey *et al.* in [4, Section 5]. This approach was applied by Conrey and Snaith in [5] to study the one-level density function for zeros of quadratic Dirichlet L -functions. The general n -level density of the same family was examined by Mason and Snaith in [27], and further enabled them to show in [26] that the result agrees with the density conjecture when the Fourier transforms of test functions are supported in $(-2, 2)$.

It is then highly desirable and interesting to compare the expressions for the n -level density functions conditional on the ratios conjecture to those obtained without the conjecture. For the family of quadratic Dirichlet L -functions, a result of Miller [29] matches the lower-order terms of the one-level density function obtained with or without the assumption of the ratios conjecture, when the Fourier transforms of test functions are supported in $(-1, 1)$. When the support is enlarged to $(-2, 2)$, Fiorilli *et al.* obtained the lower-order terms of the one-level density function in [10, 11] by applying either the ratios conjecture or otherwise. Their work assumes the GRH, and the results obtained are further shown to match up to the first lower-order term in [11].

Motivated by the above works, our next objective in the paper is to evaluate $D(\phi; w, X)$ using the ratios conjecture. We formulate the appropriate version of the ratios conjecture concerning our family \mathcal{F} in Conjecture 5.1, and use it to prove in Section 5 the following asymptotic expression of $D(\phi; w, X)$.

THEOREM 1.2. *Assume the truth of the GRH for the family of L -functions in \mathcal{F} as well as for $\zeta_K(s)$ and Conjecture 5.1. Let $w(t)$ be an even, nonzero and nonnegative Schwartz function and $\phi(x)$ an even Schwartz function whose Fourier transform $\widehat{\phi}(u)$ has compact support. Then, for any $\varepsilon > 0$,*

$$\begin{aligned}
 D(\phi; w, X) = & \frac{1}{W(X)} \sum_{(c, 1+i)=1}^* w\left(\frac{N(c)}{X}\right) \frac{1}{2\pi} \int_{\mathbb{R}} \left(2 \frac{\zeta'_K(1+2it)}{\zeta_K(1+2it)} + 2A_\alpha(it, it) \right. \\
 & + \log\left(\frac{32N(c)}{\pi^2}\right) + \frac{\Gamma'}{\Gamma}\left(\frac{1}{2} - it\right) + \frac{\Gamma'}{\Gamma}\left(\frac{1}{2} + it\right) \\
 & - \frac{8}{\pi} X_c \left(\frac{1}{2} + it\right) \zeta_K(1-2it) A(-it, it) \left. \right) \phi\left(\frac{t\mathcal{L}}{2\pi}\right) dt \\
 & + O_\varepsilon(X^{-1/2+\varepsilon}), \tag{1-3}
 \end{aligned}$$

where the functions X_c , A and A_α are given in (5-3), (5-7) and Lemma 5.1, respectively.

Our next goal is to compare the expression given for $D(\phi; w, X)$ in Theorem 1.2 with the one obtained in Theorem 1.1. To this end, we prove (see Lemma 2.6) the following expression for $D(\phi; w, X)$ when ϕ is an even Schwartz test function with compactly

supported Fourier transform:

$$\begin{aligned}
 D(\phi; w, X) &= \frac{\widehat{\phi}(0)}{\mathcal{L}W(X)} \sum_{(c,1+i)=1}^* w\left(\frac{N(c)}{X}\right) \log N(c) \\
 &\quad + \frac{\widehat{\phi}(0)}{\mathcal{L}} \left(\log \frac{32}{\pi^2} + 2 \frac{\Gamma'}{\Gamma} \left(\frac{1}{2} \right) \right) \\
 &\quad - \frac{2}{\mathcal{L}W(X)} \sum_{(c,1+i)=1}^* w\left(\frac{N(c)}{X}\right) \sum_{j \geq 1} S_j(\chi_{i(1+i)^5 c}, \mathcal{L}; \widehat{\phi}) \\
 &\quad + \frac{2}{\mathcal{L}} \int_0^\infty \frac{e^{-x/2}}{1 - e^{-x}} \left(\widehat{\phi}(0) - \widehat{\phi}\left(\frac{x}{\mathcal{L}}\right) \right) dx, \tag{1-4}
 \end{aligned}$$

where

$$S_j(\chi_{i(1+i)^5 c}, \mathcal{L}; \widehat{\phi}) = \sum_{\varpi \equiv 1 \pmod{(1+i)^3}} \frac{\log N(\varpi)}{\sqrt{N(\varpi^j)}} \chi_{i(1+i)^5 c}(\varpi^j) \widehat{\phi}\left(\frac{\log N(\varpi^j)}{\mathcal{L}}\right),$$

with the sum over ϖ running over primes in O_K . Here we note that in O_K , every ideal coprime to $(1+i)$ has a unique generator congruent to 1 modulo $(1+i)^3$, and such a generator is called primary. We use $\sum_{\varpi \equiv 1 \pmod{(1+i)^3}}$ (or sum over other variables) to indicate a sum over primary elements in O_K .

For any function W , we denote its Mellin transform by MW , so that

$$MW(s) = \int_0^\infty W(t) t^s \frac{dt}{t}. \tag{1-5}$$

We further note that around $s = 1$,

$$\zeta_K(s) = \frac{\pi}{4} \cdot \frac{1}{s-1} + \gamma_K + O(|s-1|), \tag{1-6}$$

where γ_K is a constant. We write $\gamma = 0.57 \dots$ for the Euler constant.

The following result shows the agreement of the two expressions for $D(\phi; w, X)$ given in (1-3) and (1-4) up to the first lower-order term.

THEOREM 1.3. *Assume the truth of the GRH for the family of L-functions in \mathcal{F} as well as for $\zeta_K(s)$ and Conjecture 5.1. Let $w(t)$ be an even, nonzero and nonnegative Schwartz function and $\phi(x)$ an even Schwartz function whose Fourier transform $\widehat{\phi}(u)$ has compact support. Then expression (1-3) gives that*

$$\begin{aligned}
 D(\phi; w, X) &= \widehat{\phi}(0) + \int_1^\infty \widehat{\phi}(\tau) d\tau + \frac{\widehat{\phi}(0)}{\mathcal{L}} \left(\log \frac{32}{\pi^2} + 2 \frac{\Gamma'}{\Gamma} \left(\frac{1}{2} \right) + \frac{2}{\widehat{w}(0)} \int_0^\infty w(x) \log x dx \right) \\
 &\quad + \frac{2}{\mathcal{L}} \int_0^\infty \frac{e^{-t/2}}{1 - e^{-t}} \left(\widehat{\phi}(0) - \widehat{\phi}\left(\frac{t}{\mathcal{L}}\right) \right) dt
 \end{aligned}$$

$$\begin{aligned}
 & - \frac{2}{\mathcal{L}} \sum_{\substack{\varpi \equiv 1 \pmod{(1+i)^3} \\ j \geq 1}} \frac{\log N(\varpi)}{N(\varpi)^j} \left(1 + \frac{1}{N(\varpi)}\right)^{-1} \widehat{\phi}\left(\frac{2j \log N(\varpi)}{\mathcal{L}}\right) \\
 & + \frac{\widehat{\phi}(1)}{\mathcal{L}} \left(2\gamma + \log\left(\frac{\pi^2}{27^3}\right) + 2\frac{\zeta'_K(2)}{\zeta_K(2)} - \frac{8}{\pi}\gamma_K - \frac{Mw'(1)}{Mw(1)}\right) + O(\mathcal{L}^{-2}). \tag{1-7}
 \end{aligned}$$

Also, when $\sup(\text{supp}(\widehat{\phi}(u))) < 2$, the above expression agrees with that given in (1-4).

We give the proof of Theorem 1.3 in Section 6. Our approach is inspired by the proof of [11, Theorems 1.1 and 1.4], although the computation in our situation is more involved.

2. Preliminaries

2.1. Number fields background. Recall that in this paper, $K = \mathbb{Q}(i)$ is the Gaussian field. We denote by U_K the group of units in \mathcal{O}_K , so that $U_K = \{\pm 1, \pm i\}$. As it is well known that K has class number one, we do not distinguish n and (n) when this causes no confusion from the context. We therefore write $\mu_{[i]}(n)$ to mean the Möbius function $\mu_{[i]}((n))$. We use ϖ to denote a prime (or prime ideal) in K and write $\Lambda(n)$ for the von Mangoldt function on \mathcal{O}_K so that

$$\Lambda(n) = \begin{cases} \log N(\varpi) & n = \varpi^k, \varpi \text{ prime, } k \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let $(\cdot/n)_4$ stand for the quartic residue symbol on \mathcal{O}_K . For a prime $\varpi \in \mathcal{O}_K$ with $N(\varpi) \neq 2$, the quartic symbol is defined for $a \in \mathcal{O}_K$, $(a, \varpi) = 1$ by $(a/\varpi)_4 \equiv a^{(N(\varpi)-1)/4} \pmod{\varpi}$, with $(a/\varpi)_4 \in \{\pm 1, \pm i\}$. When $\varpi|a$, we define $(a/\varpi)_4 = 0$. Then the quartic symbol can be extended to any composite n with $(N(n), 2) = 1$ multiplicatively. We further define $(\cdot/n) = (\cdot/n)_4^2$ to be the quadratic residue symbol for these n .

We say that an element $c \in \mathcal{O}_K$ (or the ideal (c)) is odd if $(c, 1+i) = 1$. Note that in \mathcal{O}_K , every odd ideal has a unique generator congruent to 1 modulo $(1+i)^3$. Such a generator is called primary. For two primary integers $m, n \in \mathcal{O}_K$, we note that the quadratic reciprocity law (see [15, formula (2.1)]) gives

$$\left(\frac{m}{n}\right) = \left(\frac{n}{m}\right). \tag{2-1}$$

Let χ denote a Hecke character of K . We say that χ is of trivial infinite type if its component at the infinite place of K is trivial. In particular, χ_c defined earlier is a Hecke character of trivial infinite type. It is further shown in [15, Section 2.1] that when c is square-free and coprime to $1+i$, $\chi_{i(1+i)^5c}$ defines a primitive Hecke character $\text{mod}((1+i)^5c)$ of trivial infinite type.

For any primitive Hecke character $\chi \pmod{m}$ of trivial infinite type, let

$$\Lambda(s, \chi) = (|D_K|N(m))^{s/2} (2\pi)^{-s} \Gamma(s) L(s, \chi),$$

where $L(s, \chi)$ is the L -function attached to χ and $D_K = -4$ is the discriminant of K . In particular, we use $\zeta_K(s)$ to denote the Dedekind zeta function of K .

It was shown by Hecke that $\Lambda(s, \chi)$ is an entire function and satisfies the functional equation [22, Theorem 3.8]

$$\Lambda(s, \chi) = W(\chi)(N(m))^{-1/2} \Lambda(1 - s, \bar{\chi}),$$

where $|W(\chi)| = (N(m))^{1/2}$.

2.2. Poisson summation. For any $r, n \in \mathcal{O}_K$ with n odd, we define the Gauss sum $g(r, n)$ as

$$g(r, n) = \sum_{x \pmod n} \left(\frac{x}{n}\right) \tilde{e}\left(\frac{rx}{n}\right)$$

where $\tilde{e}(z) = \exp(2\pi i((z/2i) - (\bar{z}/2i)))$. It is shown in [15, Lemma 2.2] that, for a primary prime ϖ ,

$$g(r, \varpi) = \left(\frac{ir}{\varpi}\right) N(\varpi)^{1/2}.$$

We quote the following Poisson summation formula from [15, Lemma 2.7].

LEMMA 2.1. *Let $n \in \mathcal{O}_K$ be primary and χ a quadratic character (mod n) of trivial infinite type. For any Schwartz class function W , we have*

$$\sum_{m \in \mathcal{O}_K} \chi(m) W\left(\frac{N(m)}{X}\right) = \frac{X}{N(n)} \sum_{k \in \mathcal{O}_K} g(k, n) \tilde{W}\left(\sqrt{\frac{N(k)X}{N(n)}}\right) \tag{2-2}$$

and

$$\sum_{m \in \mathcal{O}_K} W\left(\frac{N(m)}{X}\right) = X \sum_{k \in \mathcal{O}_K} \tilde{W}(\sqrt{N(k)X}), \tag{2-3}$$

where

$$\tilde{W}(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W(N(x + yi)) \tilde{e}(-t(x + yi)) dx dy, \quad t \geq 0. \tag{2-4}$$

We include here our conventions for various transforms used in this paper. For any function W , we write \tilde{W} for the Fourier transform of W and recall that its Mellin transform $\mathcal{M}w$ is defined in (1-5). We note that, for any Schwartz class function W and any integer $E \geq 0$, integration by parts $E + 1$ times yields that for $\text{Re}(s) > 0$,

$$\mathcal{M}W(s) \ll \frac{1}{|s|(1 + |s|)^E}. \tag{2-5}$$

Furthermore, we set

$$\tilde{\tilde{W}}(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{W}(N(u + vi)) \tilde{e}(-t(u + vi)) du dv, \quad t \geq 0.$$

When $W(t)$ is a real smooth function, one follows the arguments that lead to the bounds given in [15, formula (2.12)] to get that both \widetilde{W} and \widetilde{W} are real and

$$\widetilde{W}^{(\mu)}(t), \widetilde{W}^{(\mu)}(t) \ll_j \min\{1, |t|^{-j}\} \tag{2-6}$$

for all integers $\mu \geq 0, j \geq 1$ and all real t .

2.3. Some consequences of the GRH. In this section we state a few results that are derived by using the GRH. The first one is for sums over primes.

LEMMA 2.2. *Suppose that the GRH is true. For any Hecke character $\chi(\bmod m)$ of trivial infinite type, we have for $X \geq 1$,*

$$\sum_{\substack{N(\varpi) \leq X \\ \varpi \equiv 1 \pmod{(1+i)^3}}} \chi(\varpi) \log N(\varpi) = \delta_\chi X + O(X^{1/2} \log^2(2X) \log N(m)), \tag{2-7}$$

where $\delta_\chi = 1$ if χ is principal and $\delta_\chi = 0$ otherwise. Moreover, we have

$$\sum_{\substack{N(\varpi) \leq X \\ \varpi \equiv 1 \pmod{(1+i)^3}}} \frac{\log N(\varpi)}{N(\varpi)} = \log X + O(1). \tag{2-8}$$

PROOF. The formula in (2-7) follows directly from [22, Theorem 5.15] and (2-8) is derived from (2-7) by taking χ to be the principal character modulo 1 and using partial summation. \square

Our next two lemmas provide estimations on certain weighted quadratic character sums. The following one is a generalization of [9, Lemma 2.10].

LEMMA 2.3. *Suppose that the GRH is true. For any even, nonzero and nonnegative Schwartz function w , we have for any primary $n \in O_K$ and $\epsilon > 0$,*

$$\begin{aligned} \sum_{(c, 1+i)=1}^* w\left(\frac{N(c)}{X}\right) \left(\frac{i(1+i)^5 c}{n}\right) &= \delta_{\chi_n} \frac{\pi X}{3\zeta_K(2)} \widehat{w}(0) \prod_{\varpi|n} \left(1 + \frac{1}{N(\varpi)}\right)^{-1} \\ &+ O(N(n)^{3(1-\delta_{\chi_n})/8+\epsilon} X^{1/4+\epsilon}). \end{aligned}$$

Here we recall that \sum^* denotes the sum over square-free elements in O_K .

PROOF. Since each c coprime to $1+i$ can be uniquely written as the product of a unit and a primary element, it follows that

$$\begin{aligned} &\sum_{(c, 1+i)=1}^* w\left(\frac{N(c)}{X}\right) \left(\frac{(1+i)^5 c}{n}\right) \\ &= \left(1 + \left(\frac{-1}{n}\right) + \left(\frac{i}{n}\right) + \left(\frac{i^3}{n}\right)\right) \sum_{c \equiv 1 \pmod{(1+i)^3}}^* w\left(\frac{N(c)}{X}\right) \left(\frac{(1+i)c}{n}\right) \end{aligned}$$

$$= 2\left(1 + \left(\frac{i}{n}\right)\right)\left(\frac{1+i}{n}\right) \sum_{c \equiv 1 \pmod{(1+i)^3}}^* w\left(\frac{N(c)}{X}\right)\left(\frac{c}{n}\right).$$

Note further that the quadratic reciprocity law (2-1) allows us to write $\chi_n(c)$ for (c/n) , and χ_n is a Hecke character. We then apply the Mellin inversion formula and get

$$\begin{aligned} & \sum_{c \equiv 1 \pmod{(1+i)^3}}^* w\left(\frac{N(c)}{X}\right)\chi_n(c) \\ &= \frac{1}{2\pi i} \int_{(2)} \sum_{c \equiv 1 \pmod{(1+i)^3}} \frac{\mu_{|i|}^2(c)\chi_n(c)}{N(c)^s} X^s \mathcal{M}w(s) ds \\ &= \frac{1}{2\pi i} \int_{(2)} \prod_{\substack{\varpi \equiv 1 \pmod{(1+i)^3} \\ (\varpi, 1+i)=1}} \left(1 + \frac{\chi_n(\varpi)}{N(\varpi)^s}\right) \mathcal{M}w(s) X^s ds \\ &= \frac{1}{2\pi i} \int_{(2)} \frac{L(s, \chi_n)}{L(2s, \chi_n^2)} \left(1 + \frac{\chi_n(1+i)}{N(1+i)^s}\right)^{-1} \mathcal{M}w(s) X^s ds. \end{aligned}$$

Here and henceforth, we write $\int_{(c)}$ for the integral over the vertical line with $\text{Re}(s) = c$.

