SOME UNIQUENESS THEOREMS FOR FUNCTIONAL EQUATIONS

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The generalized Pexider equation

$$g(F(x, y)) = H(f(x), f(y), x, y) \tag{1}$$

where f and g are unknown and x, y, are real, has been discussed by J. Aczél [1] and J. Aczél and M. Hosszú [2]. In [2] it is shown that if F is continuous and F and H are strictly increasing in their first variables and strictly decreasing in their second variables, then two initial conditions suffice to determine at most one continuous solution f of (1). We extend these results to strictly increasing and strictly decreasing functions F and derive results for strictly monotonic F and H.

As in [2] we call F reflexive at a if $F(a, a) = a$, i.e. if $x = a$ is a fixed-point of $F(x, x)$. We sometimes omit parentheses where the meaning is clear, e.g. $fx = f(x)$. We use the standard notation for iterates, e.g. $\alpha^0 x = x$, $\alpha^{n+1}x = \alpha\alpha^n x$, $n = 0, 1, \cdots$.

THEOREM 1. *Let I be an interval. Let F be continuous and strictly increasing (or decreasing) in $I \times I$. Let N be a Hausdorff space. Let H be defined in $N \times N \times I \times I$ such that*

(I) $\quad H(u_1, u_1, x, x) = H(u_2, u_2, x, x)$ *implies* $u_1 = u_2$;

and either

(II) $\quad H(u, u_1, x, y) = H(u, u_2, x, y)$ *implies* $u_1 = u_2$,

or

(III) $\quad H(u_1, u, x, y) = H(u_2, u, x, y)$ *implies* $u_1 = u_2$.

Let $g = g_1$ and $f = f_1$, also $g = g_2$ and $f = f_2$, satisfy (1) in $I \times I$. Let f_1 and f_2 be continuous maps of I into N, and $a, b \in I$ such that $a \neq b$, $f_1(a) = f_2(a)$, $f_1(b) = f_2(b)$. Then $f_1 = f_2$ in I and $g_1 = g_2$ in the range of F on $I \times I$.

Let $g = f$ in (1). If F is reflexive everywhere in I the condition (I) is redundant. If $F(a, a) \in I$ and F is not reflexive at a the condition $f_1(b) = f_2(b)$

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is redundant. If I is the interval $0 \leqq x < d$ and $F(x, y) = x+y$ the conditions (II) *and* (III) *are redundant; and provided $a > 0$ the condition $f_1(b) = f_2(b)$ is redundant.*

Proof. Let $px = F(x, x), x \in I$. Then p is continuous and strictly monotonic and has a unique inverse p^{-1}. We may define $G = p^{-1}F$. Then G has domain $I \times I$ and range I, and is continuous, strictly increasing in both variables, and reflexive in I.

Let (II) hold. We may suppose $a < b$. Let $\alpha x = G(a, x)$, $\beta x = G(b, x)$, $x \in I$. Then α and β are continuous, strictly increasing, and have unique inverses. Also $\alpha a = a$, $\beta b = b$, $\alpha x < x$ for $x > a$, $\beta x > x$ for $x < b$.

We may substitute $G(x, y)$ for x and y in (1), then f_1 and f_2 satisfy the functional equation

$$H(fx, fy, x, y) = H(fG(x, y), fG(x, y), G(x, y), G(x, y)) \tag{2}$$

in $I \times I$. Substitute $x = a$ and $y = b, \alpha b, \alpha^2 b, \cdots$, successively in (2), then (I) implies $f_1 \alpha^n b = f_2 \alpha^n b$ $(n = 1, 2, \cdots)$.

Let I' be the interval $a \leqq x \leqq b$. Assume that $f_1 \neq f_2$ in I'. Then it is not true that $f_1 = f_2$ in a set which is dense in I'. Hence there exist $a_1, b_1 \in I'$ such that f_1 and f_2 intersect at a_1 and b_1 but are different everywhere in $a_1 < x < b_1$. Let $c_n = G(\alpha^n b, a_1)$. Then (I) and (2) imply $f_1 c_n = f_2 c_n$. But $\alpha^n b \to a$, $c_n \to \alpha a_1$, and since these sequences strictly decrease, there exists an integer m such that

$$\alpha a_1 < c_m < \alpha b_1, \; a_1 < \alpha^{-1} c_m < b_1.$$

But (II) and (2) with $x = a$ and $y = \alpha^{-1} c_m$ imply $f_1 \alpha^{-1} c_m = f_2 \alpha^{-1} c_m$, which contradicts our assumption. Hence $f_1 = f_2$ in I'.

