Canad. J. Math. Vol. **63** (3), 2011 pp. 689–720
doi:10.4153/CJM-2011-012-1

Higher Rank Wavelets

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Abstract. A theory of higher rank multiresolution analysis is given in the setting of abelian multiscalings. This theory enables the construction, from a higher rank MRA, of finite wavelet sets whose multidilations have translates forming an orthonormal basis in $L^2(\mathbb{R}^d)$. While tensor products of uniscaled MRAs provide simple examples we construct many nonseparable higher rank wavelets. In particular we construct *Latin square wavelets* as rank 2 variants of Haar wavelets. Also we construct nonseparable scaling functions for rank 2 variants of Meyer wavelet scaling functions, and we construct the associated nonseparable wavelets with compactly supported Fourier transforms. On the other hand we show that compactly supported scaling functions for biscaled MRAs are necessarily separable.

1 Introduction

The term multiscaling in wavelet theory commonly refers to the various scaling levels present in a nest of subspaces

$$\cdots \subseteq V_{-1} \subseteq V_0 \subseteq V_1 \subseteq \cdots$$

of a multiresolution analysis in $L^2(\mathbb{R}^d)$. See, for example, Dutkay and Jorgensen [5]. However such subspaces are associated with the powers of a single dilation matrix. In contrast we develop here a theory of wavelets for higher rank multiresolution analyses that are generated by several independent commuting dilation matrices.

Recall that a wavelet set, or multiwavelet, is generally taken to be a set of functions $\psi_1(x), \dots, \psi_t(x)$ in $L^2(\mathbb{R}^d)$ for which certain translates of dilates form an orthonormal basis of the form

$$\{(\det A)^{\frac{m}{2}}\psi_i(A^m x + k) : m \in \mathbb{Z}, k \in \mathbb{Z}^d, 1 \leq i \leq t\},$$

where A is a scaling matrix in $\mathrm{GL}(\mathbb{R}^n)$. Wavelet theory is concerned with identifying constructions for which the wavelets ψ_i exhibit forms of directionality and smoothness. In particular it has been of interest to obtain multivariable wavelets which are in some sense nonseparable with respect to the coordinates of $\mathbb{R}^d$. See for example [1], [3], [7], [9], [11], [12], [16]. In particular Belogay and Wang [1] construct nonseparable wavelets in $\mathbb{R}^2$, for some dilation matrices with determinant 2, which are arbitrarily smooth.

Most commonly, particularly in multiresolution analysis, the dilation group is singly generated, as above. Some recent studies with multiscalings that go beyond

Received by the editors December 18, 2008.
Published electronically February 25, 2011.
The first author is supported by an EPSRC grant. The second author is partially supported by EPSRC grant EP/E002625/1.
AMS subject classification: **42C40**, 42A65, 42A16, 43A65.
Keywords: wavelet, multi-scaling, higher rank, multiresolution, Latin squares.

this are summarised in Gu, Labate, Lim, Weiss and Wilson [8]. In these settings volume-preserving sheering unitaries combine with a single strict dilation to generate the nonabelian dilation groups of interest, and associated wavelet sets are constructed. Also these so-called affine orthonormal systems exhibit specific forms of directionality and nonseparability. In contrast to this our wavelets are constructed for a dilation representation of the abelian group $\mathbb{Z}^r$.

There are two main approaches to constructing wavelet sets, namely the multiresolution analysis approach of Mallat [13], on the one hand, and constructions associated with self-similar tilings of $\mathbb{R}^d$ on the other. See Wang [16], for example, for connections with tilings. As far as the authors are aware there has been little development of traditional multiresolution analysis wavelet theory for multiscaled settings in which the role of dilation matrices A^m is played by an abelian group of dilation matrices $A_1^{m_1}A_2^{m_2}\cdots A_r^{m_s}$. The simplest such multiscaled context of this kind is the biscaled dyadic case for wavelets in $L^2(\mathbb{R}^2)$ associated with the dilation matrices

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

If $\psi_A(x)$ and $\psi_B(x)$ are univariate dyadic wavelets in $L^2(\mathbb{R})$ then the separable function $\psi(x, y) = \psi_A(x)\psi_B(y)$ is a biscaled wavelet, that is, the set

$$\{2^{(m+n)/2}\psi(A^m B^n x + k) : (m, n) \in \mathbb{Z}^2, k \in \mathbb{Z}^2\}$$

is an orthonormal basis in $L^2(\mathbb{R}^2)$.

The evident nonlocality of such separable wavelets has possibly not encouraged the elaboration of a multiscaled wavelet theory. (See, for example, the discussion in [17].) However, we shall show that even in this apparently adverse setting of separated coordinates it is possible to construct nonseparable wavelets and even nonseparable scaling functions. Moreover we develop what might be termed a theory of higher rank wavelets and multiresolution analysis.

We define a higher rank multiresolution analysis $(\phi, \mathfrak{V})$, of rank r, where $\mathfrak{V} = \{V_{\underline{i}} : \underline{i} \in \mathbb{Z}^r\}$ is a commuting lattice of closed subspaces of $L^2(\mathbb{R}^d)$ with appropriate inclusions. See Definition 2.3 and Section 2.8. In our first main result, Theorem 3.4, we show how one may construct multiscaled wavelet sets from a higher rank multiresolution of rank two.

