# AN INEQUALITY CONCERNING ANALYTIC FUNCTIONS WITH A POSITIVE REAL PART

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This paper contains an inequality about functions which are analytic and have a positive real part in the unit disk. A first consequence of the inequality is the fact that $\sum_{n=1}^{\infty} |a_n|^2 \leqq \delta^2/2$ if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is analytic for $|z| < 1$ and has values lying in a strip of width $\delta$. This result is known and was first proved by Tammi (**1**).

Our second theorem is a generalization of this. Namely, *if* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *is analytic for* $|z| < 1$ *and satisfies* $\operatorname{Re}\{z^m f^{(m)}(z)\} \geqq A$ *and*

$$\operatorname{Re}\{z^{m+k} f^{(m+k)}(z)\} \leqq B,$$

*then* $\sum_{n=1}^{\infty} n^{2m+k}|a_n|^2$ *converges.*

Another application of our fundamental inequality is the following. Let $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ be analytic for $|z| < 1$ and satisfy $\operatorname{Re} p(z) \geqq 0$ and set $p_n(z) = 1 + \sum_{k=1}^{n} p_k z^k$ and $p_n^*(z) = 1 + \sum_{k=1}^{n} z^k$. To each function $p_n$ there is a complex number $\epsilon$ such that

$$|\epsilon| = 1 \quad \text{and} \quad \max_{|z| \leqq 1} |p_n(z) - p_n^*(\epsilon z)| \leqq n.$$

This paper also formulates some more general problems based upon the last mentioned result. Also included is a remark on the possibility of applying our inequality to univalent, starlike mappings. It is expected that this inequality will be useful in other situations.

Lemma. *Let* $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ *be analytic for* $|z| < 1$ *and satisfy* $\operatorname{Re} p(z) \geqq 0$, *and let* $\{\lambda_n\}$ *be a sequence of non-negative real numbers such that* $q(z) = \sum_{n=1}^{\infty} \lambda_n p_n z^n$ *is analytic for* $|z| < 1$. *If* $\operatorname{Re} q(z) \leqq M$ *for* $|z| < 1$, *then*

$$\sum_{n=1}^{\infty} \lambda_n |p_n|^2 \leqq 2M.$$

*Proof.* Let $0 < r < 1$, $u(r, \theta) = \operatorname{Re} p(re^{i\theta})$, $v(r, \theta) = \operatorname{Re} q(re^{i\theta})$, and $p_n = a_n + ib_n$. Then,

$$u(r, \theta) = 1 + \sum_{n=1}^{\infty} (a_n \cos n\theta - b_n \sin n\theta) r^n$$

and

$$v(r, \theta) = \sum_{n=1}^{\infty} \lambda_n (a_n \cos n\theta - b_n \sin n\theta) r^n.$$

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If these two series are multiplied and then integrated term-by-term, one obtains

$$\int_0^{2\pi} u(r, \theta)v(r, \theta)\, d\theta = \pi \sum_{n=1}^{\infty} \lambda_n(a_n^2 + b_n^2)r^{2n}.$$

This uses the orthogonality of the trigonometric functions and the results

$$\int_0^{2\pi} \cos^2 n\theta\, d\theta = \int_0^{2\pi} \sin^2 n\theta\, d\theta = \pi$$

as well as the absolute and uniform convergence of the two series in the interval $0 \leqq \theta \leqq 2\pi$. We have obtained the formula

$$\sum_{n=1}^{\infty} \lambda_n |p_n|^2 r^{2n} = \frac{1}{\pi} \int_0^{2\pi} u(r, \theta)v(r, \theta)\, d\theta.$$

Since $u(r, \theta) \geqq 0$ and $v(r, \theta) \leqq M$, we further deduce that

$$\sum_{n=1}^{\infty} \lambda_n |p_n|^2 r^{2n} \leqq \frac{M}{\pi} \int_0^{2\pi} u(r, \theta)\, d\theta = 2M.$$

The conclusion now follows since the last inequality holds for each $r$ in the interval $0 < r < 1$.

THEOREM 1. *If the analytic function* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *maps* $|z| < 1$ *into a strip of width* $\delta$, *then* $\sum_{n=1}^{\infty} |a_n|^2 \leqq \frac{1}{2}\delta^2$.