If $x \in I$, $x \geqq a$, then $\alpha^q x \in I'$ for some positive integer q; hence $f_1 \alpha^q x = f_2 \alpha^q x$, and (II) and (2) with $x = a$ and $\langle y = \alpha^{q-1}x, \alpha^{q-2}x, \cdots, x, \rangle$ successively, imply $f_1 x = f_2 x$. If $x \in I$, $x \leqq b$, then $\beta^r x \in I'$ for some positive integer r; hence $f_1 \beta^r x = f_2 \beta^r x$, and (II) and (2) with $x = b$ and

$$y = \beta^{r-1}x, \beta^{r-2}x, \cdots, x,$$

successively, imply $f_1 x = f_2 x$. Hence $f_1 = f_2$ in I.

If instead of (II), (III) holds, the above process may be repeated with $G(a, x)$ and $G(b, x)$ replaced by $G(x, a)$ and $G(x, b)$, respectively.

Let $g = f$ in (1). If $F(x, x) \equiv x, x \in I$, then f_1 and f_2 satisfy the equation $fx = H(fx, fx, x, x)$ in I, which is sufficient for the proof of the theorem, instead of (I). If $F(a, a) \in I$ and $F(a, a) \neq a$ then $f_1 a = f_2 a$ implies $f_1 F(a, a) = f_2 F(a, a)$, and we may take $b = F(a, a)$.

Let $g = f$ in (1), I be the interval $0 \leqq x < d$, and $F(x, y) = x+y$. If $t \in I$ such that $f_1 t = f_2 t$ then $f_k t = H(f_k \frac{1}{2}t, f_k \frac{1}{2}t, \frac{1}{2}t, \frac{1}{2}t)$ $(k = 1, 2)$, and (I) implies $f_1 \frac{1}{2}t = f_2 \frac{1}{2}t$. If also $nt \in I$ for some positive integer n, then

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$$f_1 2t = H(f_1 t, f_1 t, t, t) = f_2 2t,$$
$$f_1 3t = H(f_1 2t, f_1 t, 2t, t) = f_2 3t,$$

and by induction $f_1 nt = f_2 nt$. Now, either $a > 0$ or $b > 0$, and if $a > 0$ the points $m2^{-n}a$ (m and n positive integers) are dense in I. But $f_1 a = f_2 a$ implies $f_1 m2^{-n}a = f_2 m2^{-n}a$, hence $f_1 = f_2$ in I.

THEOREM 2. *Let $g = f$ in* (1). *Let F be reflexive at a, continuous and strictly monotonic in each variable in a neighbourhood of (a, a). For (x, y) in a neighbourhood of (a, a) and (u, v) in a neighbourhood of (c, c) let $H(u, v, x, y)$ be strictly monotonic in u and v. Let f_1 and f_2 be continuous solutions of* (1) *in a neighbourhood of (a, a), and $f_1(a) = f_2(a) = c$. Then there exists a neighbourhood of a in which either f_1 and f_2 have only the one point in common or are identical.*

PROOF. f_1 and f_2 will satisfy

$$fE(x, y) = J(fx, fy, x, y) \tag{3}$$

for (x, y) in a neighbourhood of (a, a), where

$$E(x, y) = F\{F(x, a), F(a, y)\}$$

and

$$J(u, v, x, y) = H\{H(u, c, x, a), H(c, v, a, y), F(x, a), F(a, y)\}.$$

But $E(a, a) = a$ and E is continuous and strictly increasing in a neighbourhood of (a, a). Also for (x, y) in a neighbourhood of (a, a) and (u, v) in a neighbourhood of (c, c), $J(u, v, x, y)$ is strictly increasing in u and v. Hence there exists a neighbourhood N of c and a neighbourhood I of a such that f_1 and f_2, E and J satisfy the hypotheses of Theorem 1. Then I is the required neighbourhood of a.

EXAMPLE. The equation

$$f(\sqrt{|xy|}) = \sqrt{f(x)f(y)} \tag{4}$$

illustrates both theorems. The function $F(x, y) = \sqrt{|xy|}$ is not strictly monotonic in either variable in any region including the origin; indeed there is an infinite number of continuous solutions of (4) of the form $A|x|^B$ which pass through the origin and any other point with positive y coordinate. If we consider (4) only for negative x and y we may apply Theorem 1. However although F is then not reflexive anywhere both initial conditions are necessary; in this case the condition $F(a, a) \in I$ in Theorem 1 is not fulfilled for any $a \in I$.

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NOTE. The reflexive case in Theorem 1 is already contained in [1] as a special case.

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