The distinguished novelties of higher rank multiresolution analysis are already present in the simplest setting of biscaling ($r = 2$) and dimension 2 ($d = 2$). For such a biscaled multiresolution analysis (BMRA) the scaling function $\phi(x, y)$ possesses two marginal filter functions $m_\phi^A(\xi), m_\phi^B(\xi)$ and these must satisfy the intertwining relation

$$m_\phi^A(B\xi)m_\phi^B(\xi) = m_\phi^A(\xi)m_\phi^B(A\xi). \tag{1}$$

This filter relation follows from the coincidence of the triple subspace inclusions associated with the dilation pairs B, AB and A, BA. That is, the intertwining relation is a consequence of the lattice structure of $\mathfrak{V}$. In addition to this the orthogonality structure, or *commuting* (projection lattice) structure, of the BMRA leads to a filter identity of some complexity, namely,

$$\begin{aligned}
&m_\phi^A(\xi_1+\pi,2\xi_2)m_\phi^A(\xi_1,\xi_2)\overline{m_\phi^B(\xi_1,\xi_2)m_\phi^B(2\xi_1,\xi_2+\pi)}\\
&\quad - m_\phi^A(\xi_1,2\xi_2)m_\phi^A(\xi_1+\pi,\xi_2)\overline{m_\phi^B(\xi_1+\pi,\xi_2)m_\phi^B(2\xi_1,\xi_2+\pi)}\\
&\quad - m_\phi^A(\xi_1+\pi,2\xi_2)m_\phi^A(\xi_1,\xi_2+\pi)\overline{m_\phi^B(\xi_1,\xi_2+\pi)m_\phi^B(2\xi_1,\xi_2)}\\
&\quad + m_\phi^A(\xi_1,2\xi_2)m_\phi^A(\xi_1+\pi,\xi_2+\pi)\overline{m_\phi^B(\xi_1+\pi,\xi_2+\pi)m_\phi^B(2\xi_1,\xi_2)} = 0.
\end{aligned}$$

These two necessary conditions present challenges for the construction of nonseparable scaling functions. Nevertheless, in another of our main results, Theorem 7.2, we construct examples of BMRAs for which the scaling function is indeed nonseparable. In this case the derived wavelets are similarly nonseparable. Furthermore, they inherit smoothness from ϕ. In this way we obtain singleton bidyadic wavelets whose Fourier transforms have compact support and these wavelets are in fact higher ranks variants of the well-known Meyer wavelets [14].

On the other hand, even for separable rank 2 multiresolutions and separable scaling functions we show that there is sufficient freedom in the nondyadic case

$$(\det A - 1)(\det B - 1) \geq 2$$

to derive nonseparable wavelets. This is achieved by constructing filter matrix functions that are not elementary tensor products. The construction of these filter functions parallels the well-known method of unitary matrix completion although our arguments, given in the proof of Theorem 3.4, involve a nesting of several Gram–Schmidt completion processes. The wavelet sets here include some very interesting and computable examples of what we term *Latin square wavelet sets*, for evident reasons. (See Theorem 4.3.) These are multiscaled versions of the classical wavelets associated with Haar bases.

In another of our main results we reveal a striking constraint for compactly supported scaling functions in the biscaled theory in $L^2(\mathbb{R}^2)$, namely that such functions are necessarily separable. Equivalently put, a BMRA $(\phi, \mathfrak{B})$ with compactly supported scaling function ϕ is equivalent to an elementary tensor of two uniscaled MRAs. This fact may have been a further implicit obstacle to the development of multiscaled multiresolution wavelets.

There are many intriguing wavelet directions that now seem to beckon. For example, we have not particularly addressed smoothness and approximation properties in this article. Although we have shown that the generalised Meyer context, with scaling functions with compactly supported Fourier transforms, allows the appearance of nonseparable wavelets, the context is nevertheless a constraining one. Also, we have shown that the intertwining relation is an exactitude that rules out compactly supported higher rank scaling functions. It is plausible that the setting of frames and higher rank GMRAs, which relaxes this condition in some way, may allow for the construction of compactly supported higher rank frames. Moreover it will also be of interest to develop this theory further for dilation representations of $\mathbb{Z}^d$ on $\mathbb{R}^n$ other than the basic ones we consider.

Even in theoretical articles such as this one it is customary to make a few remarks concerning the potential efficiency of any new species of wavelets or frames. In the current climate of diverse and burgeoning applications this hardly seems necessary. However, we remark that nature often presents pairs of partially independent features with their own scaling aspects. Space-time scales are one obvious source of this. A purely spatial context can be found in the statistical theory of textures. This theory is suitable for localised analysis through wavelets (see [6]) and, evidently, many natural textures have a rectangular emphasis.

Our account is self-contained with complete proofs apart from a few basic standard lemmas. In particular in Section 5 we reprove the construction and formula of a uniscaled dyadic wavelet associated with an MRA, and in Section 7 we give the construction scheme for classical Meyer wavelets. We refer the reader to the books of Wojtaszczyk [17] and Brattelli and Jorgensen [2], which are excellent sources for diverse wavelet theory.