We shift the line of integration to $\text{Re}(s) = 1/4 + \varepsilon$ and we encounter a pole at $s = 1$ only when χ_n is a principal character. In that case, the residue is easily seen (recall that the residue of $\zeta_K(s)$ at $s = 1$ is $\pi/4$) to be

$$\begin{aligned} & \frac{\pi}{6} \zeta_K^{-1}(2) \prod_{\varpi|n} \left(1 + \frac{1}{N(\varpi)}\right)^{-1} \mathcal{M}w(1)X \\ &= \frac{\pi}{12} \zeta_K^{-1}(2) \prod_{\varpi|n} \left(1 + \frac{1}{N(\varpi)}\right)^{-1} \widehat{w}(0)X, \end{aligned}$$

by noting that $\mathcal{M}w(1) = \widehat{w}(0)/2$ when w is even. The remaining integral over the line $\text{Re}(s) = 1/4 + \varepsilon$ can be estimated by using (2-5) for a suitable E and the bound

$$L^{-1}(2s, \chi_n) \ll (N(n)(1 + \text{Im}(s))^\varepsilon, \tag{2-9}$$

assuming the GRH, which follows from [22, Theorem 5.19]. This gives the result when χ_n is a principal character.

When χ_n is not principal, we apply the convexity bound [22, formula (5.20)] for L -functions attached to nonprincipal characters, such that

$$L(s, \chi_n) \ll_\varepsilon (N(n)(|s| + 1)^2)^{(1-\text{Re}(s))/2+\varepsilon/2} \quad (0 \leq \text{Re}(s) \leq 1).$$

Combining this with (2-9) allows us to readily deduce the assertion of the lemma for χ_n being nonprincipal. This completes the proof. □

By taking $n = 1$ in Lemma 2.3, we immediately obtain that

$$W(X) = \frac{\pi X}{3\zeta_K(2)} \widehat{w}(0) + O(X^{1/4+\varepsilon}). \tag{2-10}$$

LEMMA 2.4. *Suppose that the GRH is true. For any even, nonzero and nonnegative Schwartz function w , we have*

$$\begin{aligned} & \frac{1}{W(X)} \sum_{(c,1+i)=1}^* w\left(\frac{N(c)}{X}\right) \log N(c) \\ &= \log X + \frac{2}{\widehat{w}(0)} \int_0^\infty w(x) \log x \, dx + O(X^{-1/2+\varepsilon}). \end{aligned}$$

PROOF. We have

$$\begin{aligned} & \sum_{(c,1+i)=1}^* w\left(\frac{N(c)}{X}\right) \log N(c) \\ &= -\frac{4}{2\pi i} \int_{(2)} \frac{d}{ds} \left(\sum_{c \equiv 1 \pmod{(1+i)^3}} \frac{\mu_{|i|}^2(c)}{N(c)^s} \right) \mathcal{M}w(s) X^s \, ds \\ &= -\frac{4}{2\pi i} \int_{(2)} \frac{d}{ds} \left(\frac{\zeta_K(s)}{\zeta_K(2s)(1+2^{-s})} \right) \mathcal{M}w(s) X^s \, ds. \end{aligned}$$

We shift the contour of integration to the line $\text{Re}(s) = 1/4 + \varepsilon$. Note that under the GRH, the only poles of the function

$$\begin{aligned} & \frac{d}{ds} \left(\frac{\zeta_K(s)}{\zeta_K(2s)(1+2^{-s})} \right) \\ &= \frac{\zeta'_K(s)}{\zeta_K(2s)(1+2^{-s})} - \frac{2\zeta'_K(2s)\zeta_K(s)}{\zeta_K^2(2s)(1+2^{-s})} + \frac{\log 2 \cdot \zeta_K(s)}{2^s \zeta_K(2s)(1+2^{-s})^2} \end{aligned}$$

in the region $1/4 + \varepsilon \leq \text{Re}(s) \leq 2$ are at $s = 1$ and $s = 1/2$. Only the contribution of the residue at $s = 1$ is much greater than X . It is then easy to compute the contribution of the residues to be

$$\frac{2\pi}{3\zeta_K(2)} \mathcal{M}w(1)X \log X + \frac{2\pi}{3\zeta_K(2)} (\mathcal{M}w)'(1)X + O(X^{1/2+\varepsilon}).$$

The assertion of the lemma follows from this and (2-10), by noting that $\mathcal{M}w(1) = \widehat{w}(0)/2$. □

2.4. The explicit formula. Let f be an even, positive Schwartz function whose Fourier transform \hat{f} is a smooth function with compact support. Let χ be a primitive Hecke character $\chi(\text{mod } m)$ of trivial infinite type. In this section we derive an explicit formula which allows us to convert the evaluation of f at the nontrivial zeros of $L(s, \chi)$ to a sum over powers of prime ideals. Note that the nontrivial zeros of $L(s, \chi)$ are

precisely those of the corresponding $\Lambda(s, \chi)$. For some $c > 1$, consider the integral

$$\frac{1}{2\pi i} \int_{(c)} \frac{\Lambda'}{\Lambda}(s, \chi) f\left(\frac{s-1/2}{2\pi i}\right) ds.$$

By moving the line of integration to $1 - c$, we obtain

$$\begin{aligned} & \frac{1}{2\pi i} \int_{(c)} \frac{\Lambda'}{\Lambda}(s, \chi) f\left(\frac{s-1/2}{2\pi i}\right) ds \\ &= \sum_j f\left(\frac{\gamma_{\chi j}}{2\pi}\right) + \frac{1}{2\pi i} \int_{(1-c)} \frac{\Lambda'}{\Lambda}(s, \chi) f\left(\frac{s-1/2}{2\pi i}\right) ds. \end{aligned}$$

Now the functional equation $\Lambda(s, \chi) = W(\chi)(N(m))^{-1/2} \Lambda(1-s, \bar{\chi})$ implies

$$\frac{\Lambda'}{\Lambda}(s, \chi) = -\frac{\Lambda'}{\Lambda}(1-s, \bar{\chi}).$$

It follows that

$$\begin{aligned} \sum_j f\left(\frac{\gamma_{\chi j}}{2\pi}\right) &= \frac{1}{2\pi i} \int_{(c)} \frac{\Lambda'}{\Lambda}(s, \chi) f\left(\frac{s-1/2}{2\pi i}\right) ds \\ &+ \frac{1}{2\pi i} \int_{(c)} \frac{\Lambda'}{\Lambda}(s, \bar{\chi}) f\left(\frac{1/2-s}{2\pi i}\right) ds. \end{aligned}$$

Using

$$\frac{\Lambda'}{\Lambda}(s, \chi) = \frac{1}{2} \log \frac{|D_K|N(m)}{(2\pi)^2} + \frac{\Gamma'}{\Gamma}(s) + \frac{L'}{L}(s, \chi),$$

we obtain that

$$\frac{1}{2\pi i} \int_{(c)} \frac{\Lambda'}{\Lambda}(s, \chi) f\left(\frac{s-1/2}{2\pi i}\right) ds + \frac{1}{2\pi i} \int_{(c)} \frac{\Lambda'}{\Lambda}(s, \bar{\chi}) f\left(\frac{1/2-s}{2\pi i}\right) ds = T_1 + T_2,$$

where

$$\begin{aligned} T_1 &= \frac{1}{2\pi i} \int_{(c)} \left(\frac{1}{2} \log \frac{|D_K|N(m)}{(2\pi)^2} + \frac{\Gamma'}{\Gamma}(s) \right) f\left(\frac{s-1/2}{2\pi i}\right) ds \\ &+ \frac{1}{2\pi i} \int_{(c)} \left(\frac{1}{2} \log \frac{|D_K|N(m)}{(2\pi)^2} + \frac{\Gamma'}{\Gamma}(s) \right) f\left(\frac{1/2-s}{2\pi i}\right) ds \end{aligned}$$

and

$$T_2 = \frac{1}{2\pi i} \int_{(c)} \frac{L'}{L}(s, \chi) f\left(\frac{s-1/2}{2\pi i}\right) ds + \frac{1}{2\pi i} \int_{(c)} \frac{L'}{L}(s, \bar{\chi}) f\left(\frac{1/2-s}{2\pi i}\right) ds.$$

For T_1 , we move the line of integration to $1/2$ and apply the change of variables $s = 1/2 + 2\pi it$. In so doing, we obtain

$$\begin{aligned} T_1 &= \int_{-\infty}^{\infty} \left(\log \frac{|D_K|N(m)}{(2\pi)^2} + \frac{\Gamma'}{\Gamma}\left(\frac{1}{2} + 2\pi it\right) + \frac{\Gamma'}{\Gamma}\left(\frac{1}{2} - 2\pi it\right) \right) f(t) dt \\ &= \int_{-\infty}^{\infty} \left(\log \frac{|D_K|N(m)}{(2\pi)^2} + 2\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}\right) \right) f(t) dt \\ &\quad + \int_0^{\infty} \frac{e^{-t/2}}{1 - e^{-t}} (2\hat{f}(0) - \hat{f}(t) - \hat{f}(-t)) dt, \end{aligned} \quad (2-11)$$

where the second equality above follows from [33, Lemma 12.14].

We express T_2 as

$$\begin{aligned} T_2 &= - \sum_{\substack{(n) \\ n \in O_K}} \chi(n) \Lambda(n) \frac{1}{2\pi i} \int_{(c)} \frac{1}{N(n)^s} f\left(\frac{s-1/2}{2\pi i}\right) ds \\ &\quad - \sum_{\substack{(n) \\ n \in O_K}} \bar{\chi}(n) \Lambda(n) \frac{1}{2\pi i} \int_{(c)} \frac{1}{N(n)^s} f\left(\frac{1/2-s}{2\pi i}\right) ds. \end{aligned}$$

Moving the lines of integration for I_2 to $\text{Re}(s) = 1/2$ and setting $s = 1/2 + 2\pi it$, the integrations become

$$\int_{-\infty}^{\infty} \frac{1}{N(n)^{1/2 \pm 2\pi it}} f(t) dt = \frac{1}{\sqrt{N(n)}} \hat{f}(\mp \log N(n)).$$

Thus,

$$T_2 = - \sum_{\substack{(n) \\ n \in O_K}} \frac{\chi(n) \Lambda(n)}{\sqrt{N(n)}} \hat{f}(-\log N(n)) - \sum_{\substack{(n) \\ n \in O_K}} \frac{\bar{\chi}(n) \Lambda(n)}{\sqrt{N(n)}} \hat{f}(\log N(n)). \quad (2-12)$$

We then derive from (2-11) and (2-12) that

$$\begin{aligned} \sum_j f\left(\frac{\gamma_{\chi,j}}{2\pi}\right) &= \int_{-\infty}^{\infty} \left(\log \frac{|D_K|N(m)}{(2\pi)^2} + 2\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}\right) \right) f(t) dt \\ &\quad + \int_0^{\infty} \frac{e^{-t/2}}{1 - e^{-t}} (2\hat{f}(0) - \hat{f}(t) - \hat{f}(-t)) dt \\ &\quad - \sum_{\substack{(n) \\ n \in O_K}} \frac{\chi(n) \Lambda(n)}{\sqrt{N(n)}} \hat{f}(-\log N(n)) - \sum_{\substack{(n) \\ n \in O_K}} \frac{\bar{\chi}(n) \Lambda(n)}{\sqrt{N(n)}} \hat{f}(\log N(n)). \end{aligned}$$

Now an easy computation in Fourier transforms gives

$$\begin{aligned} \sum_j f\left(\frac{\log X \gamma_{\chi,j}}{2\pi}\right) &= \frac{1}{\log X} \int_{-\infty}^{\infty} \left(\log \frac{|D_K|N(m)}{(2\pi)^2} + 2\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}\right) \right) f(t) dt \\ &\quad + \frac{1}{\log X} \int_0^{\infty} \frac{e^{-t/2}}{1 - e^{-t}} \left(2\hat{f}(0) - \hat{f}\left(\frac{t}{\log X}\right) - \hat{f}\left(-\frac{t}{\log X}\right) \right) dt \\ &\quad - \frac{1}{\log X} \sum_{\substack{(n) \\ n \in \mathcal{O}_K}} \frac{\chi(n)\Lambda(n)}{\sqrt{N(n)}} \hat{f}\left(-\frac{\log N(n)}{\log X}\right) \\ &\quad - \frac{1}{\log X} \sum_{\substack{(n) \\ n \in \mathcal{O}_K}} \frac{\bar{\chi}(n)\Lambda(n)}{\sqrt{N(n)}} \hat{f}\left(\frac{\log N(n)}{\log X}\right). \end{aligned}$$

Recall that $\mathcal{L} = \log X$ and note that as f is even, so is \hat{f} . Thus, when χ is a quadratic Hecke character, we can simplify the above expression as

$$\begin{aligned} \sum_j f\left(\frac{\mathcal{L}\gamma_{\chi,j}}{2\pi}\right) &= \frac{1}{\mathcal{L}} \left(\log \frac{|D_K|N(m)}{(2\pi)^2} + 2\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}\right) \right) \hat{f}(0) \\ &\quad + \frac{2}{\mathcal{L}} \int_0^{\infty} \frac{e^{-t/2}}{1 - e^{-t}} \left(\hat{f}(0) - \hat{f}\left(\frac{t}{\mathcal{L}}\right) \right) dt \\ &\quad - \frac{2}{\mathcal{L}} \sum_{\substack{(n) \\ n \in \mathcal{O}_K}} \frac{\chi(n)\Lambda(n)}{\sqrt{N(n)}} \hat{f}\left(\frac{\log N(n)}{\mathcal{L}}\right). \end{aligned} \tag{2-13}$$

Recall that every odd prime $\varpi \in \mathcal{O}_K$ has a primary generator. In our paper, we work explicitly with the Hecke characters $\chi_{i(1+i)^5c}$ for odd, square-free c . Our choice for such characters is inspired by the Dirichlet characters χ_{8d} for odd, square-free rational integers d considered by Soundararajan [40] in his work on nonvanishing of quadratic Dirichlet L -functions at the central value. The advantage of using the characters $\chi_{i(1+i)^5c}$ is that, besides their primitivity, the presence of the factors $i(1+i)^3$ makes the resulting expression much neater after applying the Poisson summation formula given in Lemma 2.1. We now apply the formula given in (2-13) to the special case $\chi_{i(1+i)^5c}$ for odd, square-free c to arrive at the following lemma.

LEMMA 2.5 (Explicit formula). *Let $\phi(x)$ be an even Schwartz function whose Fourier transform $\hat{\phi}(u)$ has compact support. Let c be an odd and square-free element of \mathcal{O}_K satisfying $(c, 1+i) = 1$. We have*

$$\begin{aligned} S(\chi_{i(1+i)^5c}, \phi) &= \frac{1}{\mathcal{L}} \left(\log \frac{32N(c)}{\pi^2} + 2\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}\right) \right) \hat{\phi}(0) \\ &\quad + \frac{2}{\mathcal{L}} \int_0^{\infty} \frac{e^{-t/2}}{1 - e^{-t}} \left(\hat{\phi}(0) - \hat{\phi}\left(\frac{t}{\mathcal{L}}\right) \right) dt - \frac{2}{\mathcal{L}} \sum_{j \geq 1} S_j(\chi_{i(1+i)^5c}, \mathcal{L}; \hat{\phi}), \end{aligned}$$

where

$$S_j(\chi_{i(1+i)^5c}, \mathcal{L}; \hat{\phi}) = \sum_{\varpi \equiv 1 \pmod{(1+i)^3}} \frac{\log N(\varpi)}{\sqrt{N(\varpi^j)}} \chi_{i(1+i)^5c}(\varpi^j) \hat{\phi}\left(\frac{\log N(\varpi^j)}{\mathcal{L}}\right),$$

with the sum over ϖ running over primes in O_K .

Using Lemma 2.5, upon summing over c against the weight function w , we obtain the following result (the formula in (1-4)) for $D(\phi; w, X)$.