*Proof.* There are complex numbers $a$ and $b$ with $|a| = 1$ such that

$$g(z) = af(z) + b = \sum_{n=0}^{\infty} b_n z^n$$

satisfies $0 \leqq \operatorname{Re} g(z) \leqq \delta$ for $|z| < 1$. Let $b_0 = \alpha + i\beta$ and assume that $\alpha \neq 0$ as otherwise $f$ is constant and the theorem is obvious. Let

$$p(z) = \frac{g(z) - i\beta}{\alpha} = 1 + \sum_{n=1}^{\infty} \frac{b_n}{\alpha} z^n \quad \text{and} \quad q(z) = p(z) - 1,$$

and note that $\operatorname{Re} p(z) \geqq 0$ and $\operatorname{Re} q(z) \leqq \delta/\alpha - 1$. The Lemma is applicable where $\lambda_n = 1$ for each $n$, and this produces the inequality

$$\sum_{n=1}^{\infty} \frac{|b_n|^2}{\alpha^2} \leqq 2\left(\frac{\delta}{\alpha} - 1\right).$$

Therefore,

$$\sum_{n=1}^{\infty} |a_n|^2 = \sum_{n=1}^{\infty} |b_n|^2 \leqq 2(\alpha\delta - \alpha^2) \leqq \tfrac{1}{2}\delta^2.$$

The last inequality maximizes $2(\alpha\delta - \alpha^2)$ at $\alpha = \frac{1}{2}\delta$, and this completes the proof.

*Remarks.* (1) The function

$$f(z) = \frac{\delta}{\pi} \log \frac{1+z}{1-z}$$

is extremal for Theorem 1. It maps $|z| < 1$ one-to-one onto the strip $|Iw| < \frac{1}{2}\delta$, and writing $f(z) = \sum_{n=0}^{\infty} a_n z^n$ we find that

$$\sum_{n=1}^{\infty} |a_n|^2 = \frac{4\delta^2}{\pi} \sum_{m=1}^{\infty} \frac{1}{(2m+1)^2} = \tfrac{1}{2}\delta^2.$$

(2) Another simple consequence of the Lemma is the following. In addition to the hypotheses of the first sentence of the Lemma, suppose that $\operatorname{Re} p(z)$ is not bounded from above and $\liminf \lambda_n > 0$. Then $\sum_{n=1}^{\infty} \lambda_n |p_n|^2$ as well as $\sum_{n=1}^{\infty} |p_n|^2$ diverges, and consequently the real part of $q(z)$ cannot be bounded from above. This is moderately interesting since $\lambda_n$ still can be chosen fairly much at random. Our assertion about $q$ is quite simple with certain regular sequences $\{\lambda_n\}$, such as the one defined by the relation $q(z) = p(z^k) - 1$, where $k$ is a positive integer.

*Definition.* If $A$ is a set of complex numbers and if $a$ and $b$ are two given complex numbers, then $aA + b = \{w: w = az + b, z \in A\}$. We will say that two sets $A$ and $B$ are bounded on opposite sides if there are two complex numbers $a$ and $b$ and a real number $M$ such that $a \neq 0$, $\operatorname{Re} z \geqq 0$ if $z \in aA + b$ and $\operatorname{Re} z \leqq M$ if $z \in aB + b$. The relation between two such sets has an evident geometric meaning.

THEOREM 2. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be analytic for $|z| < 1$ and let $D_n$ $(n \geqq 0)$ denote the image of $|z| < 1$ under $z^n f^{(n)}(z)$. If $D_m$ and $D_{m+k}$ are bounded on opposite sides, then $\sum_{n=1}^{\infty} n^{2m+k} |a_n|^2$ converges. In particular, $a_n = o(n^{-m-k/2})$ as $n \to \infty$.*

*Proof.* There are numbers $a$, $b$, and $M$ such that $a \neq 0$, $g(z) = az^m f^{(m)}(z) + b$ satisfies $\operatorname{Re} g(z) \geqq 0$, and $h(z) = az^{m+k} f^{(m+k)}(z) + b$ satisfies $\operatorname{Re} h(z) \leqq M$. Let $g(0) = \alpha + i\beta$ and note that we may assume that $\alpha \neq 0$, for otherwise $f$ is a polynomial, and the theorem is immediate. Set