2 Higher Rank Multi-resolution Analysis

In this section we formulate the structure of a higher rank multiresolution analysis although we focus attention on the biscaled case of Definition 2.3. We obtain the frequency domain identification of a function $f(x)$ in $V_{i,j}$ in terms of a filter function $m_f(\xi)$ and the Fourier transform $\hat{\phi}(\xi)$. As preparation for the construction of wavelets from a BMRA we obtain a filter function matrix criterion in order that a given set $\{f_1, \dots, f_r\}$ of unit vectors in $L^2(\mathbb{R}^d)$ should generate an orthonormal basis under the operations of translation and bidilation.

2.1 Preliminaries

We take the Fourier transform $\hat{f}$ of an integrable function $f(x)$ on $\mathbb{R}^d$ in the form

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-i\langle \xi, x\rangle} f(x)\, dx,$$

and define $\hat{f}$ for f in $L^2(\mathbb{R}^d)$ by the usual unitary extension of the map $f \to \hat{f}$. The domain of Fourier transforms is referred to as the frequency domain and for emphasis is denoted $\hat{\mathbb{R}}^d$. For a matrix $A \in \mathrm{GL}_d(\mathbb{R})$ and a point x in the time domain $\mathbb{R}^d$ write Ax the product with x viewed as a column vector and define the unitary operator $D_A \colon L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ by

$$(D_A f)(x) = |\det A|^{1/2} f(Ax). \tag{2}$$

Recall that A is said to be an *expansive* matrix if all eigenvalues of A have absolute value greater than one. We say here that A is a *dilation matrix* if in addition $A(\mathbb{Z}^d) \subset \mathbb{Z}^d$, that is, if A is a matrix of integers.

https://doi.org/10.4153/CJM-2011-012-1 Published online by Cambridge University Press

A *commuting dilation pair* (A, B) is a pair of matrices in $\mathrm{GL}_d(\mathbb{R})$ such that after a suitable permutation of the standard basis of $\mathbb{R}^d$ we have

$$A = \begin{bmatrix} A' & 0 \\ 0 & I_{d-p} \end{bmatrix}, \qquad B = \begin{bmatrix} I_{d-q} & 0 \\ 0 & B' \end{bmatrix} \tag{3}$$

where A' and B' are dilation matrices in $\mathrm{GL}_p(\mathbb{R})$ and $\mathrm{GL}_q(\mathbb{R})$. Note that $T = A^n B^m$ is a dilation matrix if and only if $n \geq 1$ and $m \geq 1$, while the matrices A^i and B^j are *weak dilation matrices* in the sense that their eigenvalues are not less than 1. In Sections 5, 6, and 7 we confine attention to wavelets in $L^2(\mathbb{R}^d)$ for a fundamental dilation pair

$$A = \begin{bmatrix} \alpha I & 0 \\ 0 & I \end{bmatrix}, \qquad B = \begin{bmatrix} I & 0 \\ 0 & \beta I \end{bmatrix}$$

where $\alpha, \beta \geq 2$ are integers.

2.2 Biscaled Multiresolution Analysis

We now define a biscaled analogue of a multiresolution analysis associated with a single dilation matrix.

Definition 2.3 A biscaled multiresolution analysis, or BMRA, with respect to the commuting dilation pair (A, B) in $\mathrm{GL}_d(\mathbb{R})$ is a pair $(\phi, \mathfrak{V})$ where $\phi \in L^2(\mathbb{R}^d)$ and $\mathfrak{V} = \{V_{i,j} : (i, j) \in \mathbb{Z}^2\}$ is a family of closed subspaces of $L^2(\mathbb{R}^d)$ such that

(i) $V_{i,j} \subseteq V_{l,m}$, for $i \leq l$, $j \leq m$,

(ii) $D_A^i D_B^j V_{0,0} = V_{i,j}$, for all i, j,

(iii) $\bigcap_{(i,j)\in\mathbb{Z}^2} V_{i,j} = \{0\}$,

(iv) $\overline{\bigcup_{(i,j)\in\mathbb{Z}^2} V_{i,j}} = L^2(\mathbb{R}^d)$,

(v) $P_{i,j} P_{l,m} = P_{l,m} P_{i,j}$ for all orthogonal projections $P_{s,t} : L^2(\mathbb{R}^d) \to V_{s,t}$.

(vi) $\{\phi(x - k) : k \in \mathbb{Z}^d\}$ is an orthonormal basis for $V_{0,0}$.

It is helpful to make the following distinctions.

If we have merely conditions (i), (iii), (iv) then we say that $\mathfrak{V}$ is a *grid* of subspaces. If, additionally, $V_{i,j} \cap V_{k,l} = V_{m,n}$ with $m = \min\{i, k\}$, $n = \min\{j, l\}$, for all i, j, k, l, then we say that $\mathfrak{V}$ is a *lattice* of subspaces. Finally, if, additionally, the stronger commuting condition (v) holds, then we say that $\mathfrak{V}$ is a *commuting lattice* of subspaces. Thus a BMRA is a commuting lattice of type $\mathbb{Z} \times \mathbb{Z}$ which is generated by a single function ϕ and the dilation operators.

We begin the analysis of BMRAs by identifying the various frequency domain descriptions of functions in $V_{i,j}$.