LEMMA 2.6. *Assume that ϕ is an even Schwartz test function whose Fourier transform has compact support. Then we have*

$$\begin{aligned} D(\phi; w, X) &= \frac{\widehat{\phi}(0)}{\mathcal{L}W(X)} \sum_{(c,1+i)=1}^* w\left(\frac{N(c)}{X}\right) \log N(c) + \frac{\widehat{\phi}(0)}{\mathcal{L}} \left(\log \frac{32}{\pi^2} + 2 \frac{\Gamma'}{\Gamma}\left(\frac{1}{2}\right) \right) \\ &\quad - \frac{2}{\mathcal{L}W(X)} \sum_{(c,1+i)=1}^* w\left(\frac{N(c)}{X}\right) \sum_{j \geq 1} S_j(\chi_{i(1+i)^5c}, \mathcal{L}; \hat{\phi}) \\ &\quad + \frac{2}{\mathcal{L}} \int_0^\infty \frac{e^{-x/2}}{1 - e^{-x}} \left(\widehat{\phi}(0) - \widehat{\phi}\left(\frac{x}{\mathcal{L}}\right) \right) dx. \end{aligned}$$

Now let $w(t)$ be an even, nonzero and nonnegative Schwartz function as in the theorems. We define

$$g(y) = \widetilde{w}(\sqrt{2}y), \quad g_1(y) = \widetilde{g}(\sqrt{y}), \tag{2-14}$$

where we recall that for any function W , the definition of \widetilde{W} is given in (2-4).

Our next lemma establishes a relation between the Mellin transforms of g and g_2 . This is a generalization of [11, formula (2.3)].

LEMMA 2.7. *For any $z \in \mathbb{C}, z \neq 0, -1$, we have*

$$\zeta_K(z+1) \mathcal{M}g_1(z+1) = \zeta_K(-z) \mathcal{M}g(-z).$$

PROOF. Our proof of this lemma is motivated by Riemann’s proof of the functional equation of the Riemann zeta function $\zeta(s)$ (see [6, Section 8]). We first note that, for $\text{Re}(z) > 1$,

$$\begin{aligned} \zeta_K(z) \mathcal{M}g(z) &= \frac{1}{4} \sum_{\substack{k \in O_K \\ k \neq 0}} \int_0^\infty g(t) \left(\frac{t}{N(k)}\right)^z \frac{dt}{t} = \frac{1}{4} \sum_{\substack{k \in O_K \\ k \neq 0}} \int_0^\infty g(N(k)t) t^z \frac{dt}{t} \\ &= -\frac{1}{4} \frac{g(0)}{z} + \frac{1}{4} \int_1^\infty \sum_{\substack{k \in O_K \\ k \neq 0}} g(N(k)t) t^z \frac{dt}{t} + \frac{1}{4} \int_0^1 \sum_{k \in O_K} g(N(k)t) t^z \frac{dt}{t}. \end{aligned}$$

Now applying (2-3) to the second of the above sums and another change of variables, we get

$$\begin{aligned} \zeta_K(z)\mathcal{M}g(z) &= -\frac{1}{4} \frac{g(0)}{z} + \frac{1}{4} \frac{\tilde{g}(0)}{z-1} + \frac{1}{4} \int_1^\infty \sum_{\substack{k \in O_K \\ k \neq 0}} g(N(k)t) t^z \frac{dt}{t} \\ &\quad + \frac{1}{4} \int_1^\infty \sum_{\substack{k \in O_K \\ k \neq 0}} \tilde{g}(\sqrt{N(k)t}) t^{-z+1} \frac{dt}{t}. \end{aligned}$$

Note that the last two integrals converge absolutely for all $z \in \mathbb{C}$, by applying estimate (2-6) to both g and \tilde{g} . The last expression above thus gives an analytical extension of $\zeta_K(z)\mathcal{M}g(z)$ to all $z \in \mathbb{C}, z \neq 0, 1$.

Similarly, we also deduce from (2-3) that for $\text{Re}(z) > 0$,

$$\begin{aligned} \zeta_K(z+1)\mathcal{M}g_1(z+1) &= \frac{1}{4} \sum_{\substack{k \in O_K \\ k \neq 0}} \int_0^\infty \tilde{g}(\sqrt{t}) \left(\frac{t}{N(k)}\right)^{z+1} \frac{dt}{t} \\ &= \frac{1}{4} \frac{g(0)}{z} - \frac{1}{4} \frac{\tilde{g}(0)}{z+1} + \frac{1}{4} \int_1^\infty \sum_{\substack{k \in O_K \\ k \neq 0}} \tilde{g}(\sqrt{N(k)t}) t^{z+1} \frac{dt}{t} \\ &\quad + \frac{1}{4} \int_1^\infty \sum_{\substack{k \in O_K \\ k \neq 0}} g(N(k)t) t^{-z} \frac{dt}{t}. \end{aligned}$$

Once again by applying estimate (2-6) to both g and \tilde{g} , we see that the last two integrals above converge absolutely for all $z \in \mathbb{C}$, so the last expression above gives an analytical extension of $\zeta_K(z+1)\mathcal{M}g_1(z+1)$ to all $z \in \mathbb{C}, z \neq 0, 1$. Now, by comparing the above expressions for $\zeta_K(z)\mathcal{M}g(z)$ and $\zeta_K(z+1)\mathcal{M}g_1(z+1)$, we readily deduce the assertion of the lemma. □

3. Analyzing sums over primes

We devote this section to the analysis of the sum over primes in (1-4). We first separate the odd and the even prime powers by writing

$$S_{\text{odd}} = -\frac{2}{\mathcal{L}W(X)} \sum_{(c,1+i)=1}^* w\left(\frac{N(c)}{X}\right) \sum_{\substack{j \geq 1 \\ j \equiv 1 \pmod{2}}} S_j(\chi_{i(1+i)^5 c}, \mathcal{L}; \hat{\phi}), \tag{3-1}$$

and similarly for S_{even} . Moreover, it follows from Lemma 2.3, (2-8) and (2-10) that

$$S_{\text{even}} = -\frac{2}{\mathcal{L}} \sum_{\substack{\varpi \equiv 1 \pmod{(1+i)^3} \\ j \geq 1}} \frac{\log N(\varpi)}{N(\varpi)^j} \left(1 + \frac{1}{N(\varpi)}\right)^{-1} \tilde{\phi}\left(\frac{2j \log N(\varpi)}{\mathcal{L}}\right) + O(X^{-3/4+\varepsilon}). \tag{3-2}$$

3.1. Estimation of S_{even} . We first expand S_{even} into descending powers of \mathcal{L} . We generalize [10, Lemma 3.7] to obtain the following result.

LEMMA 3.1. *Suppose that $\sigma = \sup(\text{supp } \widehat{\phi}) < \infty$. Then, for any integer $M \geq 1$, we have the expansion*

$$S_{\text{even}} = -\frac{\phi(0)}{2} + \sum_{m=1}^M \frac{d_m \widehat{\phi}^{(m-1)}(0)}{\mathcal{L}^m} + O\left(\frac{1}{\mathcal{L}^{M+1}}\right), \tag{3-3}$$

where the coefficients d_m are real numbers that can be given explicitly.

PROOF. It suffices to show that the expansion given in (3-3) is valid if we ignore the $O(X^{-3/4+\varepsilon})$ term in (3-2). As σ is finite, the sum in (3-2) is finite as we must have $N(\varpi)^{2j} \leq X^\sigma$. It follows that the sum of the terms with $j \geq 2$ can be expanded as follows:

$$\begin{aligned} & -\frac{2}{\mathcal{L}} \sum_{\substack{\varpi \equiv 1 \pmod{(1+i)^3} \\ j \geq 2, N(\varpi)^{2j} \leq X^\sigma}} \frac{\log N(\varpi)}{N(\varpi)^j} \left(1 + \frac{1}{N(\varpi)}\right)^{-1} \\ & \quad \times \left(\sum_{m=0}^M \frac{\widehat{\phi}^{(m)}(0)}{m!} \left(\frac{2j \log N(\varpi)}{\mathcal{L}}\right)^m + O\left(\left(\frac{2j \log N(\varpi)}{\mathcal{L}}\right)^{M+1}\right) \right) \\ & = -\frac{2}{\mathcal{L}} \sum_{m=0}^M \frac{\widehat{\phi}^{(m)}(0)}{m! \mathcal{L}^m} \sum_{\substack{\varpi \equiv 1 \pmod{(1+i)^3} \\ j \geq 2, N(\varpi)^{2j} \leq X^\sigma}} \frac{\log N(\varpi) (2j \log N(\varpi))^m}{N(\varpi)^j} \\ & \quad \times \left(1 + \frac{1}{N(\varpi)}\right)^{-1} + O(\mathcal{L}^{-M-2}) \\ & = -\frac{2}{\mathcal{L}} \sum_{m=0}^M \frac{\widehat{\phi}^{(m)}(0)}{m! \mathcal{L}^m} \sum_{\substack{\varpi \equiv 1 \pmod{(1+i)^3} \\ j \geq 2}} \frac{\log N(\varpi) (2j \log N(\varpi))^m}{N(\varpi)^j} \\ & \quad \times \left(1 + \frac{1}{N(\varpi)}\right)^{-1} + O(\mathcal{L}^{-M-2}), \end{aligned} \tag{3-4}$$

by noting that the inner sum of the last expression above converges.

It remains to expand the terms with $j = 1$. For this, we first note that, using the Taylor expansion of $\widehat{\phi}$ around the origin and rewriting $(1 + N(\varpi)^{-1})^{-1}$ as a geometric

series,

$$\begin{aligned}
 & -\frac{2}{\mathcal{L}} \sum_{\varpi \equiv 1 \pmod{(1+i)^3}} \frac{\log N(\varpi)}{N(\varpi)} \left(1 + \frac{1}{N(\varpi)}\right)^{-1} \widehat{\phi}\left(\frac{2 \log N(\varpi)}{\mathcal{L}}\right) \\
 &= -\frac{2}{\mathcal{L}} \sum_{\varpi \equiv 1 \pmod{(1+i)^3}} \frac{\log N(\varpi)}{N(\varpi)} \left(1 + \frac{1}{N(\varpi)}\right)^{-1} \\
 & \quad \times \left(\sum_{m=0}^M \frac{\widehat{\phi}^{(m)}(0)}{m!} \left(\frac{2 \log N(\varpi)}{\mathcal{L}}\right)^m + O\left(\left(\frac{\log N(\varpi)}{\mathcal{L}}\right)^{M+1}\right) \right) \\
 &= -\frac{2}{\mathcal{L}} \sum_{\varpi \equiv 1 \pmod{(1+i)^3}} \frac{\log N(\varpi)}{N(\varpi)} \left(\sum_{m=0}^M \frac{\widehat{\phi}^{(m)}(0)}{m!} \left(\frac{2 \log N(\varpi)}{\mathcal{L}}\right)^m \right. \\
 & \quad \left. + O\left(\left(\frac{\log N(\varpi)}{\mathcal{L}}\right)^{M+1}\right) \right) - \sum_{m=0}^M \frac{\widehat{\phi}^{(m)}(0)C_1(m)}{m! \mathcal{L}^{m+1}} + O(\mathcal{L}^{-M-1}) \\
 &= -\frac{2}{\mathcal{L}} \sum_{\varpi \equiv 1 \pmod{(1+i)^3}} \frac{\log N(\varpi)}{N(\varpi)} \widehat{\phi}\left(\frac{2 \log N(\varpi)}{\mathcal{L}}\right) - \sum_{m=0}^M \frac{\widehat{\phi}^{(m)}(0)C_1(m)}{m! \mathcal{L}^{m+1}} + O_K(\mathcal{L}^{-M-1})
 \end{aligned} \tag{3-5}$$

where

$$C_1(m) = \sum_{\varpi \equiv 1 \pmod{(1+i)^3}} \sum_{l \geq 1} (-1)^l \frac{(2 \log N(\varpi))^{m+1}}{N(\varpi)^{l+1}} < \infty.$$

Next, we note that

$$E(t) := \sum_{\substack{N(\varpi) \leq t \\ \varpi \equiv 1 \pmod{(1+i)^3}}} \log N(\varpi) - t \ll t^{1/2+\varepsilon} \tag{3-6}$$

from (2-7).

We then apply partial summation to arrive at

$$\begin{aligned}
 & -\frac{2}{\mathcal{L}} \sum_{\varpi \equiv 1 \pmod{(1+i)^3}} \frac{\log N(\varpi)}{N(\varpi)} \widehat{\phi}\left(\frac{2 \log N(\varpi)}{\mathcal{L}}\right) \\
 &= -\frac{2}{\mathcal{L}} \int_1^\infty \frac{1}{t} \widehat{\phi}\left(\frac{2 \log t}{\mathcal{L}}\right) d(t + E(t)) \\
 &= -\int_0^\infty \widehat{\phi}(u) du + \frac{2}{\mathcal{L}} \int_1^\infty E(t) \frac{d}{dt} \left(\frac{1}{t} \widehat{\phi}\left(\frac{2 \log t}{\mathcal{L}}\right) \right) dt \\
 &= -\frac{1}{2} \phi(0) + \frac{2}{\mathcal{L}} \int_1^\infty E(t) \frac{d}{dt} \left(\frac{1}{t} \widehat{\phi}\left(\frac{2 \log t}{\mathcal{L}}\right) \right) dt,
 \end{aligned} \tag{3-7}$$

with $E(t)$ defined in (3-6). We can now expand the derivative in the last integrand in (3-7) into Taylor expansions involving powers of $2 \log t / \mathcal{L}$ and use the corresponding series to calculate the last integral above. Note that the new integrals emerging from this process are all convergent because of the bound (3-6). The assertion of the lemma now follows by combining (3-4), (3-5) and (3-7). \square

3.2. Estimation of S_{odd} : Poisson summation. Starting from this section, we concentrate on the estimation of S_{odd} . First note that the contribution from the terms with $j \geq 3$ in (3-1) is $O(X^{-3/4+\epsilon})$ by Lemma 2.3. It thus remains to treat the case for $j = 1$. For this case, we use the Möbius function to detect the condition that c is square-free to get

$$S_{\text{odd}} = -\frac{2}{\mathcal{L}W(X)} \sum_{l \equiv 1 \pmod{(1+i)^3}} \mu_{[i]}(l) \sum_{\varpi \equiv 1 \pmod{(1+i)^3}} \frac{\log N(\varpi)}{\sqrt{N(\varpi)}} \hat{\phi}\left(\frac{\log N(\varpi)}{\log X}\right) \times \sum_{(c,1+i)=1} \left(\frac{i(1+i)cl^2}{\varpi}\right)_w \left(\frac{N(cl^2)}{X}\right) + O(X^{-3/4+\epsilon}).$$

We divide the sum over l above into two parts, one over $l \leq Z$ and the other over $l > Z$, with Z to be chosen optimally later. Note that if c is odd, then $i(1+i)^5 c l^2$ is never a square. Similarly to the treatment of $S_R(X, Y; \hat{\phi}, \Phi)$ in [15, Section 3.3] (except that we use Lemma 2.2 here instead of [15, Lemma 2.5]), we get that the terms with $l > Z$ are

$$\ll X^\epsilon (\log Z)^3 Z^{-1}.$$

For the terms with $l \leq Z$, we apply the Poisson summation (2-2) given in Lemma 2.1 and argue as in [15] (the treatment here is essentially the same as that of $S_M(X, Y; \hat{\phi}, \Phi)$ in [15, Section 3.2]) to arrive at the following lemma.

LEMMA 3.2. *Suppose that the GRH is true. We have for any $Z \geq 1$ and any $\epsilon > 0$,*

$$S_{\text{odd}} = -\frac{X}{\mathcal{L}W(X)} \sum_{\substack{N(l) \leq Z \\ l \equiv 1 \pmod{(1+i)^3}}} \frac{\mu_{[i]}(l)}{N(l^2)} \sum_{k \in \mathbb{Z}[i]} (-1)^{N(k)} \times \sum_{\varpi \equiv 1 \pmod{(1+i)^3}} \frac{\log N(\varpi)}{N(\varpi)} \left(\frac{kl^2}{\varpi}\right) \hat{\phi}\left(\frac{\log N(\varpi)}{\log X}\right) \tilde{w}\left(\sqrt{\frac{N(k)X}{2N(l^2\varpi)}}\right) + O(X^{-3/4+\epsilon} + X^\epsilon (\log Z)^3 Z^{-1}). \tag{3-8}$$

We now generalize [10, Lemma 2.7] to further analyze the sums in (3-8), obtaining the following result.