$$p(z) = \frac{g(z) - i\beta}{\alpha} = 1 + \sum_{n=1}^{\infty} p_n z^n$$

so that $p_1 = p_2 = \ldots = p_{m-1} = 0$, $p_{m+l} = (a/\alpha)(l+1)(l+2)\ldots(l+m)a_{m+l}$ for $l = 0, 1, 2, \ldots$ and $\operatorname{Re} p(z) \geqq 0$. Letting $q(z) = (h(z) - h(0))/\alpha$, we see that $\operatorname{Re} q(z) \leqq (M - \operatorname{Re} h(0))/\alpha$ and it is possible to put $q(z)$ into the form $\sum_{n=1}^{\infty} \lambda_n p_n z^n$, where $\lambda_n$ are non-negative real numbers. Specifically, we choose $\lambda_n$ as follows: $\lambda_1, \lambda_2, \ldots, \lambda_{m-1}$ may be chosen arbitrarily and for convenience may be set equal to zero, $\lambda_m = \lambda_{m+1} = \ldots = \lambda_{m+k-1} = 0$, and $\lambda_{m+k+l} = (l+1)(l+2)\ldots(l+k)$ for $l = 0, 1, 2, \ldots$.

The Lemma is applicable and this yields the inequality

$$\sum_{l=0}^{\infty} \lambda_{m+k+l}|p_{m+k+l}|^2 \leqq 2\left(\frac{M - \operatorname{Re} h(0)}{\alpha}\right),$$

which is the same as

$$\sum_{l=0}^{\infty} (l+1)(l+2)\dots(l+k)(l+1)^2(l+2)^2\dots(l+m)^2|a_{m+k+l}|^2$$
$$\leqq \frac{2\alpha}{|a|^2}\,[M - \operatorname{Re} h(0)].$$

The series on the left-hand side of the last inequality converges simultaneously with the series $\sum_{n=1}^{\infty} n^{2m+k}|a_n|^2$, and this proves the theorem.

THEOREM 3. *Let* $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ *be analytic for* $|z| < 1$ *and satisfy* $\operatorname{Re} p(z) \geqq 0$, *and set* $p_n(z) = 1 + \sum_{k=1}^{n} p_k z^k$ *and* $p_n^*(z) = 1 + \sum_{k=1}^{n} z^k$. *There exists a complex number* $\epsilon$ *depending on* $p_n(z)$ *such that* $|\epsilon| = 1$ *and*

$$\max_{|z|\leqq 1} |p_n(z) - p_n^*(\epsilon z)| \leqq n.$$

*Proof.* Let $M = \max_{|z|\leqq 1} \operatorname{Re} p_n(z)$ and choose $z_0$ such that $|z_0| = 1$ and $\operatorname{Re} p_n(z_0) = M$. Applying the Lemma, where $q(z) = p_n(z)$ (so that

$$\lambda_1 = \lambda_2 = \dots = \lambda_n = 1$$

and $\lambda_k = 0$ for $k > n$), we conclude that $\sum_{k=1}^{n} |p_k|^2 \leqq 2M = 2\operatorname{Re} p_n(z_0)$. Set $\epsilon = \bar{z}_0$ and suppose that $|z| \leqq 1$; then

$$|p_n(z) - p_n^*(\epsilon z)| = \left|\sum_{k=1}^{n} (p_k - \epsilon^k)z^k\right| \leqq \sum_{k=1}^{n} |p_k - \epsilon^k|$$
$$\leqq \sum_{k=1}^{n} \{\tfrac{1}{2}|p_k - \epsilon^k|^2 + \tfrac{1}{2}\} = n + \frac{1}{2}\left\{\sum_{k=1}^{n} |p_k|^2 - 2\operatorname{Re}\sum_{k=1}^{n} p_k\bar{\epsilon}^k\right\} \leqq n.$$

*Remarks.* (1) If $p$ is the function $p(z) = 1$, then for each number $\epsilon$, $|\epsilon| = 1$,

$$\max_{|z|\leqq 1} |p_n(z) - p_n^*(\epsilon z)| = \max_{|z|\leqq 1}\left|\sum_{k=1}^{n} \epsilon^k z^k\right| \geqq \left|\sum_{k=1}^{n} \epsilon^k \bar{\epsilon}^k\right| = n.$$

The function

$$p(z) = \frac{1+z}{1-z} = 1 + \sum_{n=1}^{\infty} 2z^n$$

also satisfies $\operatorname{Re} p(z) > 0$ for $|z| < 1$, and for each $\epsilon$, $|\epsilon| = 1$,

$$\max_{|z|\leqq 1} |p_n(z) - p_n^*(\epsilon z)| \geqq |p_n(1) - p_n^*(\epsilon)| = \left|2n - \sum_{k=1}^{n} \epsilon^k\right| \geqq 2n - n = n.$$

Thus, the number $n$ in the conclusion of Theorem 3 cannot be decreased. Moreover, we have exhibited two quite different examples of functions $p$ which are extremal and each one serves for every $n$.