Let $f \in V_{i,j}$, $T = A^i B^j$ and let $t = \det T$. Then $f \circ T^{-1}$ is in $V_{0,0}$ and so there is an expansion

$$f(T^{-1}x) = \sum_{k\in\mathbb{Z}^d} c_f(k)\phi(x - k)$$

in $L^2(\mathbb{R}^d)$ for some square summable sequence $\{c_f(k)\}_{k\in\mathbb{Z}^d}$. This sequence provides the *filter coefficients* of f for the dilation T. Define the $2\pi\mathbb{Z}^d$-periodic function $m_f(\xi)$

in $L^2(\hat{\mathbb{R}}^d)$ by

$$m_f(\xi) = \frac{1}{t} \sum_{k \in \mathbb{Z}^d} c_f(k) e^{-i\langle \xi, k\rangle}.$$

This function is the *filter function* for f associated with T. Note that $\widehat{f \circ T^{-1}}(\xi) = t\hat{f}(T^*\xi)$ and on the other hand

$$(\widehat{f \circ T^{-1}})(\xi) = \sum_{k \in \mathbb{Z}^d} c_f(k) e^{-i\langle \xi, k\rangle} \hat{\phi}(\xi).$$

Thus $\hat{f}(T^*\xi) = m_f(\xi)\hat{\phi}(\xi)$ and so the filter function determines f in $TV_{0,0}$ by means of the formula

$$\hat{f}(\xi) = m_f(T^{*-1}\xi)\hat{\phi}(T^{*-1}\xi). \tag{4}$$

We may view the filter function $m_f(\xi)$ as being defined on $2\pi\mathbb{T}^d$, in which case one readily sees that

$$\|m_f\|_{L^2(2\pi\mathbb{T}^d)} = \frac{1}{\sqrt{t}}\|f\|_2.$$

In view of our applications and for notational simplicity we assume henceforth that $A = A^*, B = B^*$ and hence $T = T^*$.

Note that $m_f(T^{-1}\xi)$ is periodic by translates from $2\pi T\mathbb{Z}^d$ and that we have the following converse. If $g(\xi)$ is $2\pi T\mathbb{Z}^d$ periodic and has square integrable restriction to $2\pi\mathbb{T}^d$ then the function f in $L^2(\mathbb{R}^d)$ with $\hat{f}(\xi) = g(\xi)\hat{\phi}(T^{-1}\xi)$ lies in $V_{i,j}$. In particular we have the following criterion for membership of $V_{0,0}$ which we state explicitly as it will play a role in establishing the lattice property of a BMRA.

Proposition 2.4 *Let $(\phi, \mathfrak{B})$ be a BMRA. Then a function f in $L^2(\mathbb{R}^d)$ lies in $V_{0,0}$ if and only if $\hat{f}(\xi) = g(\xi)\hat{\phi}(\xi)$ for some $2\pi\mathbb{Z}^d$ periodic function g which has square summable restriction to $2\pi\mathbb{T}^d$.*

Consider now the cosets of $T\mathbb{Z}^d$ in the abelian group $\mathbb{Z}^d$, of which there are t in number, say $E_0, \dots, E_{t-1}$. (See Proposition 5.5 in [17] for example.) Representative elements $\Gamma_0, \dots, \Gamma_{t-1}$ of the cosets are often known as *digits* and comprise a *set of digits for T*.

For $0 \le i \le t-1$ define the i-th translate function for the filter $m_f(\xi)$ as the function

$$m_f^i(\xi) = m_f(\xi + 2\pi T^{-1}\Gamma_i). \tag{5}$$

Note that from the proof of Lemma 2.7 it follows that if f is a unit vector then the row vector formed by the t translates $m_f^i(\xi)$ is a unit vector almost everywhere on $2\pi\mathbb{T}^d$.

Definition 2.5 Let $(\phi, \mathfrak{B})$ be a biscaled multiresolution analysis, let $(i, j) \in \mathbb{Z}^2$, and let $\mathcal{F} = \{f_0, \dots, f_s\}$ be an ordered set of functions in $V_{i,j}$. The (translation form) filter matrix for $\mathcal{F}$ is the function matrix $U_{\mathcal{F}}(\xi)$ where

$$U_{\mathcal{F}}(\xi) = [m_{f_l}^n(\xi)]_{l=0,n=0}^{s,t-1}$$

and $t = \det A^i B^j$.

We remark that in Section 4 we define a related filter matrix $U'_{\mathcal{F}}(\xi)$ whose rows are formed by the partial sums over cosets of the filter functions in the first column of $U_{\mathcal{K}}$.

The key filter matrix lemma below will be used in the construction of biscaled wavelets and in this connection it will be applied to various pairs X, Y from the spaces $V_{0,0}, V_{1,0}, V_{0,1}, V_{1,1}$. The proof is similar to well-known constructions in the the monoscaled case.

Let $X \subseteq Y$ be closed subspaces of $L^2(\mathbb{R}^d)$ with $Y = D_T X$ and let ϕ be a scaling function for X, Y in the sense that the translates of ϕ form an orthonormal basis for X. In the light of our applications later we make the simplifying assumption that the digits are chosen so that the set $T[0, 2\pi]^d$ is the essentially disjoint union of the sets $[0, 2\pi]^d + 2\pi\Gamma_i$, $i = 0, \ldots, t-1$.