LEMMA 3.3. *Suppose that the GRH is true and that $\sigma = \sup(\text{supp } \widehat{\phi}) < \infty$. Then, we have for any $1 \leq Z \leq X^2$ and any $\varepsilon > 0$,*

$$S_{\text{odd}} = \frac{X}{W(X)} \sum_{\substack{N(l) \leq Z \\ l \equiv 1 \pmod{(1+i)^3}}} \frac{\mu_{[i]}(l)}{N(l^2)} \left(\frac{1}{2} I_{(1+i)l}(X) - I_l(X) \right) + O(X^{-3/4+\varepsilon} + X^\varepsilon (\log Z)^3 Z^{-1} + ZX^{\sigma/2-1+\varepsilon} + X^{-1/2+\varepsilon} Z^\varepsilon), \tag{3-9}$$

where

$$I_l(X) = \int_0^\infty \widehat{\phi}(u) \sum_{\substack{k \in \mathbb{Z}[i] \\ k \neq 0}} \widetilde{w} \left(2N(k) \sqrt{\frac{X^{1-u}}{2N(l^2)}} \right) du. \tag{3-10}$$

PROOF. Note that as in [15, Section 3.4] the inner sum in (3-8) corresponding to $k = 0$ is zero. It also follows from the treatment of [15, Section 3.5] by setting $U = 1$ and dividing the estimation obtained in [15, formula (3.9)] by X (since our definition of S_{odd} differs from $S_M(X, Y; \widehat{\phi}, \Phi)$ defined in [15] by an extra factor $W^{-1}(X)$) that the contribution of $k \neq \square$ (k is not a square) to the expression for S_{odd} given in (3-8) is

$$\ll ZX^{\sigma/2-1+\varepsilon}.$$

We are left to consider the contribution from $k = \square$ (k is a square), $k \neq 0$, to the expression for S_{odd} given in (3-8). For this, we make a change of variables $k \mapsto k^2$, while noting that $k_1^2 = k_2^2$ if and only if $k_1 = \pm k_2$, and deduce that

$$S_{\text{odd}} = -\frac{X}{2\mathcal{L}W(X)} \sum_{\substack{N(l) \leq Z \\ l \equiv 1 \pmod{(1+i)^3}}} \frac{\mu_{[i]}(l)}{N(l^2)} \sum_{\substack{(\varpi, l) = 1 \\ \varpi \equiv 1 \pmod{(1+i)^3}}} \frac{\log N(\varpi)}{N(\varpi)} \widehat{\phi} \left(\frac{\log N(\varpi)}{\log X} \right) \times \sum_{\substack{k \in \mathbb{Z}[i], k \neq 0 \\ (k, \varpi) = 1}} (-1)^{N(k)} \widetilde{w} \left(N(k) \sqrt{\frac{X}{2N(l^2 \varpi)}} \right) + O(X^{-3/4+\varepsilon} + X^\varepsilon (\log Z)^3 Z^{-1} + ZX^{\sigma/2-1+\varepsilon}).$$

In view of the rapid decay property of \widetilde{w} implied by (2-6), we now remove the condition that $(\varpi, l) = 1$ at the cost of an error

$$\ll \frac{1}{\mathcal{L}} \sum_{\substack{N(l) \leq Z \\ l \equiv 1 \pmod{(1+i)^3}}} \frac{1}{N(l^2)} \sum_{\substack{\varpi | l \\ \varpi \equiv 1 \pmod{(1+i)^3}}} \frac{\log N(\varpi)}{N(\varpi)} \sum_{\substack{k \in \mathbb{Z}[i], k \neq 0 \\ N(k) \leq \sqrt{2N(l^2 \varpi)}/X}} 1 \ll X^{-1/2} \sum_{\substack{N(l) \leq Z \\ l \equiv 1 \pmod{(1+i)^3}}} \frac{1}{N(l)} \sum_{\substack{\varpi | l \\ \varpi \equiv 1 \pmod{(1+i)^3}}} \frac{\log N(\varpi)}{\sqrt{N(\varpi)}} \ll X^{-1/2} Z^\varepsilon,$$

where we use the well-known fact that for $N(l) \geq 3$, the number, $\omega(l)$, of distinct primes in $\mathbb{Z}[i]$ dividing l can be bounded as

$$\omega(l) \ll \frac{\log N(l)}{\log \log N(l)}.$$

One can show similarly that removing the condition $(k, \varpi) = 1$ introduces an error of size much less than $X^{-1/2}Z^\varepsilon$. We thus derive, using (2-7), that

$$\begin{aligned} S_{\text{odd}} &= -\frac{X}{2\mathcal{L}W(X)} \sum_{\substack{N(l) \leq Z \\ l \equiv 1 \pmod{(1+i)^3}}} \frac{\mu_{[i]}(l)}{N(l^2)} \sum_{\varpi \equiv 1 \pmod{(1+i)^3}} \frac{\log N(\varpi)}{N(\varpi)} \\ &\quad \times \hat{\phi}\left(\frac{\log N(\varpi)}{\log X}\right) \sum_{\substack{k \in \mathbb{Z}[i] \\ k \neq 0}} (-1)^{N(k)} \tilde{w}\left(N(k) \sqrt{\frac{X}{2N(l^2\varpi)}}\right) \\ &\quad + O(X^{-3/4+\varepsilon} + X^\varepsilon(\log Z)^3 Z^{-1} + ZX^{\sigma/2-1+\varepsilon} + X^{-1/2+\varepsilon} Z^\varepsilon) \\ &= -\frac{X}{2\mathcal{L}W(X)} \sum_{\substack{N(l) \leq Z \\ l \equiv 1 \pmod{(1+i)^3}}} \frac{\mu_{[i]}(l)}{N(l^2)} \sum_{\substack{k \in \mathbb{Z}[i] \\ k \neq 0}} (-1)^{N(k)} \\ &\quad \times \int_1^\infty \frac{1}{y} \hat{\phi}\left(\frac{\log y}{\mathcal{L}}\right) \tilde{w}\left(N(k) \sqrt{\frac{X}{2N(l^2)y}}\right) d(y+h(y)) \\ &\quad + O(X^{-3/4+\varepsilon} + X^\varepsilon(\log Z)^3 Z^{-1} + ZX^{\sigma/2-1+\varepsilon} + X^{-1/2+\varepsilon} Z^\varepsilon) \\ &= S_1 + S_2 + O(X^{-3/4+\varepsilon} + X^\varepsilon(\log Z)^3 Z^{-1} + ZX^{\sigma/2-1+\varepsilon} + X^{-1/2+\varepsilon} Z^\varepsilon), \end{aligned}$$

where h is a function satisfying $h(y) = O(y^{1/2} \log^2(2y))$,

$$\begin{aligned} S_1 &= -\frac{X}{2\mathcal{L}W(X)} \sum_{\substack{N(l) \leq Z \\ l \equiv 1 \pmod{(1+i)^3}}} \frac{\mu_{[i]}(l)}{N(l^2)} \sum_{\substack{k \in \mathbb{Z}[i] \\ k \neq 0}} (-1)^{N(k)} \\ &\quad \times \int_1^\infty \frac{1}{y} \hat{\phi}\left(\frac{\log y}{\mathcal{L}}\right) \tilde{w}\left(N(k) \sqrt{\frac{X}{2N(l^2)y}}\right) dy \end{aligned}$$

and

$$\begin{aligned} S_2 &= -\frac{X}{2\mathcal{L}W(X)} \sum_{\substack{N(l) \leq Z \\ l \equiv 1 \pmod{(1+i)^3}}} \frac{\mu_{[i]}(l)}{N(l^2)} \sum_{\substack{k \in \mathbb{Z}[i] \\ k \neq 0}} (-1)^{N(k)} \\ &\quad \times \int_1^\infty \frac{1}{y} \hat{\phi}\left(\frac{\log y}{\mathcal{L}}\right) \tilde{w}\left(N(k) \sqrt{\frac{X}{2N(l^2)y}}\right) dh(y). \end{aligned}$$

Note that

$$\begin{aligned}
 S_2 &\ll \frac{1}{\mathcal{L}} \sum_{\substack{N(l) \leq Z \\ l \equiv 1 \pmod{(1+i)^3}}} \frac{1}{N(l^2)} \\
 &\quad \times \sum_{\substack{k \in \mathbb{Z}[i] \\ k \neq 0}} \int_1^\infty y^{1/2} \log^2(2y) \frac{d}{dy} \left(\frac{1}{y} \hat{\phi} \left(\frac{\log y}{\mathcal{L}} \right) \tilde{w} \left(N(k) \sqrt{\frac{X}{2N(l^2)y}} \right) \right) dy \\
 &\ll S_{2,1} + S_{2,2},
 \end{aligned}$$

where

$$S_{2,1} = \frac{1}{\mathcal{L}} \sum_{\substack{N(l) \leq Z \\ l \equiv 1 \pmod{(1+i)^3}}} \frac{1}{N(l^2)} \sum_{\substack{k \in \mathbb{Z}[i] \\ k \neq 0}} \left| \tilde{w} \left(N(k) \sqrt{\frac{X}{2N(l^2)y}} \right) \right|$$

and

$$\begin{aligned}
 S_{2,2} &= \frac{1}{\mathcal{L}} \sum_{\substack{N(l) \leq Z \\ l \equiv 1 \pmod{(1+i)^3}}} \frac{1}{N(l^2)} \int_1^\infty \sum_{\substack{k \in \mathbb{Z}[i] \\ k \neq 0}} y^{1/2} \log^2(2y) \\
 &\quad \times \left(\left| \left(\frac{1}{y^2 \mathcal{L}} \hat{\phi}' \left(\frac{\log y}{\mathcal{L}} \right) \tilde{w} \left(N(k) \sqrt{\frac{X}{2N(l^2)y}} \right) \right) \right| + \left| \frac{1}{y^2} \hat{\phi} \left(\frac{\log y}{\mathcal{L}} \right) \tilde{w}' \left(N(k) \sqrt{\frac{X}{2N(l^2)y}} \right) \right| \right. \\
 &\quad \left. + \left| \hat{\phi} \left(\frac{\log y}{\mathcal{L}} \right) N(k) \sqrt{\frac{X}{8N(l^2)y}} y^{-5/2} \tilde{w}' \left(N(k) \sqrt{\frac{X}{2N(l^2)y}} \right) \right| \right) dy.
 \end{aligned}$$

Using (2-6), we deduce that

$$\sum_{\substack{k \in \mathbb{Z}[i] \\ k \neq 0}} \left| \tilde{w} \left(N(k) \sqrt{\frac{X}{2N(l^2)y}} \right) \right| \ll \sqrt{\frac{N(l^2)y}{X}}$$

and

$$\sum_{\substack{k \in \mathbb{Z}[i] \\ k \neq 0}} \left| N(k) \tilde{w}' \left(N(k) \sqrt{\frac{X}{2N(l^2)y}} \right) \right| \ll \frac{N(l^2)y}{X}.$$

Thus, it follows that

$$\begin{aligned}
 S_{2,2} &\ll \frac{1}{\mathcal{L}} \sum_{\substack{N(l) \leq Z \\ l \equiv 1 \pmod{(1+i)^3}}} \frac{1}{N(l^2)} \int_1^\infty y^{1/2} \log^2(2y) \\
 &\quad \times \left(\frac{1}{y^2 \mathcal{L}} \left| \hat{\phi}'\left(\frac{\log y}{\mathcal{L}}\right) \right| \sqrt{\frac{N(l^2)y}{X}} + \frac{1}{y^2} \left| \hat{\phi}\left(\frac{\log y}{\mathcal{L}}\right) \right| \sqrt{\frac{N(l^2)y}{X}} \right) \\
 &\quad + \left| \hat{\phi}\left(\frac{\log y}{\mathcal{L}}\right) \right| \sqrt{\frac{X}{8N(l^2)}} y^{-5/2} \frac{N(l^2)y}{X} dy \\
 &\ll X^{-1/2} Z^\varepsilon \int_1^\infty \log^2(2y) \left(\left| \hat{\phi}'\left(\frac{\log y}{\mathcal{L}}\right) \right| + \left| \hat{\phi}\left(\frac{\log y}{\mathcal{L}}\right) \right| \right) d\left(\frac{\log y}{\mathcal{L}}\right) \\
 &\ll X^{-1/2+\varepsilon} Z^\varepsilon,
 \end{aligned}$$

where the last estimation above follows by using a change of variable $u = \log y/\mathcal{L}$ to evaluate the proceeding integral and noting that the integrand has compact support. Similarly, we have that $S_{2,1} \ll X^{-1/2+\varepsilon} Z^\varepsilon$ so that

$$S_2 \ll X^{-1/2+\varepsilon} Z^\varepsilon.$$

It follows from the above bound that S_2 can be absorbed into the O -term in (3-9). It remains to evaluate S_1 , and applying exactly the same change of variable as above now leads to

$$\begin{aligned}
 S_1 &= -\frac{X}{2W(X)} \sum_{\substack{N(l) \leq Z \\ l \equiv 1 \pmod{(1+i)^3}}} \frac{\mu_{[i]}(l)}{N(l^2)} \\
 &\quad \times \int_0^\infty \widehat{\phi}(u) \sum_{k \in \mathbb{Z}[i], k \neq 0} (-1)^{N(k)} \widetilde{w}\left(N(k) \sqrt{\frac{X^{1-u}}{2N(l^2)}}\right) du. \tag{3-11}
 \end{aligned}$$

We note that, for any $l \in \mathcal{O}_K$, we have

$$\begin{aligned}
 &\sum_{\substack{k \in \mathbb{Z}[i] \\ k \neq 0}} (-1)^{N(k)} \widetilde{w}\left(N(k) \sqrt{\frac{X^{1-u}}{2N(l^2)}}\right) \\
 &= \sum_{\substack{k \in \mathbb{Z}[i] \\ k \neq 0 \\ 1+ik}} \widetilde{w}\left(N(k) \sqrt{\frac{X^{1-u}}{2N(l^2)}}\right) - \sum_{\substack{k \in \mathbb{Z}[i] \\ k \neq 0 \\ (1+i,k)=1}} \widetilde{w}\left(N(k) \sqrt{\frac{X^{1-u}}{2N(l^2)}}\right)
 \end{aligned}$$

$$\begin{aligned}
 &= 2 \sum_{\substack{k \in \mathbb{Z}[i] \\ k \neq 0 \\ 1+ik}} \tilde{w}\left(N(k) \sqrt{\frac{X^{1-u}}{2N(l^2)}}\right) - \sum_{\substack{k \in \mathbb{Z}[i] \\ k \neq 0}} \tilde{w}\left(N(k) \sqrt{\frac{X^{1-u}}{2N(l^2)}}\right) \\
 &= 2 \sum_{\substack{k \in \mathbb{Z}[i] \\ k \neq 0}} \tilde{w}\left(2N(k) \sqrt{\frac{X^{1-u}}{2N(l^2)}}\right) - \sum_{\substack{k \in \mathbb{Z}[i] \\ k \neq 0}} \tilde{w}\left(N(k) \sqrt{\frac{X^{1-u}}{2N(l^2)}}\right).
 \end{aligned}$$

Applying the above in (3-11), we derive that

$$S_1 = \frac{X}{W(X)} \sum_{\substack{N(l) \leq X \\ l \equiv 1 \pmod{(1+i)^3}}} \frac{\mu_{[i]}(l)}{N(l^2)} \left(\frac{1}{2} I_{(1+i)l}(X) - I_l(X) \right).$$

The above gives precisely the main term in (3-9) for S_{odd} , and this completes the proof. □

3.3. Estimation of S_{odd} : small support. In this section, we apply Lemma 3.3 to show that there are no new lower-order terms in powers of \mathcal{L}^{-1} when $\sigma = \text{sup}(\text{supp } \widehat{\phi}) < 1$. We generalize [10, Proposition 3.1] and state our result in the following proposition.