Notice that $p^*(z) = (1-z)^{-1} = \sum_{n=0}^{\infty} z^n$ satisfies the hypotheses on $p$ in Theorem 3 as $w = (1-z)^{-1}$ maps $|z| < 1$ one-to-one onto $\operatorname{Re} w > \frac{1}{2}$. Therefore, this theorem asserts that each $n$th ($n$ fixed) partial sum of a function $p$ may be "best approximated" by some rotation in $z$ of the $n$th partial sum of a particular function under consideration.

(2) We now raise the question of what general result is exhibited by Theorem 3. Suppose that $\mathscr{F}$ is a family of functions each of which is defined for $|z| < 1$ such that if $f \in \mathscr{F}$, then $f(\epsilon z) \in \mathscr{F}$ for each complex number $\epsilon$, $|\epsilon| = 1$. An equivalence relation is defined on $\mathscr{F}$ by the condition $f(z) = g(\epsilon z)$ for some $\epsilon$, $|\epsilon| = 1$. If $d$ is a metric on $\mathscr{F}$ we define $D$ by $D(F, G) = \inf d(f, g)$, where $f$ and $g$ vary over the equivalence classes $F$ and $G$.

Theorem 3 deals with the family $\{p_n(z)\}$ ($n$ fixed) and the metric

$$d(f, g) = \max_{|z| \leqq 1} |f(z) - g(z)|$$

and asserts that the equivalence classes lie in the disk $\{F: D(F, F^*) \leqq n\}$, where $F^*$ is the equivalence class containing $p_n^*$. It is easy to show that this disk is filled in the sense that to each number $m$, $0 \leqq m \leqq n$, there is an equivalence class $F$ such that $D(F, F^*) = m$. Moreover, this disk contains diametrically opposite points $F_1$ and $F_2$ in the sense that $D(F_1, F^*) = D(F_2, F^*) = n$ and $D(F_1, F_2) = 2n$, as was shown in Remark 1.

We ask: In the general situation, under what conditions on $\mathscr{F}$ and $d$ do the equivalence classes fill up a disk and when are there diametrically opposite points?

We also raise the problem of considering the kind of result of Theorem 3, where $\mathscr{F}$ and $d$ are quite specific. One then may be interested in determining exactly the centre and radius of the disk which the equivalence class covers when that is the situation.

(3) Finally, we would like to indicate another possible direction in which the Lemma may be useful. Suppose that $f(z) = z + a_2 z^2 + \ldots$ is analytic and maps $|z| < 1$ one-to-one onto a domain starlike with respect to the origin. This is equivalent to the hypothesis that

$$p(z) = \frac{zf'(z)}{f(z)} = 1 + \sum_{n=1}^{\infty} p_n z^n$$

is analytic in $|z| < 1$ and satisfies $\operatorname{Re} p(z) \geqq 0$. This relation between $f$ and $p$ can be expressed in the equivalent form

$$f(z) = z \exp\left\{ \int_0^z \frac{p(w) - 1}{w} \, dw \right\}.$$

If we let

$$q(z) = \int_0^z \frac{p(w) - 1}{w} \, dw = \sum_{n=1}^{\infty} \frac{p_n}{n} z^n \quad \text{and} \quad M(r) = \max_{|z| \leqq r} |f(z)|,$$

then

$$\operatorname{Re} q(z) \leqq \log \frac{M(r)}{r} \quad \text{for } |z| \leqq r.$$

The Lemma is applicable to the functions $p(rz)$ and $q(rz)$ (with $\lambda_n = n^{-1}$) and this yields the inequality

$$\sum_{n=1}^{\infty} \frac{|p_n|^2}{n} r^{2n} \leqq 2 \log \frac{M(r)}{r}.$$

This inequality represents an area theorem for the functions $q$, and one wonders whether there are any important applications of the inequality to the study of starlike and related functions.

## Reference

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