First we note the following well known simple property of scaling functions, which is equivalent to the orthonormality of translates. This will feature in the proof of Lemma 2.7 and in the construction of higher rank Meyer wavelets.

Lemma 2.6 *The set of $\mathbb{Z}^d$-translates of a function ϕ in $L^2(\mathbb{R}^d)$ forms an orthonormal set if and only if*

$$\sum_{k \in \mathbb{Z}^d} |\hat{\phi}(\eta - 2\pi k)|^2 = \frac{1}{(2\pi)^d}.$$

Lemma 2.7 *Let $(\phi, X, T, Y, \{\Gamma_i\})$ be as above and let $\mathcal{F} = \{f_0, \ldots, f_s\}$ be a finite ordered set in Y with filter matrix $U_{\mathcal{F}}(\xi)$. Then the system $\{f_l(x-k)\}_{k \in \mathbb{Z}^d, l=0,\ldots,s}$ is an orthonormal set in Y if and only if for almost every ξ the matrix $U_{\mathcal{F}}(\xi)$ is a partial isometry with full range. Furthermore the system is an orthonormal basis for Y if and only if $s = t-1$ and for almost every ξ the matrix $U_{\mathcal{F}}(\xi)$ is unitary.*

Proof Let f, g be two functions in $\mathcal{F}$. We evaluate the inner product

$$I = \langle f(x-k_1), g(x-k_2) \rangle$$

for $k_1, k_2 \in \mathbb{Z}^d$. Since $f(x-k)$ has Fourier transform $e^{-i\langle \xi, k \rangle} \hat{f}(\xi)$ the unitarity of the Fourier transform implies

$$I = \int_{\hat{\mathbb{R}}^2} \hat{f}(\xi)\overline{\hat{g}(\xi)} e^{-i\langle \xi, k_1 - k_2 \rangle}\, d\xi. \tag{6}$$

Thus using (4), the substitution $\eta = T^{-1}\xi$, and the $2\pi\mathbb{Z}^d$ periodicity of the filter functions, we have

$$I = \int_{\hat{\mathbb{R}}^d} m_f(T^{-1}\xi)\overline{m_g(T^{-1}\xi)} |\hat{\phi}(T^{-1}\xi)|^2 e^{-i\langle \xi, k_1 - k_2 \rangle}\, d\xi \tag{7}$$

$$= t \int_{\hat{\mathbb{R}}^d} m_f(\eta)\overline{m_g(\eta)} |\hat{\phi}(\eta)|^2 e^{-i\langle T\eta, k_1 - k_2 \rangle}\, d\eta \tag{8}$$

$$= \sum_{k \in \mathbb{Z}^d} t \int_{[0,2\pi]^d} m_f(\eta)\overline{m_g(\eta)} |\hat{\phi}(\eta - 2\pi k)|^2 e^{-i\langle T\eta, k_1 - k_2 \rangle}\, d\eta. \tag{9}$$

By the last lemma,

$$\sum_{k\in\mathbb{Z}^d} |\hat{\phi}(\eta - 2\pi k)|^2 = \frac{1}{(2\pi)^d} \tag{10}$$

and so

$$I = \frac{t}{(2\pi)^d} \int_{[0,2\pi]^d} m_f(\eta)\overline{m_g(\eta)} e^{-i\langle T\eta, k_1-k_2\rangle}\, d\eta, \tag{11}$$

thus

$$I = \frac{1}{(2\pi)^d} \int_{T[0,2\pi]^d} m_f(T^{-1}\xi)\overline{m_g(T^{-1}\xi)} e^{-i\langle \xi, k_1-k_2\rangle}\, d\xi. \tag{12}$$

Considering translates by $2\pi\Gamma_i$ and using periodicity in the exponential factor, we conclude that the inner product $\langle f(x-k_1), g(x-k_2)\rangle$ is equal to

$$\frac{1}{(2\pi)^d} \int_{[0,2\pi]^d} \Big(\sum_{i=0}^{t-1} m_f^i(T^{-1}\xi)\overline{m_g^i(T^{-1}\xi)} \Big) e^{-i(\langle \xi, k_1-k_2\rangle)}\, d\xi. \tag{13}$$

Now the sum function in the integrand is not merely $2\pi T\mathbb{Z}^d$ periodic (in view of its terms) but is $2\pi\mathbb{Z}^d$ periodic by virtue of being a sum over all translates. Thus the integral is the (k_1-k_2)-th Fourier coefficient of the sum function. In particular, with $f = g$ we deduce that the row vector function

$$[m_f^0(\xi) \quad m_f^1(\xi) \quad \cdots \quad m_f^{t-1}(\xi)]$$

is a unit vector almost everywhere. Now the lemma follows exactly as in the mono-scaled case. ■

2.8 Separability and Higher Rank

We define a general *higher rank multiresolution analysis* for a commuting r-tuple $(A_1,\dots,A_r)$ in $\mathrm{GL}_d(\mathbb{R})$ to be a pair $(\phi,\mathfrak{V})$, where $\phi \in L^2(\mathbb{R}^d)$ and $\mathfrak{V} = \{V_k : k \in \mathbb{Z}^r\}$ is a family of closed subspaces satisfying the r-fold version of the conditions (i)–(vi) of Definition 2.3. The r-tuple is assumed to take the form

$$A_1 = A_1' \oplus I_{d_2} \oplus \cdots \oplus I_{d_r}, \dots, A_r = I_{d_1} \oplus \cdots \oplus I_{d_{r-1}} \oplus A_r',$$

where $d_1 + d_2 + \cdots + d_r = d$ and $A_i' \in \mathrm{GL}_{d_i}(\mathbb{R})$ are dilation matrices (with integer entries).