PROPOSITION 3.4. *Suppose that the GRH is true and that $\sigma = \text{sup}(\text{supp } \widehat{\phi}) < 1$. Then, we have for any $\varepsilon > 0$,*

$$S_{\text{odd}} \ll X^{\sigma/4-1/2+\varepsilon} + X^{3\sigma/4-3/4+\varepsilon}.$$

PROOF. We let

$$\Phi(X) = \sum_{\substack{N(l) \leq X \\ l \equiv 1 \pmod{(1+i)^3}}} \frac{\mu_{[i]}(l)}{N(l)^2} = \frac{4}{3\zeta_K(2)} + O(X^{-3/2+\varepsilon}), \tag{3-12}$$

where the last equality above follows from the observation that under the GRH we have

$$\sum_{\substack{N(l) \leq X \\ (l, 1+i)=1}} \mu_{[i]}(l) \ll X^{1/2+\varepsilon}.$$

We then deduce that for $0 \leq u \leq 1$,

$$\begin{aligned}
 \sum_{\substack{N(l) \leq Z \\ l \equiv 1 \pmod{(1+i)^3}}} \frac{\mu_{[i]}(l)}{N(l^2)} \tilde{w}\left(N(k) \sqrt{\frac{X^{1-u}}{2N(l^2)}}\right) &= \int_{0^+}^Z \tilde{w}\left(\frac{N(k)}{t} \sqrt{\frac{X^{1-u}}{2}}\right) d\Phi(t) \\
 &= \int_{0^+}^Z \tilde{w}\left(\frac{N(k)}{t} \sqrt{\frac{X^{1-u}}{2}}\right) d\left(\frac{4}{3\zeta_K(2)} + O(t^{-3/2+\varepsilon})\right)
 \end{aligned}$$

$$\begin{aligned}
 &= \widetilde{w}\left(\frac{N(k)}{Z} \sqrt{\frac{X^{1-u}}{2}}\right) O(Z^{-3/2+\varepsilon}) + N(k) \sqrt{\frac{X^{1-u}}{2}} \int_{0^+}^Z \widetilde{w}'\left(\frac{N(k)}{t} \sqrt{\frac{X^{1-u}}{2}}\right) O(t^{-3/2+\varepsilon}) \frac{dt}{t^2} \\
 &\ll Z^{-3/2+\varepsilon} \left| \widetilde{w}\left(\frac{N(k)}{Z} \sqrt{\frac{X^{1-u}}{2}}\right) \right| + N(k) X^{(1-u)/2} \int_{0^+}^Z \left| \widetilde{w}'\left(\frac{N(k)}{t} \sqrt{\frac{X^{1-u}}{2}}\right) \right| \frac{dt}{t^{7/2-\varepsilon}}.
 \end{aligned}$$

Note that the part of the last integral for $t \in (0, X^{(1-u)/2-\varepsilon}]$ is $O((N(k)X)^{-A})$ for any $A \geq 1$, by the rapid decay of \widetilde{w}' . Summing over k and integrating over u , we obtain that

$$\begin{aligned}
 &\sum_{\substack{N(l) \leq Z \\ l \equiv 1 \pmod{(1+i)^3}}} \frac{\mu_{[l]}(l)}{N(l^2)} I_l(X) \\
 &\ll \int_0^\infty |\widehat{\phi}(u)| \sum_{\substack{k \in \mathbb{Z}[i] \\ k \neq 0}} \left(Z^{-3/2+\varepsilon} \left| \widetilde{w}\left(\frac{N(k)}{Z} \sqrt{\frac{X^{1-u}}{2}}\right) \right| \right. \\
 &\quad \left. + N(k) X^{(1-u)/2} \int_{0^+}^Z \left| \widetilde{w}'\left(\frac{N(k)}{t} \sqrt{\frac{X^{1-u}}{2}}\right) \right| \frac{dt}{t^{7/2-\varepsilon}} \right) du + X^{-1} \\
 &\ll \int_0^\infty |\widehat{\phi}(u)| \left(Z^{-3/2+\varepsilon} \frac{Z}{X^{(1-u)/2}} + \frac{1}{X^{(1-u)/2}} \int_{X^{(1-u)/2-\varepsilon}}^Z \frac{dt}{t^{3/2-\varepsilon}} \right) du + X^{-1} \\
 &\ll \frac{X^{(\sigma-1)/2}}{Z^{1/2-\varepsilon}} + X^{3\sigma/4-3/4+\varepsilon}.
 \end{aligned}$$

Similarly, we have

$$\sum_{\substack{N(l) \leq Z \\ l \equiv 1 \pmod{(1+i)^3}}} \frac{\mu_{[l]}(l)}{N(l^2)} I_{(1+i)l}(X) \ll \frac{X^{(\sigma-1)/2}}{Z^{1/2-\varepsilon}} + X^{3\sigma/4-3/4+\varepsilon}.$$

Hence, it follows from Lemma 3.3 that for $Z \leq X^2$,

$$S_{\text{odd}} \ll \frac{X^{(\sigma-1)/2}}{Z^{1/2-\varepsilon}} + X^{3\sigma/4-3/4+\varepsilon} + X^{-3/4+\varepsilon} + X^\varepsilon Z^{-1} + ZX^{\sigma/2-1+\varepsilon} + X^{-1/2+\varepsilon} Z^\varepsilon.$$

The result follows by taking $Z = X^{1/2-\sigma/4}$. □

3.5. Estimation of S_{odd} : extended support. In this section, we analyze the lower-order terms of S_{odd} when $\sigma = \sup(\text{supp } \widehat{\phi}) \geq 1$.

LEMMA 3.5. *Suppose that $\sigma = \sup(\text{supp } \widehat{\phi}) < \infty$. Then, concerning the function $I_l(X)$ defined in (3-10), we have*

$$\begin{aligned}
 I_l(X) &= -\widetilde{w}(0) \int_1^\infty \widehat{\phi}(u) du + \frac{\widetilde{g}(0)}{\mathcal{L}} \int_0^\infty \phi(1 + \tau/\mathcal{L}) e^{\tau/2} N(l) d\tau \\
 &\quad + \frac{1}{\mathcal{L}} \int_0^\infty \widehat{\phi}\left(1 + \frac{\tau}{\mathcal{L}}\right) e^{\tau/2} N(l) \sum_{\substack{j \in \mathbb{Z}[i] \\ j \neq 0}} \widetilde{g}(\sqrt{N(j)e^{\tau/2}N(l)}) d\tau \\
 &\quad + \frac{1}{\mathcal{L}} \int_0^\infty \widehat{\phi}\left(1 - \frac{\tau}{\mathcal{L}}\right) \sum_{\substack{k \in \mathbb{Z}[i] \\ k \neq 0}} g(N(k) \sqrt{\frac{e^\tau}{N(l^2)}}) d\tau + O(N(l)X^{-1/2}) \tag{3-13}
 \end{aligned}$$

and

$$\begin{aligned}
 I_{(1+i)l}(X) &= -\widetilde{w}(0) \int_1^\infty \widehat{\phi}(u) du + \frac{2\widetilde{g}(0)}{\mathcal{L}} \int_0^\infty \phi\left(1 + \frac{\tau}{\mathcal{L}}\right) e^{\tau/2} N(l) d\tau \\
 &\quad + \frac{1}{\mathcal{L}} \int_0^\infty \widehat{\phi}\left(1 + \frac{\tau}{\mathcal{L}}\right) 2e^{\tau/2} N(l) \sum_{\substack{j \in \mathbb{Z}[i] \\ j \neq 0}} \widetilde{g}(2\sqrt{N(j)e^{\tau/2}N(l)}) d\tau \\
 &\quad + \frac{1}{\mathcal{L}} \int_0^\infty \widehat{\phi}\left(1 - \frac{\tau}{\mathcal{L}}\right) \sum_{\substack{k \in \mathbb{Z}[i] \\ k \neq 0}} g\left(\frac{N(k)}{2} \sqrt{\frac{e^\tau}{N(l^2)}}\right) d\tau, \tag{3-14}
 \end{aligned}$$

where $g(y)$ is given as in (2-14).

PROOF. We first extend the integral in (3-10) to \mathbb{R} and make the substitution $\tau = \mathcal{L}(u - 1)$ to obtain that

$$\begin{aligned}
 I_l(X) &= \frac{1}{\mathcal{L}} \int_{-\infty}^\infty \widehat{\phi}\left(1 + \frac{\tau}{\mathcal{L}}\right) \sum_{\substack{k \in \mathbb{Z}[i] \\ k \neq 0}} \widetilde{w}(2N(k) \sqrt{\frac{e^{-\tau}}{2N(l^2)}}) d\tau \\
 &\quad + O(N(l)X^{-1/2}), \tag{3-15}
 \end{aligned}$$

since for $u \leq 0$, we have

$$\begin{aligned}
 &\int_{-\infty}^0 \widehat{\phi}(u) \sum_{\substack{k \in \mathbb{Z}[i] \\ k \neq 0}} \widetilde{w}\left(2N(k) \sqrt{\frac{X^{1-u}}{2N(l^2)}}\right) du \\
 &\ll \int_{-\infty}^0 \widehat{\phi}(u) \sqrt{\frac{N(l^2)}{X^{1-u}}} du \ll N(l)X^{-1/2}.
 \end{aligned}$$

We then break the integral in (3-15) into integrals over $(-\infty, 0]$ and $[0, \infty)$ and denote them, respectively, by $I_l^-(X)$ and $I_l^+(X)$. By applying the Poisson summation

formula (2-3) given in Lemma 2.1, we see that

$$\begin{aligned}
 I_l^+(X) &= \frac{1}{\mathcal{L}} \int_0^\infty \widehat{\phi}\left(1 + \frac{\tau}{\mathcal{L}}\right) \left(-\widetilde{w}(0) + \sum_{k \in \mathbb{Z}[i]} g\left(N(k) \sqrt{\frac{e^{-\tau}}{N(l^2)}}\right)\right) d\tau \\
 &= \frac{1}{\mathcal{L}} \int_0^\infty \widehat{\phi}\left(1 + \frac{\tau}{\mathcal{L}}\right) \left(-\widetilde{w}(0) + e^{\tau/2} N(l) \sum_{j \in \mathbb{Z}[i]} \widetilde{g}\left(\sqrt{N(j)e^{\tau/2} N(l)}\right)\right) d\tau \\
 &= \frac{1}{\mathcal{L}} \int_0^\infty \widehat{\phi}\left(1 + \frac{\tau}{\mathcal{L}}\right) \left(-\widetilde{w}(0) + e^{\tau/2} N(l) \widetilde{g}(0) + e^{\tau/2} N(l) \sum_{\substack{j \in \mathbb{Z}[i] \\ j \neq 0}} \widetilde{g}\left(\sqrt{N(j)e^{\tau/2} N(l)}\right)\right) d\tau.
 \end{aligned}$$

Moreover, substituting τ by $-\tau$ in $I_l^-(X)$, we obtain that

$$I_l^-(X) = \frac{1}{\mathcal{L}} \int_0^\infty \widehat{\phi}\left(1 + \frac{\tau}{\mathcal{L}}\right) \sum_{\substack{k \in \mathbb{Z}[i] \\ k \neq 0}} g\left(N(k) \sqrt{\frac{e^\tau}{N(l^2)}}\right) d\tau.$$

Combining the above expressions for $I_l^-(X)$ and $I_l^+(X)$, we readily derive the expression for $I_l(X)$ in (3-13). The expression for $I_{(1+i)l}(X)$ in (3-14) can be similarly obtained via the expression for $I_l(X)$ with the function $g(y)$ replaced by $g(y/2)$, and this completes the proof. □

We define the functions

$$h_1(x) = \frac{3\zeta_K(2)}{\pi\widetilde{w}(0)} \sum_{l \equiv 1 \pmod{(1+i)^3}} \frac{\mu_{[i]}(l)}{N(l)} (\widetilde{g}(\sqrt{2N(l)x}) - \widetilde{g}(\sqrt{N(l)x}))$$

and

$$h_2(x) = \frac{3\zeta_K(2)}{\pi\widetilde{w}(0)} \sum_{l \equiv 1 \pmod{(1+i)^3}} \frac{\mu_{[i]}(l)}{N(l^2)} \left(\frac{1}{2} g\left(\frac{x}{2N(l)}\right) - g\left(\frac{x}{N(l)}\right)\right).$$

It is easy to see that $h_1(x)$ and $h_2(x)$ are smooth on $(0, \infty)$ and $[0, \infty)$, respectively. Moreover, we have the bounds $h_1(x) \ll x^{-A}$ for any $A \geq 1$ and $h_2(x) \ll x^{-3/2+\varepsilon}$ for any $\varepsilon > 0$ under the GRH. We point out here that the above notation and bounds are inspired by the corresponding notation introduced on page 1206 of [10].

We now apply Lemma 3.4 to derive the following generalization of [10, Corollary 3.4].

LEMMA 3.6. *Suppose that the GRH is true. Then, we have for $\sigma < 2$,*

$$S_{\text{odd}} = \int_1^\infty \widehat{\phi}(u) du + J(X) + O(X^{\sigma/6-1/3+\varepsilon}),$$

where

$$J(X) = \frac{1}{\mathcal{L}} \int_0^\infty \left(\widehat{\phi} \left(1 + \frac{\tau}{\mathcal{L}} \right) e^{\tau/2} \sum_{\substack{j \in \mathbb{Z}[i] \\ j \neq 0}} h_1(N(j)e^{\tau/2}) + \widehat{\phi} \left(1 - \frac{\tau}{\mathcal{L}} \right) \sum_{\substack{k \in \mathbb{Z}[i] \\ k \neq 0}} h_2(N(k)e^{\tau/2}) \right) d\tau. \tag{3-16}$$

PROOF. First note that by following the arguments that lead to [15, formula (2.13)], we have

$$\widetilde{w}(0) = \frac{\pi}{2} \widehat{w}(0). \tag{3-17}$$

It follows from (2-10), (3-12) and (3-17) that

$$\frac{X}{W(X)} \sum_{\substack{N(l) \leq Z \\ l \equiv 1 \pmod{(1+i)^3}}} \frac{\mu_{[i]}(l)}{N(l^2)} \widetilde{w}(0) \int_1^\infty \widehat{\phi}(u) du = 2 \int_1^\infty \widehat{\phi}(u) du + O(Z^{-3/2+\varepsilon}). \tag{3-18}$$

We now combine Lemmas 3.3, 3.4 and (3-18) to obtain that if $Z \leq X^2$, then

$$\begin{aligned} S_{\text{odd}} &= \frac{X}{W(X)} \sum_{\substack{N(l) \leq Z \\ l \equiv 1 \pmod{(1+i)^3}}} \frac{\mu_{[i]}(l)}{N(l^2)} \left(\int_0^\infty \widehat{\phi} \left(1 + \frac{\tau}{\mathcal{L}} \right) e^{\tau/2} N(l) \right. \\ &\quad \times \sum_{\substack{j \in \mathbb{Z}[i] \\ j \neq 0}} \left(\tilde{g}(2\sqrt{N(j)e^{\tau/2}N(l)}) - \tilde{g}(\sqrt{N(j)e^{\tau/2}N(l)}) \right) d\tau \\ &\quad + \int_0^\infty \widehat{\phi} \left(1 - \frac{\tau}{\mathcal{L}} \right) \sum_{\substack{k \in \mathbb{Z}[i] \\ k \neq 0}} \left(\frac{1}{2} g \left(\frac{N(k)}{2} \sqrt{\frac{e^\tau}{N(l^2)}} \right) - g \left(N(k) \sqrt{\frac{e^\tau}{N(l^2)}} \right) \right) d\tau \\ &\quad \left. + \int_1^\infty \widehat{\phi}(u) du + O(X^{-3/4+\varepsilon} + X^\varepsilon Z^{-1} + ZX^{\sigma/2-1+\varepsilon} + X^{-1/2+\varepsilon} Z^\varepsilon) \right). \tag{3-19} \end{aligned}$$

Note that for the first two integrals in the above expression, we have

$$\begin{aligned} &\frac{1}{\mathcal{L}} \int_0^\infty \widehat{\phi} \left(1 + \frac{\tau}{\mathcal{L}} \right) e^{\tau/2} N(l) \sum_{\substack{j \in \mathbb{Z}[i] \\ j \neq 0}} \left(\tilde{g}(2\sqrt{N(j)e^{\tau/2}N(l)}) - \tilde{g}(\sqrt{N(j)e^{\tau/2}N(l)}) \right) d\tau \\ &\ll \frac{1}{\mathcal{L}} \int_0^\infty \widehat{\phi} \left(1 + \frac{\tau}{\mathcal{L}} \right) d\tau \ll 1 \end{aligned}$$

and

$$\begin{aligned} & \frac{1}{\mathcal{L}} \int_0^\infty \widehat{\phi}\left(1 - \frac{\tau}{\mathcal{L}}\right) \sum_{\substack{k \in \mathbb{Z}[l] \\ k \neq 0}} \left(\frac{1}{2} g\left(\frac{N(k)}{2} \sqrt{\frac{e^\tau}{N(l^2)}}\right) - g\left(N(k) \sqrt{\frac{e^\tau}{N(l^2)}}\right) \right) d\tau \\ & \ll \frac{1}{\mathcal{L}} \int_0^\infty \widehat{\phi}\left(1 - \frac{\tau}{\mathcal{L}}\right) N(l) e^{-\tau/2} d\tau \ll N(l). \end{aligned}$$

We can therefore use $\Phi(X)$ defined in (3-12) and partial summation to extend the sum over l to all odd elements in O_K by introducing an extra error term of size $O(X^\varepsilon Z^{-3/2})$. We now set $Z = X^{2/3-\sigma/3}$, change the order of summation in (3-19) and apply (2-10) to derive the desired result. \square

4. Proof of Theorem 1.1

We combine Lemmas 2.4, 2.6, 3.1 and 3.5 to arrive at the following result.