The simplest way to create such a multiresolution analysis is as the tensor product

$$(\phi_1 \otimes \cdots \otimes \phi_r, \mathfrak{V}_1 \otimes \cdots \otimes \mathfrak{V}_r)$$

https://doi.org/10.4153/CJM-2011-012-1 Published online by Cambridge University Press

of MRAs $(\phi_i, \mathfrak{V}_i)$, with $\mathfrak{V}_k = \{V_i^{(k)} : i \in \mathbb{Z}\}$, where

$$\mathfrak{V}_1 \otimes \cdots \otimes \mathfrak{V}_r = \{V_k = V_{k_1}^{(1)} \otimes \cdots \otimes V_{k_r}^{(r)} : k = (k_1, \ldots, k_r) \in \mathbb{Z}^r\}.$$

Here the scaling function $\phi = \phi_1 \otimes \cdots \otimes \phi_r$ is separable in the sense that there is an evident partition of the coordinates of $L^2(\mathbb{R}^d)$ with $\phi(x) = \phi_1(x)\phi_2(x)\cdots\phi_r(x)$, where $\phi_i(x)$ is a function of the coordinates in the i-th partition set.

We say that a higher rank MRA $(\phi, \mathfrak{V})$ of rank r is *purely separable* if it is unitarily equivalent to such an r-fold tensor product. For $r \geq 3$ there can be partial forms of separability, which we do not discuss here, while for $r = 2$ we shall simply say that a BMRA $(\phi, \mathfrak{V})$ is *separable* if it is unitarily equivalent to a 2-fold tensor product decomposition.

If $(\phi, \{V_{(i,j)}\})$ is a BMRA then selecting the diagonally labelled subspaces $V_{i,i}$ gives rise to the MRA $(\phi, \{V_{i,i}\})$. We may thus observe from known facts for MRAs that there are redundancies in the conditions of Definition 2.3. Thus condition (iii) follows from (i), (ii), (vi). See Lemma 7.1. In general, however, we have not found that the presence of MRAs in BMRAs provides any shortcuts to the construction of higher rank wavelets. One can develop formulae for filter functions of functions in V_k and pursue the theory for rank greater than two but we do not do so here.

3 Construction of Wavelets from BMRAs

We now show how to construct wavelets and wavelet sets from a given BMRA, and we elucidate the interrelationship between filter functions.

Let $(\phi, \mathfrak{V})$ be a BMRA in $L^2(\mathbb{R}^d)$ for the dilation pair (A, B). We refine some notation for filter functions as follows. If f lies in $V_{1,0}$ then we write $m_f^A(\xi)$ for the $2\pi\mathbb{Z}^d$-periodic filter function $m_f(\xi)$ arising from $T = A$, and if f lies in $V_{0,1}$ we write $m_f^B(\xi)$ for the filter function for $T = B$. Thus if f lies in $V_{1,0} \cap V_{0,1}$ then we have, from (4),

$$\hat{f}(\xi) = m_f^A(A^{-1}\xi)\hat{\phi}(A^{-1}\xi),$$

and

$$\hat{f}(\xi) = m_f^B(B^{-1}\xi)\hat{\phi}(B^{-1}\xi).$$

In particular these remarks apply to ϕ itself and so

$$\hat{\phi}(A\xi) = m_\phi^A(\xi)\hat{\phi}(\xi), \tag{14}$$

$$\hat{\phi}(B\xi) = m_\phi^B(\xi)\hat{\phi}(\xi). \tag{15}$$

Put $B\xi$ for ξ in (14) and use (15) to obtain

$$\hat{\phi}(AB\xi) = m_\phi^A(B\xi)m_\phi^B(\xi)\hat{\phi}(\xi). \tag{16}$$

Reciprocally

$$\hat{\phi}(BA\xi) = m_\phi^B(A\xi)m_\phi^A(\xi)\hat{\phi}(\xi). \tag{17}$$

Thus, if $\hat{\phi}(\xi)$ is nonvanishing almost everywhere then we obtain the fundamental intertwining relation

$$m_\phi^A(B\xi)m_\phi^B(\xi) = m_\phi^A(\xi)m_\phi^B(A\xi). \tag{18}$$

Since $\mathfrak{V}$ is a commuting lattice the subspaces $V_{1,0} \ominus V_{0,0}$ and $V_{0,1} \ominus V_{0,0}$ are orthogonal and so we may define

$$W_{0,0} = V_{1,1} \ominus \big((V_{1,0} \ominus V_{0,0}) \oplus (V_{0,1} \ominus V_{0,0}) \oplus V_{0,0}\big).$$

Moreover, let $W_{i,j} = D_A^i D_B^j W_{0,0}$ for $(i, j) \in \mathbb{Z}^2$, so that

$$W_{i,j} = V_{i+1,j+1} \ominus \big((V_{i+1,j} \ominus V_{i,j}) \oplus (V_{i,j+1} \ominus V_{i,j}) \oplus V_{i,j}\big).$$

Thus these spaces are orthogonal. Moreover since the intersection of the spaces $V_{i,j}$ is the zero space it follows that

$$V_{i+1,j+1} = \sum \bigoplus_{(m,n)\leq(i,j)} W_{m,n},$$

and since the union of the $V_{i,j}$ is dense, $L^2(\mathbb{R}^d)$ is the Hilbert space direct sum of all the $W_{m,n}$.

As in the monoscaled theory, if an explicit orthonormal basis $\{\psi_1, \dots, \psi_s\}$ can be constructed for the subspace $W_{0,0}$ then this is a wavelet set in the sense of the following definition.

Definition 3.1 Let (A, B) be a commuting dilation pair in $\mathrm{GL}_d(\mathbb{R})$. Then $\{\psi_1, \dots, \psi_s\}$ is a wavelet set for (A, B) if

$$\{|\det(A^mB^n)|^{\frac{1}{2}}\psi_i(A^mB^nx + k) : (m, n) \in \mathbb{Z}^2, k \in \mathbb{Z}^d, 1 \leq i \leq s\}$$

is an orthonormal basis in $L^2(\mathbb{R}^d)$.

We now show how one can construct a wavelet set in $W_{0,0}$ by means of a nested Gram–Schmidt orthogonalisation process and repeated applications of both directions of the equivalences given in Lemma 2.7.

It is convenient to introduce the following terminology which anticipates the construction process. As before we consider a BMRA $(\phi, \mathfrak{V})$ for the dilation pair (A, B), where $\mathfrak{V}$ satisfies the lattice condition. Also $p = \det A$, $q = \det B$.

Definition 3.2 A *wavelet family* for the BMRA $(\phi, \mathfrak{V})$ is a set

$$\mathcal{F} = \{\psi_1^A, \dots, \psi_{p-1}^A, \psi_1^B, \dots, \psi_{q-1}^B, \psi_1, \dots, \psi_s\}$$

where $s = (p-1)(q-1)$, and $\{\psi_i^A\}$ (respectively $\{\psi_j^B\}$, respectively $\{\psi_k\}$) is an orthonormal set whose $\mathbb{Z}^2$ translates form an orthonormal basis for $V_{1,0} \ominus V_{0,0}$ (respectively $V_{0,1} \ominus V_{0,0}$, respectively $W_{0,0}$).

Definition 3.3 A *filter bank* for the BMRA $(\phi, \mathfrak{V})$ is a set of functions $\tilde{\mathfrak{F}}$ in $L^2(\mathbb{R}^d)$,

$$\tilde{\mathfrak{F}} = \{\phi, \psi_1^A, \dots, \psi_{p-1}^A, \psi_1^B, \dots, \psi_{q-1}^B, \psi_1, \dots, \psi_s\},$$

where the functions $\psi_i^A \in V_{1,0} \ominus V_{0,0}$, $\psi_j^B \in V_{0,1} \ominus V_{0,0}$, and $\psi_k \in W_{0,0}$ are such that

(i) the associated $(pq-1) \times (pq-1)$ filter matrix $U_{\tilde{\mathfrak{F}}}(\xi)$ for $T = AB$ is unitary almost everywhere,

(ii) the $p \times p$ (respectively $q \times q$) filter matrix $U^A(\xi)$ (respectively $U^B(\xi)$) for the set $\{\phi, \psi_1^A, \dots, \psi_{p-1}^A\}$ and $T = A$ (respectively $\{\phi, \psi_1^B, \dots, \psi_{q-1}^B\}$ and $T = B$) is unitary almost everywhere.

If $\tilde{\mathfrak{F}}$ is a filter bank as above then by (i) and Lemma 2.7 the functions of $\tilde{\mathfrak{F}}$ and their $\mathbb{Z}^d$ translates provide an orthonormal basis for $V_{1,1}$. Also, by (ii) and Lemma 2.7 applied twice, the set $\{\phi, \psi_1^A, \dots, \psi_{p-1}^A\}$ (resp. $\{\psi_1^B, \dots, \psi_{q-1}^B\}$) has translates forming an orthonormal basis for $V_{1,0}$ (resp. $V_{0,1}$). It follows that the set $\{\psi_1, \dots, \psi_s\}$ has translates which form an orthonormal basis of $W_{0,0}$ and so this set is a wavelet set.

Although Theorem 3.4 is stated as an existence theorem, the proof provides a recipe for construction which we shall carry out in the next section.

Theorem 3.4 *Let $(\phi, \mathfrak{V})$ be a rank 2 MRA with respect to the commuting dilation pair (A, B). Then there exists a wavelet set for (A, B).*

Proof From the preceding discussion it suffices to construct a filter bank $\tilde{\mathfrak{F}} = \{\phi\} \cup \mathcal{F}$.