LEMMA 4.1. *Suppose that the GRH is true. Let $\phi(x)$ be an even Schwartz function whose Fourier transform $\widehat{\phi}(u)$ has compact support in $(-2, 2)$ and let w be an even nonzero and nonnegative Schwartz function. For any integer $M \geq 1$, the one-level density of low-lying zeros in the family \mathcal{F} of quadratic Hecke L -functions is given by*

$$\begin{aligned} \mathcal{D}(\phi; w, X) &= \widehat{\phi}(0) - \frac{1}{2} \int_{-1}^1 \widehat{\phi}(u) du \\ &+ \frac{\widehat{\phi}(0)}{\mathcal{L}} \left(\log \frac{32}{\pi^2} + 2 \frac{\Gamma'}{\Gamma}\left(\frac{1}{2}\right) + \frac{2}{w(0)} \int_0^\infty w(x) \log x dx \right) \\ &+ J(X) + \frac{2}{\mathcal{L}} \int_0^\infty \frac{e^{-x/2}}{1 - e^{-x}} \left(\widehat{\phi}(0) - \widehat{\phi}\left(\frac{x}{\mathcal{L}}\right) \right) dx \\ &+ \sum_{m=1}^M \frac{d_m \widehat{\phi}^{(m-1)}(0)}{\mathcal{L}^m} + O\left(\frac{1}{\mathcal{L}^{M+1}}\right), \end{aligned} \tag{4-1}$$

where $J(X)$ is given as in Lemma 3.5 and the coefficients d_k are explicitly computable numbers given in Lemma 3.1.

The next lemma allows us to expand $J(X)$ in descending powers of $\mathcal{L} = \log X$. This is a generalization of [10, Lemma 3.6].

LEMMA 4.2. *Suppose that the GRH is true and suppose that*

$$\sigma = \sup(\text{supp } \widehat{\phi}) < 2.$$

Then, for any integer $M \geq 1$, we have the expansion

$$J(X) = \sum_{m=1}^M \frac{c_{w,m} \widehat{\phi}^{(m-1)}(1)}{\mathcal{L}^m} + O\left(\frac{1}{\mathcal{L}^{M+1}}\right), \tag{4-2}$$

where the constants $c_{w,m}$ can be given explicitly.

PROOF. Note that as $\sigma = \sup(\text{supp } \widehat{\phi}) < 2$, we have

$$J(X) = \frac{1}{\mathcal{L}} \int_0^{\mathcal{L}} \left(\widehat{\phi}(1 + \tau/\mathcal{L}) \sqrt{2} e^{\tau/2} \sum_{\substack{j \in \mathbb{Z}[i] \\ j \neq 0}} h_1(N(j) e^{\tau/2}) + \widehat{\phi}(1 - \tau/\mathcal{L}) \sum_{\substack{k \in \mathbb{Z}[i] \\ k \neq 0}} h_2(N(k) e^{\tau/2}) \right) d\tau.$$

Recall that we have the bounds $h_1(x) \ll x^{-N}$ for any $N \geq 1$ and $h_2(x) \ll x^{-3/2+\varepsilon}$ for any $\varepsilon > 0$ under the GRH. It follows from this that we can expand $\widehat{\phi}$ in Taylor series to obtain that

$$J(X) = \sum_{m=1}^M \frac{\widehat{\phi}^{(m-1)}(1)}{(m-1)! \mathcal{L}^m} \int_0^{\mathcal{L}} \left(\tau^{m-1} e^{\tau/2} \sum_{\substack{j \in \mathbb{Z}[i] \\ j \neq 0}} h_1(N(j) e^{\tau/2}) + (-\tau)^{m-1} \sum_{\substack{k \in \mathbb{Z}[i] \\ k \neq 0}} h_2(N(k) e^{\tau/2}) \right) d\tau + O(\mathcal{L}^{-M-1}), \tag{4-3}$$

since here the error term introduced can be estimated as

$$\begin{aligned} &\ll \mathcal{L}^{-M-1} \int_0^{\mathcal{L}} (\tau^M e^{-\tau/2} + (-\tau)^M e^{-(3/4+\varepsilon)\tau}) d\tau \\ &\ll \mathcal{L}^{-M-1} \int_0^{\infty} (\tau^M e^{-\tau/2} + (-\tau)^M e^{-(3/4+\varepsilon)\tau}) d\tau \ll \mathcal{L}^{-M-1}. \end{aligned}$$

We now extend the integral in (4-3) to infinity and note that the error introduced by this extension can be easily shown to be

$$\ll \sum_{m=1}^M \frac{\widehat{\phi}^{(m-1)}(1)}{(m-1)! \mathcal{L}^m} \int_{\mathcal{L}}^{\infty} (\tau^{m-1} e^{-\tau} + (-\tau)^{m-1} e^{-(3/4+\varepsilon)\tau}) d\tau \ll \mathcal{L}^{-M-1}.$$

We then deduce from this and (4-3) that

$$J(X) = \sum_{m=1}^M \frac{\widehat{\phi}^{(m-1)}(1)}{(m-1)! \mathcal{L}^m} \int_0^{\infty} \left(\tau^{m-1} \sqrt{2} e^{\tau/2} \sum_{\substack{j \in \mathbb{Z}[i] \\ (j,1+i)=1}} h_1(N(j) e^{\tau/2}) + (-\tau)^{m-1} \sum_{k \in \mathbb{Z}[i], k \neq 0} (-1)^{N(k)} h_2(N(k) e^{\tau/2}) \right) d\tau + O(\mathcal{L}^{-M-1}).$$

As the integral in the above expression converges, the assertion of the lemma now follows from this. \square

We now substitute (4-2) into (4-1) and expand $\widehat{\phi}(x/\mathcal{L})$ into its Taylor series around 0 so that

$$\begin{aligned} \widehat{\phi}(x/\mathcal{L}) &= \sum_{m=0}^M \frac{\widehat{\phi}^{(m)}(0)}{(m)! \mathcal{L}^m} \int_0^\infty \left(\tau^{m-1} \sqrt{2} e^{\tau/2} \sum_{\substack{j \in \mathbb{Z}[i] \\ (j, 1+i)=1}} h_1(N(j)e^{\tau/2}) \right. \\ &\quad \left. + (-\tau)^{m-1} \sum_{k \in \mathbb{Z}[i], k \neq 0} (-1)^{N(k)} h_2(N(k)e^{\tau/2}) \right) d\tau + O(\mathcal{L}^{-M-1}), \end{aligned}$$

and an interchange of the series and the integral allow us to deduce (1-2). In particular, we see that we have for $m \geq 2$,

$$R_{w,m}(\phi) = c_{w,m} \widehat{\phi}^{(m-1)}(1) + d_m \widehat{\phi}^{(m-1)}(0) - \frac{2\widehat{\phi}^{(m-1)}(0)}{(m-1)!} \int_0^\infty \frac{e^{-x/2} x^{m-1}}{1 - e^{-x}} dx. \tag{4-4}$$

This completes the proof of Theorem 1.1.

5. Proof of Theorem 1.2

We first follow the recipe given in [4] to derive a suitable version of the ratios conjecture for the family \mathcal{F} . We start by considering the expression

$$R(\alpha, \beta) = \frac{1}{W(X)} \sum_{(c, 1+i)=1}^* w\left(\frac{N(c)}{X}\right) \frac{L(1/2 + \alpha, \chi_{i(1+i)^5 c})}{L(1/2 + \beta, \chi_{i(1+i)^5 c})}. \tag{5-1}$$

Similarly to the treatment in [16, Section 4.1], we use the approximation

$$L(s, \chi_{i(1+i)^5 c}) \approx \sum_{n \neq 0} \frac{\chi_{i(1+i)^5 c}(n)}{N(n)^s} + X_c(s) \sum_{n \neq 0} \frac{\chi_{i(1+i)^5 c}(n)}{N(n)^{1-s}}, \tag{5-2}$$

where $\sum_{n \neq 0}$ denotes a sum over nonzero integral ideals in \mathcal{O}_K and

$$X_c(s) = \frac{\Gamma(1-s)}{\Gamma(s)} \left(\frac{\pi^2}{32N(c)} \right)^{s-1/2}. \tag{5-3}$$

Writing $\mu_{[i]}$ for the Möbius function on K , we obtain that for $\text{Re}(s) > 1$,

$$\frac{1}{L(s, \chi_{i(1+i)^5 c})} = \sum_{m \neq 0} \frac{\mu_{[i]}(m) \chi_{i(1+i)^5 c}(m)}{N(m)^s}. \tag{5-4}$$

Applying (5-2) and (5-4) to (5-1), we see that

$$R(\alpha, \beta) \approx R_1(\alpha, \beta) + R_2(\alpha, \beta), \tag{5-5}$$

where

$$R_1(\alpha, \beta) = \frac{1}{W(X)} \sum_{(c,1+i)=1}^* w\left(\frac{N(c)}{X}\right) \sum_{m, n \neq 0} \frac{\mu_{[i]}(m) \chi_{i(1+i)^5 c}(nm)}{N(m)^{1/2+\beta} N(n)^{1/2+\alpha}}$$

and

$$R_2(\alpha, \beta) = \frac{1}{W(X)} \sum_{(c,1+i)=1}^* w\left(\frac{N(c)}{X}\right) X_c\left(\frac{1}{2} + \alpha\right) \sum_{m, n \neq 0} \frac{\mu_{[i]}(m) \chi_{i(1+i)^5 c}(nm)}{N(m)^{1/2+\beta} N(n)^{1/2-\alpha}}.$$

When nm is an odd square, we expect to gain a main contribution to both R_1 and R_2 . Applying Lemma 2.3, we have in this case

$$\frac{1}{W(X)} \sum_{(c,1+i)=1}^* w\left(\frac{N(c)}{X}\right) \chi_{i(1+i)^5 c}(nm) \approx \prod_{\substack{\varpi \equiv 1 \pmod{(1+i)^3} \\ \varpi | nm}} \left(1 + \frac{1}{N(\varpi)}\right)^{-1}.$$

We then deduce that, upon writing \square for a perfect square,

$$\begin{aligned} R_1(\alpha, \beta) &\sim \widetilde{R}_1(\alpha, \beta) = \sum_{nm=\text{odd } \square} \frac{\mu_{[i]}(m)}{N(m)^{1/2+\beta} N(n)^{1/2+\alpha}} \\ &\times \prod_{\substack{\varpi \equiv 1 \pmod{(1+i)^3} \\ \varpi | nm}} \left(1 + \frac{1}{N(\varpi)}\right)^{-1}. \end{aligned}$$

A computation using the Euler product of ζ_K shows that

$$\widetilde{R}_1(\alpha, \beta) = \frac{\zeta_K(1 + 2\alpha)}{\zeta_K(1 + \alpha + \beta)} A(\alpha, \beta), \tag{5-6}$$

where

$$\begin{aligned} A(\alpha, \beta) &= \left(\frac{2^{1+\alpha+\beta} - 2^{\beta-\alpha}}{2^{1+\alpha+\beta} - 1}\right) \prod_{\varpi \equiv 1 \pmod{(1+i)^3}} \left(1 - \frac{1}{N(\varpi)^{1+\alpha+\beta}}\right)^{-1} \\ &\times \left(1 - \frac{1}{(N(\varpi) + 1)N(\varpi)^{1+2\alpha}} - \frac{1}{(N(\varpi) + 1)N(\varpi)^{\alpha+\beta}}\right). \end{aligned} \tag{5-7}$$

Note that the product $A(\alpha, \beta)$ is absolutely convergent for $\text{Re}(\alpha), \text{Re}(\beta) > -1/4$.

Similarly, we obtain

$$R_2(\alpha, \beta) \approx \widetilde{R}_2(\alpha, \beta) = \frac{1}{W(X)} \sum_{(c,1+i)=1}^* w\left(\frac{N(c)}{X}\right) X_c\left(\frac{1}{2} + \alpha\right) \widetilde{R}_1(-\alpha, \beta). \tag{5-8}$$

Combining (5-5) with (5-6) and (5-8), we deduce the following appropriate version of the ratios conjecture for our family \mathcal{F} .

CONJECTURE 5.1. Let $\varepsilon > 0$ and let w be an even and nonnegative Schwartz test function on \mathbb{R} which is not identically zero. For complex numbers α and β satisfying

$|\operatorname{Re}(\alpha)| < 1/4$, $(\log X)^{-1} \ll \operatorname{Re}(\beta) < 1/4$ and $\operatorname{Im}(\alpha), \operatorname{Im}(\beta) \ll X^{1-\varepsilon}$, we have that

$$\begin{aligned} & \frac{1}{W(X)} \sum_{(c,1+i)=1}^* w\left(\frac{N(c)}{X}\right) \frac{L(1/2 + \alpha, \chi_{i(1+i)^5 c})}{L(1/2 + \beta, \chi_{i(1+i)^5 c})} \\ &= \frac{\zeta_K(1 + 2\alpha)}{\zeta_K(1 + \alpha + \beta)} A(\alpha, \beta) \\ &+ \frac{1}{W(X)} \sum_{(c,1+i)=1}^* w\left(\frac{N(c)}{X}\right) X_c \left(\frac{1}{2} + \alpha\right) \frac{\zeta_K(1 - 2\alpha)}{\zeta_K(1 - \alpha + \beta)} A(-\alpha, \beta) \\ &+ O_\varepsilon(X^{-1/2+\varepsilon}), \end{aligned}$$

where $A(\alpha, \beta)$ is defined in (5-7) and $X_c(s)$ is defined in (5-3).

Similarly to the derivation of [16, Lemma 4.3], we deduce from Conjecture 5.1 the following result needed in the calculation of the one-level density.

LEMMA 5.2. *Assuming the truth of the GRH and Conjecture 5.1, we have for any $\varepsilon > 0$, $(\log X)^{-1} \ll \operatorname{Re}(r) < 1/4$ and $\operatorname{Im}(r) \ll X^{1-\varepsilon}$,*

$$\begin{aligned} & \frac{1}{W(X)} \sum_{(c,1+i)=1}^* w\left(\frac{N(c)}{X}\right) \frac{L'(1/2 + r, \chi_{i(1+i)^5 c})}{L(1/2 + r, \chi_{i(1+i)^5 c})} \\ &= \frac{\zeta'_K(1 + 2r)}{\zeta_K(1 + 2r)} + A_\alpha(r, r) - \frac{4}{\pi} \frac{1}{W(X)} \sum_{(c,1+i)=1}^* w\left(\frac{N(c)}{X}\right) \\ &\quad \times X_c\left(\frac{1}{2} + r\right) \zeta_K(1 - 2r) A(-r, r) + O_\varepsilon(X^{-1/2+\varepsilon}), \end{aligned}$$

where

$$A_\alpha(r, r) = \left. \frac{\partial}{\partial \alpha} A(\alpha, \beta) \right|_{\alpha=\beta=r}.$$

We now proceed as in [16, Section 4.4]. Assuming the truth of the GRH, it follows from Lemma 5.1 that

$$\begin{aligned} D(\phi; w, X) &= \frac{1}{W(X)} \sum_{(c,1+i)=1}^* w\left(\frac{N(c)}{X}\right) \frac{1}{2\pi i} \int_{(a-1/2)} \left(2 \frac{\zeta'_K(1 + 2r)}{\zeta_K(1 + 2r)} \right. \\ &\quad + 2A_\alpha(r, r) - \frac{X'_c(1/2 + r)}{X_c(1/2 + r)} \\ &\quad - \left. \frac{8}{\pi} X_c\left(\frac{1}{2} + r\right) \zeta_K(1 - 2r) A(-r, r) \right) \phi\left(\frac{i\mathcal{L}r}{2\pi}\right) dr \\ &+ O_\varepsilon(X^{-1/2+\varepsilon}), \end{aligned} \tag{5-9}$$

where $D(\phi; w, X)$ is defined in (1-1). Note that the integrand in (5-9) is analytic in the region $\text{Re}(r) \geq 0$ (in particular, it is analytic at $r = 0$). The assertion of Theorem 1.2 now follows by moving the contour of integration from $\text{Re}(r) = a - 1/2$ to $\text{Re}(r) = 0$.

6. Proof of Theorem 1.3

6.1. Initial treatment. In this section, we consider the expansions of the $D(\phi; w, X)$ given in (1-3) as powers of $1/\mathcal{L}$ with $\mathcal{L} = \log X$. Recall from (5-9) that, up to an error term of size $O(\mathcal{L}^{-2})$, we have

$$D(\phi; w, X) = \frac{1}{W(X)} \sum_{(c,1+i)=1}^* w\left(\frac{N(c)}{X}\right) \frac{1}{2\pi i} \int_{(a')} \left(2 \frac{\zeta'_K(1+2r)}{\zeta_K(1+2r)} + 2A_\alpha(r, r) - \frac{X'_c(1/2+r)}{X_c(1/2+r)} - \frac{8}{\pi} X_c \left(\frac{1}{2} + r\right) \zeta_K(1-2r) A(-r, r) \right) \phi\left(\frac{i\mathcal{L}r}{2\pi}\right) dr, \tag{6-1}$$

where $\mathcal{L}^{-1} < a' < 1/4$.

We set

$$I = -\frac{8}{\pi} \frac{1}{W(X)} \sum_{(c,1+i)=1}^* w\left(\frac{N(c)}{X}\right) \frac{1}{2\pi i} \int_{(a')} \left(X_c \left(\frac{1}{2} + r\right) \zeta_K(1-2r) A(-r, r) \right) \phi\left(\frac{i\mathcal{L}r}{2\pi}\right) dr. \tag{6-2}$$

We postpone the evaluation of I to the next section and proceed here with the treatment of the other terms on the right-hand side of (6-1).

We deduce first from Lemma 2.3 and (2-10), after partial summation, that

$$\begin{aligned} & \frac{1}{W(X)} \sum_{(c,1+i)=1}^* w\left(\frac{N(c)}{X}\right) \frac{1}{2\pi} \int_{\mathbb{R}} \log\left(\frac{32N(c)}{\pi^2}\right) \phi\left(\frac{t\mathcal{L}}{2\pi}\right) dt \\ &= \frac{\widehat{\phi}(0)}{\mathcal{L}} \frac{1}{W(X)} \sum_{(c,1+i)=1}^* w\left(\frac{N(c)}{X}\right) \log\left(\frac{32N(c)}{\pi^2}\right) \\ &= \frac{\widehat{\phi}(0)}{\mathcal{L}} \left(\log \frac{32}{\pi^2} + \mathcal{L} + \frac{2}{\widehat{w}(0)} \int_0^\infty w(x) \log x \, dx \right) + O(\mathcal{L}^{-2}). \end{aligned} \tag{6-3}$$

Next note that, similarly to [16, formula (4.33)], we have

$$\begin{aligned} & \frac{1}{W(X)} \sum_{(c,1+i)=1}^* w\left(\frac{N(c)}{X}\right) \frac{1}{2\pi} \int_{\mathbb{R}} \left(\frac{\Gamma'}{\Gamma}\left(\frac{1}{2} - it\right) + \frac{\Gamma'}{\Gamma}\left(\frac{1}{2} + it\right) \right) \phi\left(\frac{t\mathcal{L}}{2\pi}\right) dt \\ &= 2 \frac{\Gamma'}{\Gamma}\left(\frac{1}{2}\right) \frac{\widehat{\phi}(0)}{\mathcal{L}} + \frac{2}{\mathcal{L}} \int_0^\infty \frac{e^{-t/2}}{1 - e^{-t}} \left(\widehat{\phi}(0) - \widehat{\phi}\left(\frac{t}{\mathcal{L}}\right) \right) dt. \end{aligned} \tag{6-4}$$

Furthermore, we follow the treatment of [11, Lemma 4.1] to obtain via a direct calculation (noting that $A(r, r) = 1$) that

$$A_\alpha(r, r) + \frac{\zeta'_K(1 + 2r)}{\zeta_K(1 + 2r)} = \frac{-1}{N(\varpi) + 1} \sum_{\varpi \equiv 1 \pmod{(1+i)^3}} \frac{N(\varpi) \log N(\varpi)}{N(\varpi)^{1+2r} - 1}.$$

It follows from this, after the substitution $u = -i\mathcal{L}r/(2\pi)$ and interchanging the summations and the integral, that

$$\begin{aligned} & \frac{1}{2\pi i} \int_{(a')} \left(2 \frac{\zeta'_K(1 + 2r)}{\zeta_K(1 + 2r)} + 2A_\alpha(r, r) \right) \phi\left(\frac{i\mathcal{L}r}{2\pi}\right) dr \\ &= -\frac{2}{\mathcal{L}} \sum_{\varpi \equiv 1 \pmod{(1+i)^3}} \frac{N(\varpi) \log N(\varpi)}{N(\varpi) + 1} \\ & \quad \times \sum_{j=1}^\infty \frac{1}{N(\varpi)^j} \int_{C'} \phi(u) \exp\left(-2\pi i u \left(\frac{2^j \log N(\varpi)}{\mathcal{L}}\right)\right) du, \end{aligned} \tag{6-5}$$

where C' denotes the horizontal line $\text{Im}(u) = -\mathcal{L}'/(2\pi)$.

As $\widehat{\phi}$ is compactly supported and $\phi(z) = \int_{\mathbb{R}} \widehat{\phi}(x) e^{2\pi i x z} dx$, it follows from integration by parts that uniformly for $-\mathcal{L}'/(2\pi) \leq t \leq 0$,

$$|\phi(T + it)| \ll \frac{1}{|T| + 1}.$$

In view of this, we can shift the contour of the last integration in (6-5) from C' to $\text{Im}(u) = 0$ to deduce that

$$\begin{aligned} & \frac{1}{2\pi i} \int_{(a')} \left(2 \frac{\zeta'_K(1 + 2r)}{\zeta_K(1 + 2r)} + 2A_\alpha(r, r) \right) \phi\left(\frac{i\mathcal{L}r}{2\pi}\right) dr \\ &= -\frac{2}{\mathcal{L}} \sum_{\substack{\varpi \equiv 1 \pmod{(1+i)^3} \\ j \geq 1}} \frac{\log N(\varpi)}{N(\varpi)^j} \left(1 + \frac{1}{N(\varpi)}\right)^{-1} \widehat{\phi}\left(\frac{2^j \log N(\varpi)}{\mathcal{L}}\right). \end{aligned} \tag{6-6}$$

6.2. Evaluation of I . In this section, we evaluate I , defined in (6-2). Our treatment here follows from the proof of [11, Lemma 4.6]. We deduce from (5-7) that

$$A(-\gamma, \gamma) = \frac{3(2 - 2^{2r})}{4 - 2^{2r}} \frac{\zeta_K(2)}{\zeta_K(2 - 2r)}.$$

Substituting the above in the right-hand side of (6-2), we deduce from the definitions of X_c given in (5-3) and a change of variable $r = 2\pi i\tau/\mathcal{L}$ that

$$\begin{aligned}
 I &= -\frac{8}{\pi} \frac{\zeta_K(2)}{\mathcal{L}} \int_{C'} \left(\frac{\Gamma(1/2 - 2\pi i\tau/\mathcal{L})}{\Gamma(1/2 + 2\pi i\tau/\mathcal{L})} \right) \left(\frac{\pi^2}{32} \right)^{2\pi i\tau/\mathcal{L}} \left(1 + \frac{2 - 2^{4\pi i\tau/\mathcal{L}+1}}{4 - 2^{4\pi i\tau/\mathcal{L}}} \right) \\
 &\quad \times \frac{\zeta_K(1 - 4\pi i\tau/\mathcal{L})}{\zeta_K(2 - 4\pi i\tau/\mathcal{L})} \phi(\tau) \frac{1}{W(X)} \sum_{(c,1+i)=1}^* w\left(\frac{N(c)}{X}\right) N(c)^{-2\pi i\tau/\mathcal{L}} d\tau, \tag{6-7}
 \end{aligned}$$

where we also use C' to denote the horizontal line $\text{Im}(\tau) = -\mathcal{L}a'/(2\pi)$.

Applying Mellin inversion to the last sum in (6-7), we obtain for $0 \leq \text{Re}(r) \leq 1/2$,

$$\sum_{(c,1+i)=1}^* w\left(\frac{N(c)}{X}\right) N(c)^{-r} = \frac{4}{2\pi i} \int_{(2)} \frac{2^{s+r}}{2^{s+r} + 1} \frac{\zeta_K(s+r)}{\zeta_K(2(s+r))} X^s \mathcal{M}w(s) ds.$$

We shift the contour of integration to the line $\text{Re}(s) = 1/2 - \text{Re}(r) + \varepsilon$ to encounter a simple pole at $s = 1 - r$. On the new line of integration, the convexity bound (see [22, Exercise 3, page 100]), together with the rapid decay of $\mathcal{M}w$, gives

$$\zeta_K(s) \ll (1 + |s|^2)^{1/4+\varepsilon}.$$

With this and recalling that the residue of $\zeta_K(s)$ at $s = 1$ is $\pi/4$, we get

$$\sum_{(c,1+i)=1}^* w\left(\frac{N(c)}{X}\right) N(c)^{-r} = \frac{2\pi}{3\zeta_K(2)} X^{1-r} \mathcal{M}w(1-r) + O_{\varepsilon,w}(|\text{Im}(r)| + 1)^{1/2+\varepsilon} X^{1/2-\text{Re}(r)+\varepsilon}.$$

Combining the above with (2-10), we deduce that for any $\varepsilon > 0$ and $0 \leq \text{Re}(r) \leq 1/2$,

$$\begin{aligned}
 &\frac{1}{W(X)} \sum_{(c,1+i)=1}^* w\left(\frac{N(c)}{X}\right) N(c)^{-r} \\
 &= \frac{2}{\widehat{w}(0)} X^{-r} \mathcal{M}w(1-r) + O_{\varepsilon,w}(|\text{Im}(r)| + 1)^{1/2+\varepsilon} X^{-1/2-\text{Re}(r)+\varepsilon}.
 \end{aligned}$$

For small $\varepsilon, \eta > 0$, we change the contour C' in (6-7) to the path

$$C = C_0 \cup C_1 \cup C_2,$$

where

$$C_0 = \{\text{Im}(\tau) = 0, |\text{Re}(\tau)| \geq \mathcal{L}^\varepsilon\}, \quad C_1 = \{\text{Im}(\tau) = 0, \eta \leq |\text{Re}(\tau)| \leq \mathcal{L}^\varepsilon\}$$

and

$$C_2 = \{|\tau| = \eta, \text{Im}(\tau) \leq 0\}.$$

As ϕ decays rapidly, the integration of I over C_0 can be shown to be negligible. We now apply the Taylor expansion to treat the integration of I over $C_1 \cup C_2$ by noting that

$$\frac{\Gamma(1/2 - 2\pi i\tau/\mathcal{L})}{\Gamma(1/2 + 2\pi i\tau/\mathcal{L})} = 1 - 2\frac{\Gamma'(1/2)}{\Gamma(1/2)} \frac{2\pi i\tau}{\mathcal{L}} + O\left(\frac{|\tau|^2}{\mathcal{L}^2}\right),$$

and that (see [17, Formula 2, Section 8.366])

$$-\frac{\Gamma'(1/2)}{\Gamma(1/2)} = 2 \log 2 + \gamma.$$

Using Taylor expansion and (1-6), we get

$$\begin{aligned} & \left(1 + \frac{2 - 2^{4\pi i\tau/\mathcal{L}+1}}{4 - 2^{4\pi i\tau/\mathcal{L}}}\right) \frac{1}{\zeta_K(2 - 4\pi i\tau/\mathcal{L})} \\ &= \frac{1}{\zeta_K(2)} + \left(\frac{1}{\zeta_K(2)}\left(-\frac{2 \log 2}{3}\right) + \frac{\zeta'_K(2)}{\zeta_K^2(2)}\right) \frac{4\pi i\tau}{\mathcal{L}} + O\left(\frac{|\tau|^2}{\mathcal{L}^2}\right) \end{aligned}$$

and

$$\zeta_K\left(1 - \frac{4\pi i\tau}{\mathcal{L}}\right) = -\frac{\pi}{4} \cdot \frac{\mathcal{L}}{4\pi i\tau} + \gamma_K + O\left(\frac{|\tau|}{\mathcal{L}}\right).$$

Using the above formulas, we get, after a short computation,

$$\begin{aligned} & -\frac{8}{\pi} \frac{\zeta_K(2)}{\mathcal{L}} \frac{\Gamma(1/2 - 2\pi i\tau/\mathcal{L})}{\Gamma(1/2 + 2\pi i\tau/\mathcal{L})} \left(\frac{\pi^2}{32}\right)^{2\pi i\tau/\mathcal{L}} \left(1 + \frac{2 - 2^{(4\pi i\tau/\mathcal{L})+1}}{4 - 2^{4\pi i\tau/\mathcal{L}}}\right) \\ & \times \frac{\zeta_K(1 - 4\pi i\tau/\mathcal{L})}{\zeta_K(2 - 4\pi i\tau/\mathcal{L})} \phi(\tau) \frac{1}{W(X)} \sum_{(c,1+i)=1}^* w\left(\frac{N(c)}{X}\right) N(c)^{-2\pi i\tau/\mathcal{L}} \\ &= \frac{1}{2\pi i\tau} \left(1 + \frac{2\pi i\tau}{\mathcal{L}} \left(2\gamma + 2 \log 4 + \log\left(\frac{\pi^2}{32}\right) + 2\frac{\zeta'_K(2)}{\zeta_K(2)} - \frac{4}{3} \log 2 - \frac{8}{\pi} \gamma_K \right. \right. \\ & \left. \left. - \frac{\mathcal{M}w'(1)}{\mathcal{M}w(1)}\right) + O\left(\frac{|\tau|^2}{\mathcal{L}^2}\right)\right) \phi(\tau) e^{-2\pi i\tau} + O_{\varepsilon,w}(X^{-1/2+\varepsilon}). \end{aligned}$$

We then deduce that

$$I = \frac{1}{2\pi i} \int_{C_1 \cup C_2} \frac{\phi(\tau)}{\tau} e^{-2\pi i\tau} d\tau + I' + O_w(\mathcal{L}^{-2}),$$

where, combining the logarithm terms,

$$\begin{aligned} I' &= \frac{1}{\mathcal{L}} \left(2\gamma + \log\left(\frac{\pi^2}{2^{7/3}}\right) + 2\frac{\zeta'_K(2)}{\zeta_K(2)} - \frac{8}{\pi} \gamma_K - \frac{\mathcal{M}w'(1)}{\mathcal{M}w(1)}\right) \int_{C_1 \cup C_2} \phi(\tau) e^{-2\pi i\tau} d\tau \\ &= \frac{1}{\mathcal{L}} \left(2\gamma + \log\left(\frac{\pi^2}{2^{7/3}}\right) + 2\frac{\zeta'_K(2)}{\zeta_K(2)} - \frac{8}{\pi} \gamma_K - \frac{\mathcal{M}w'(1)}{\mathcal{M}w(1)}\right) \end{aligned}$$

$$\begin{aligned} & \times \int_{\mathbb{R}} \phi(\tau)e^{-2\pi i\tau} d\tau + O(\mathcal{L}^{-2}) \\ & = \frac{\widehat{\phi}(1)}{\mathcal{L}} \left(2\gamma + \log\left(\frac{\pi^2}{27^{1/3}}\right) + 2\frac{\zeta'_K(2)}{\zeta_K(2)} - \frac{8}{\pi}\gamma_K - \frac{Mw'(1)}{Mw(1)} \right) + O(\mathcal{L}^{-2}). \end{aligned}$$

Similarly to the treatment of I_1 in the proof of [11, Lemma 4.6], we have

$$\int_{C_1 \cup C_2} \frac{1}{2\pi i\tau} \phi(\tau)e^{-2\pi i\tau} d\tau = \int_1^\infty \widehat{\phi}(\tau) d\tau + O(\mathcal{L}^{-2}).$$

Thus, we conclude that

$$\begin{aligned} I & = \int_1^\infty \widehat{\phi}(\tau) d\tau + \frac{\widehat{\phi}(1)}{\mathcal{L}} \left(2\gamma + \log\left(\frac{\pi^2}{27^{1/3}}\right) + 2\frac{\zeta'_K(2)}{\zeta_K(2)} - \frac{8}{\pi}\gamma_K - \frac{Mw'(1)}{Mw(1)} \right) \\ & \quad + O(\mathcal{L}^{-2}). \end{aligned}$$

Combining the above expression for I with (6-1)–(6-4) and (6-6), we deduce that expression (1-7) is valid.

6.3. Comparing terms. In this section we show that the expression given in (1-7) is in agreement with that given in (1-4) when $\sigma = \sup(\text{supp } \widehat{\phi}) < 2$. In fact, applying (6-3) and (6-4) in (1-3) and comparing it with (1-4), we see that, with the help of Lemma 2.3, it suffices to show that

$$\begin{aligned} & -\frac{2}{\mathcal{L}W(X)} \sum_{(c,1+i)=1}^* w\left(\frac{N(c)}{X}\right) \sum_{j \geq 1} S_j(\chi_{i(1+i)^5 c}, \mathcal{L}; \widehat{\phi}) \\ & = \frac{1}{W(X)} \sum_{(c,1+i)=1}^* w\left(\frac{N(c)}{X}\right) \frac{1}{2\pi} \int_{\mathbb{R}} \left(2\frac{\zeta'_K(1+2it)}{\zeta_K(1+2it)} + 2A_\alpha(it, it) \right. \\ & \quad \left. - \frac{8}{\pi} X_c \left(\frac{1}{2} + it \right) \zeta_K(1-2it) A(-it, it) \right) \phi\left(\frac{t\mathcal{L}}{2\pi}\right) dt \\ & = \frac{1}{W(X)} \sum_{(c,1+i)=1}^* w\left(\frac{N(c)}{X}\right) \frac{1}{2\pi i} \int_{(a')} \left(2\frac{\zeta'_K(1+2r)}{\zeta_K(1+2r)} + 2A_\alpha(r, r) \right. \\ & \quad \left. - \frac{8}{\pi} X_c \left(\frac{1}{2} + r \right) \zeta_K(1-2r) A(-r, r) \right) \phi\left(\frac{i\mathcal{L}r}{2\pi}\right) dr, \end{aligned} \tag{6-8}$$

where $(\log X)^{-1} < a' < 1/4$.

Now, similarly to the treatment in Section 3, we write

$$-\frac{2}{\mathcal{LW}(X)} \sum_{(c,1+i)=1}^* w\left(\frac{N(c)}{X}\right) \sum_{j \geq 1} S_j(\mathcal{X}_{i(1+i)^5 c}, \mathcal{L}; \hat{\phi}) = S_{\text{odd}} + S_{\text{even}}, \tag{6-9}$$

where

$$S_{\text{odd}} = -\frac{2}{\mathcal{LW}(X)} \sum_{(c,1+i)=1}^* w\left(\frac{N(c)}{X}\right) \sum_{\substack{j \geq 1 \\ j \equiv 1 \pmod{2}}} S_j(\mathcal{X}_{i(1+i)^5 c}, \mathcal{L}; \hat{\phi})$$

and

$$S_{\text{even}} = -\frac{2}{\mathcal{LW}(X)} \sum_{(c,1+i)=1}^* w\left(\frac{N(c)}{X}\right) \sum_{\substack{j \geq 1 \\ j \equiv 0 \pmod{2}}} S_j(\mathcal{X}_{i(1+i)^5 c}, \mathcal{L}; \hat{\phi}).$$

We then deduce from (6-6), (6-8), (6-9) and (3-2) that it remains to show that

$$S_{\text{odd}} = I + O(\mathcal{L}^{-2}), \tag{6-10}$$

where I is defined in (6-2).

6.4. Evaluation of S_{odd} , first order term. In this section we evaluate S_{odd} to the first lower-order term. Our treatment here largely follows the approach in the proof of [11, Theorem 1.1]. We recall from (3.5) that

$$S_{\text{odd}} = \int_1^\infty \widehat{\phi}(u) \overline{du} + J(X) + O(\mathcal{L}^{-2}), \tag{6-11}$$

where

$$J(X) = \frac{1}{\mathcal{L}} \int_0^\infty \left(\widehat{\phi}\left(1 + \frac{\tau}{\mathcal{L}}\right) e^{\tau/2} \sum_{\substack{j \in \mathbb{Z}[i] \\ j \neq 0}} h_1(N(j) e^{\tau/2}) \right. \\ \left. + \widehat{\phi}\left(1 - \frac{\tau}{\mathcal{L}}\right) \sum_{\substack{k \in \mathbb{Z}[i] \\ k \neq 0}} h_2(N(k) e^{\tau/2}) \right) d\tau.$$

We now evaluate $h_1(x)$ by applying the Mellin inversion to recast it as

$$h_1(x) = \frac{3\zeta_K(2)}{\pi \widehat{w}(0)} \frac{1}{2\pi i} \int_{(5/2)} \sum_{l \equiv 1 \pmod{(1+i)^3}} (2^{-z} - 1) \frac{\mu_{[l]}(l)}{N(l)^{1+z}} \mathcal{M}g_1(z) \frac{dz}{x^z} \\ = \frac{3\zeta_K(2)}{\pi \widehat{w}(0)} \frac{1}{2\pi i} \int_{(5/2)} \frac{2^{-z} - 1}{(1 - 2^{-1-z}) \zeta_K(1+z)} \mathcal{M}g_1(z) \frac{dz}{x^z}.$$

Similarly, we have, with a change of variables $z \rightarrow -z$,

$$\begin{aligned} h_2(x) &= \frac{3\zeta_K(2)}{\pi\widehat{w}(0)} \frac{1}{2\pi i} \int_{(1/2)} \sum_{l \equiv 1 \pmod{(1+i)^3}} (2^{z-1} - 1) \frac{\mu_{[l]}(l)}{N(l)^{2-z}} \mathcal{M}g(z) \frac{dz}{x^z} \\ &= \frac{3\zeta_K(2)}{\pi\widehat{w}(0)} \frac{1}{2\pi i} \int_{(-1/2)} \frac{2^{-z-1} - 1}{(1 - 2^{-2-z})\zeta_K(2+z)} \mathcal{M}g(-z)x^z dz \\ &= \frac{3\zeta_K(2)}{\pi\widehat{w}(0)} \frac{1}{2\pi i} \int_{(-5/4)} \frac{2^{-z-1} - 1}{(1 - 2^{-2-z})\zeta_K(2+z)} \mathcal{M}g(-z)x^z dz. \end{aligned}$$

With the above expressions for $h_1(x)$ and $h_2(x)$, we can write $J(x)$ given in (3-16) as

$$\begin{aligned} J(X) &= \frac{Y}{\mathcal{L}} \int_0^\infty \left(\widehat{\phi}\left(1 + \frac{\tau}{\mathcal{L}}\right) \int_{(5/2)} \frac{4(2^{-z} - 1)\zeta_K(z)}{(1 - 2^{-1-z})\zeta_K(1+z)} \mathcal{M}g_1(z) \frac{dz}{e^{(z-1)\tau/2}} \right. \\ &\quad \left. + \widehat{\phi}\left(1 - \frac{\tau}{\mathcal{L}}\right) \int_{(-5/4)} \frac{4(2^{-z-1} - 1)\zeta_K(-z)}{(1 - 2^{-2-z})\zeta_K(2+z)} \mathcal{M}g(-z)e^{z\tau/2} dz \right) d\tau \\ &= \frac{Y}{\mathcal{L}} \left(\int_{(3/2)} \frac{4(2^{-z} - 1)\zeta_K(z)}{(1 - 2^{-1-z})\zeta_K(1+z)} \mathcal{M}g_1(z) \int_0^\infty \widehat{\phi}\left(1 + \frac{\tau}{\mathcal{L}}\right) e^{-(z-1)\tau/2} d\tau dz \right. \\ &\quad \left. + \int_{(-5/4)} \frac{4(2^{-z-1} - 1)\zeta_K(-z)}{(1 - 2^{-2-z})\zeta_K(2+z)} \mathcal{M}g(-z) \int_0^\infty \widehat{\phi}\left(1 - \frac{\tau}{\mathcal{L}}\right) e^{z\tau/2} d\tau dz \right), \quad (6-12) \end{aligned}$$

where

$$Y = \frac{3\zeta_K(2)}{\pi\widehat{w}(0)} \frac{1}{2\pi i}.$$

Now we consider the Taylor expansions, centered at 1, of the terms $\widehat{\phi}(1 + \tau/\mathcal{L})$ and $\widehat{\phi}(1 - \tau/\mathcal{L})$ in (6-12). By keeping only the constant terms, we see that their contribution to $J(X)$ equals, with another change of variables $z \rightarrow z + 1$ in the first integral,

$$\begin{aligned} &\frac{3\zeta_K(2)}{\mathcal{L}\pi\widehat{w}(0)} \frac{8\widehat{\phi}(1)}{2\pi i} \left(\int_{(1/2)} \frac{(2^{-z-1} - 1)\zeta_K(z+1)}{(1 - 2^{-2-z})\zeta_K(2+z)} \mathcal{M}g_1(z+1) \frac{dz}{z} \right. \\ &\quad \left. - \int_{(-5/4)} \frac{(2^{-z-1} - 1)\zeta_K(-z)}{(1 - 2^{-2-z})\zeta_K(2+z)} \mathcal{M}g(-z) \frac{dz}{z} \right). \quad (6-13) \end{aligned}$$

We now shift the contour of the last integration to the line $\text{Re}(z) = 1/2$. We apply Lemma 2.7 to see that the quantity in (6-13) equals

$$\frac{3\zeta_K(2)}{\mathcal{L}\pi\widehat{w}(0)} \frac{8\widehat{\phi}(1)}{2\pi i} R,$$

where R is the residue of the function

$$\frac{(2^{-z-1} - 1)\zeta_K(-z)}{(1 - 2^{-2-z})\zeta_K(2+z)} \frac{\mathcal{M}g(-z)}{z}$$

at $z = 0$. We observe via integration by parts that

$$\mathcal{M}g(-z) = \int_0^\infty g(t)t^{-z} \frac{dt}{t} = \int_0^\infty g(t) d\left(\frac{t^{-z}}{-z}\right) = \frac{1}{z} \int_0^\infty t^{-z} g'(t) dt.$$

As

$$\lim_{z \rightarrow 0} \int_0^\infty t^{-z} g'(t) dt = g(0) = \widehat{w}(0) > 0,$$

it follows that $\mathcal{M}g(-z)$ has a pole at $z = 0$. We apply (3-17) and get that around $z = 0$,

$$\begin{aligned} \mathcal{M}g(-z) &= \frac{-g(0)}{z} - \int_0^\infty (\log t)g'(t) dt + O(z^2) \\ &= \frac{-\widehat{w}(0)}{z} - \int_0^\infty (\log t)g'(t) dt + O(z^2) \\ &= -\frac{\pi \widehat{w}(0)}{2z} - \int_0^\infty (\log t)g'(t) dt + O(z^2). \end{aligned} \quad (6-14)$$

Moreover, we have around $z = 0$,

$$\frac{2^{-z-1} - 1}{1 - 2^{-2-z}} = -\frac{2}{3} + \frac{-\log 2(\frac{3}{8}) + \frac{1}{2}(\log 2)\frac{1}{4}}{(1 - 2^{-2})^2} z = -\frac{2}{3} - \frac{4}{9}(\log 2)z \quad (6-15)$$

and

$$\frac{\zeta_K(-z)}{\zeta_K(2+z)} = \frac{\zeta_K(0)}{\zeta_K(2)} + \frac{-\zeta'_K(0)\zeta_K(2) - \zeta_K(0)\zeta'_K(2)}{\zeta_K^2(2)} z + O(z^2). \quad (6-16)$$

Using (6-14), (6-15) and (6-16), we get

$$\begin{aligned} R &= -\frac{\pi}{2} \widehat{w}(0) \left(-\frac{2}{3}\right) \frac{-\zeta'_K(0)\zeta_K(2) - \zeta_K(0)\zeta'_K(2)}{\zeta_K^2(2)} \\ &\quad - \frac{\pi}{2} \widehat{w}(0) \left(-\frac{4}{9} \log 2\right) \frac{\zeta_K(0)}{\zeta_K(2)} - \frac{2}{3} \frac{\zeta_K(0)}{\zeta_K(2)} \left(-\int_0^\infty (\log t)g'(t) dt\right). \end{aligned} \quad (6-17)$$

To further simplify R , we use the fact that $s\Gamma(s) = 1$ (see [6, Section 10]) when $s = 0$ and the functional equation for $\zeta_K(s)$ (see [22, Theorem 3.8]),

$$\pi^{-s}\Gamma(s)\zeta_K(s) = \pi^{-(1-s)}\Gamma(1-s)\zeta_K(1-s),$$

to obtain that $\zeta_K(0) = -1/4$.

We further use the relation (see [6, Section 10])

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$$

to derive that

$$\zeta_K(1-s) = \pi^{-2s}\Gamma(s)^2 \sin(\pi s)\zeta_K(s). \quad (6-18)$$

Applying (1-6), we see that around $s = 1$, we have

$$\zeta_K(s) \sin(\pi s) = -\frac{\pi^2}{4} - \pi\gamma_K(s - 1) + O((s - 1)^2).$$

Using the above expansion and the fact that $\Gamma'(1) = -\gamma$ (see [17, Formula 1, Section 8.366] by noting also that $\Gamma(1) = 1$), we take the derivative on both sides of (6-18) to see that

$$-\zeta'_K(0) = -\frac{1}{\pi}\gamma_K + \frac{\gamma}{2} + \frac{\log \pi}{2}.$$

Now, inserting the values of $\zeta_K(0)$, $-\zeta'_K(0)$ into (6-17), together with a short calculation, we obtain that

$$\begin{aligned} J(X) &= \frac{3\zeta_K(2)}{\mathcal{L}\pi\widehat{w}(0)} \frac{8\widehat{\phi}(1)}{2\pi i} R + O(\mathcal{L}^{-2}) \\ &= \frac{8\widehat{\phi}(1)}{\mathcal{L}} \left(-\frac{\gamma_K}{\pi} + \frac{\gamma}{2} + \frac{\log \pi}{2} + \frac{1}{4} \frac{\zeta'_K(2)}{\zeta_K(2)} \right) \\ &\quad + \left(\frac{2}{3} \log 2 \right) \zeta_K(0) - \frac{1}{2\pi\widehat{w}(0)} \left(\int_0^\infty (\log t)g'(t) dt \right) + O(\mathcal{L}^{-2}). \end{aligned} \tag{6-19}$$

We evaluate the last integral above by noticing that for small $\eta > 0$, we have

$$\begin{aligned} \int_0^\infty (\log x)g'(x) dx &= \int_0^\infty \sqrt{2} \log x \widetilde{w}'(\sqrt{2}x) dx = \int_0^\infty \log \left(\frac{x}{\sqrt{2}} \right) \widetilde{w}'(x) dx \\ &= \widetilde{w}(0) \log(\sqrt{2}) + \int_\eta^\infty (\log x) \widetilde{w}'(x) dx + O(\eta \log(\eta^{-1})). \end{aligned}$$

Now the above expression becomes, after integration by parts,

$$\widetilde{w}(0) \log(\sqrt{2}) - \int_\eta^\infty \frac{\widetilde{w}(x) - \widetilde{w}(0)I_{[0,1]}(x)}{x} dx + O(\eta \log(\eta^{-1})),$$

where $I_{[0,1]}$ is the characteristic function of the interval $[0, 1]$.

By evaluating $\widetilde{w}(x)$ in polar coordinates, we see that

$$\widetilde{w}(x) = 4 \int_0^{\pi/2} \int_0^\infty \cos(2\pi r x \sin \theta) w(r^2) r dr d\theta.$$

It follows from this, and by letting $\eta \rightarrow 0^+$ and using [41, Example (e) on page 132], that

$$\begin{aligned} \int_0^\infty (\log x)g'(x) dx &= \widetilde{w}(0) \log(\sqrt{2}) - \int_0^\infty \frac{\widetilde{w}(x) - \widetilde{w}(0)I_{[0,1]}(x)}{x} dx \\ &= \widetilde{w}(0) \log(\sqrt{2}) - 4 \int_0^\infty w(r^2)r \int_0^{\pi/2} I(\theta) d\theta dr, \end{aligned} \quad (6-20)$$

where

$$I(\theta) = \int_0^1 \frac{\cos(2\pi r x \sin \theta) - 1}{x} dx + \int_1^\infty \frac{\cos(2\pi r x \sin \theta)}{x} dx.$$

Now, with some changes of variables,

$$I(\theta) = \int_0^{2\pi r \sin \theta} \frac{\cos(u) - 1}{u} du + \int_{2\pi r \sin \theta}^\infty \frac{\cos(u)}{u} du = \gamma + \log(2\pi r \sin \theta).$$

Hence, the expression in (6-20) is

$$\begin{aligned} \widetilde{w}(0) \log(\sqrt{2}) + \frac{\pi\gamma\widehat{w}(0)}{2} + \frac{\pi \log \pi\widehat{w}(0)}{2} + \frac{\pi \log 2\widehat{w}(0)}{2} \\ + \frac{\pi}{2} \int_0^\infty w(r) \log r dr + 2 \int_0^\infty w(r) dr \int_0^{\pi/2} \log(\sin \theta) d\theta. \end{aligned}$$

As we have (see [17, Formula 3, Section 4.224])

$$\int_0^{\pi/2} \log(\sin \theta) d\theta = -\frac{\pi}{2} \log 2.$$

We thus conclude that

$$\int_0^\infty (\log x)g'(x) dx = \frac{\pi\widehat{w}(0)}{4} \log 2 + \frac{\pi\gamma\widehat{w}(0)}{2} + \frac{\pi \log \pi\widehat{w}(0)}{2} + \frac{\pi}{2} \mathcal{M}w'(1).$$

Applying this to (6-19), we see that

$$\begin{aligned} J(X) &= \frac{\widehat{\phi}(1)}{\mathcal{L}} \left(2\gamma + 2 \log 4 + \log \left(\frac{\pi^2}{32} \right) + 2 \frac{\zeta'_K(2)}{\zeta_K(2)} \right. \\ &\quad \left. - \frac{4}{3} \log 2 - \frac{8}{\pi} \gamma_K - \frac{\mathcal{M}w'(1)}{\mathcal{M}w(1)} \right) + O(\mathcal{L}^{-2}). \end{aligned}$$

With the above expression for $J(X)$ and (6-11), we conclude that the expression given in (6-10) is valid, and this completes the proof of Theorem 1.3.

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