From ϕ and the dilation $T = A$ construct the row vector valued function of ξ given by the normalised row of translated functions for $m_\phi^A(\xi)$. This has the form

$$[m_\phi^A(\xi) \quad m_\phi^A(\xi + 2\pi A^{-1} d_1) \quad \cdots \quad m_\phi^A(\xi + 2\pi A^{-1} d_{p-1})]$$

where $d_1, \dots, d_{p-1}$ in $\mathbb{Z}^p \times \{0\}$, together with $d_0 = 0$, give a set of digits for A. Precisely as in the monoscaled theory we may apply the Gram–Schmidt process to any full rank $p \times p$ completion of this row (by rows which are similarly translates of their first entry) to obtain a $p \times p$ unitary matrix-valued function $U(\xi)$. We thus obtain $U(\xi) = U_{\mathcal{G}}(\xi)$ for a family $\mathcal{G} = \{\phi, \psi_1^A, \dots, \psi_{p-1}^A\}$ where each ψ_i^A lies in $V_{1,0} \ominus V_{0,0}$. By Lemma 2.7, these functions are orthonormal, with translates forming an orthonormal basis of $V_{1,0} \ominus V_{0,0}$.

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In a similar way, using ϕ, the dilation $T = B$ and a set of digits in $\{0\} \times \mathbb{Z}^q$ for B, construct a unitary matrix $U_{\mathcal{K}}(\xi)$ and orthonormal set $\mathcal{K} = \{\phi, \psi_1^B, \ldots, \psi_{q-1}^B\}$ for which the functions ψ_j^B have translates forming an orthonormal basis for $V_{0,1} \ominus V_{0,0}$.

Consider now the union,

$$\mathcal{F}_{AB} = \{\phi, \psi_1^A, \ldots, \psi_{p-1}^A, \psi_1^B, \ldots, \psi_{q-1}^B\}.$$

Since $\mathfrak{V}$ is a lattice the ψ_i^A and the ψ_j^B are orthogonal. Moreover elements of $\mathcal{F}_{AB}$ have orthonormal translates in $V_{1,1} \ominus V_{0,0}$. It thus follows from Lemma 2.7 again that for $T = AB$ the $(p+q-1) \times pq$ filter matrix for $\mathcal{F}_{AB}$ and T, denoted $U_{\mathcal{F}_{AB}}(\xi)$, is a partial isometry almost everywhere.

As before we may complete $U_{\mathcal{F}_{AB}}(\xi)$ to a unitary $pq \times pq$ matrix which is the filter matrix of a family $\mathcal{F}_{AB} \cup \{\psi_1, \ldots, \psi_s\}$. This is the desired filter bank and, by Lemma 2.7, yet again, $\{\psi_1, \ldots, \psi_s\}$ is the desired wavelet set. ■

4 Latin Square Wavelets

4.1 Coset Filter Matrices

First we introduce a companion matrix $U'_{\mathcal{F}}(\xi)$ for the filter matrix $U_{\mathcal{F}}(\xi)$ determined by functions $\{f_0, \ldots, f_{t-1}\}$ in $V_{i,j}$ and a fixed dilation unitary T, as given in Definition 2.5. This companion $t \times t$ matrix uses cosets rather than translates and is unitary if and only if $U_{\mathcal{F}}(\xi)$ is unitary. Furthermore it takes a particularly simple form in the case of the latin square wavelets. We were unable to find a reference for this equivalence and so give the detail here.

Let T be a (possibly weak) dilation matrix, let $t = \det T$ and let $E_0, \ldots, E_{t-1}$ be the cosets of $T\mathbb{Z}^d$. Let $d_0, \ldots, d_{t-1}$ be representative digits, with $d_0 = 0$ and $E_0 = T\mathbb{Z}^2$. We are interested in the case $T = A^iB^j$ with spaces $V_{0,0}$ and $V_{i,j}$. For a function f in $V_{i,j}$ we have the coefficients $c_f(k)$ for f and T as before, determining the filter function

$$m_f(\xi) = \frac{1}{t} \sum_{k \in \mathbb{Z}^d} c_f(k) e^{-i\langle \xi, k\rangle}.$$

Recall that the matrix $U_{\mathcal{F}}(\xi)$ is determined by its first column which consists of the filter functions $m_{f_l}(\xi)$, $0 \le l \le t-1$. The rows are formed by the translates $m^i_{f_l}(\xi)$, $1 \le i \le t-1$. Consider the coset sum

$$\begin{aligned} m_{f_{l,p}}(\xi) &= \frac{1}{t} \sum_{j \in E_p} c_{f_l}(j) e^{-i\langle \xi, j\rangle} \\ &= \Big(\frac{1}{\sqrt{t}} \sum_{k \in \mathbb{Z}^2} c_{f_l}(d_p + Tk) e^{-i\langle \xi, Tk\rangle} \Big) \frac{1}{\sqrt{t}} e^{-i\langle \xi, d_p\rangle} \\ &= \mu_{f_{l,p}}(\xi) \frac{1}{\sqrt{t}} e^{-i\langle \xi, d_p\rangle}, \end{aligned}$$

where $\mu_{f_{l,p}}(\xi)$ is defined as the bracketed sum. Then

$$\begin{aligned} m^i_{f_{l,p}}(\xi) &= m_{f_{l,p}}(\xi + 2\pi T^{-1} d_i) \\ &= \mu_{f_{l,p}}(\xi + 2\pi T^{-1} d_i) \frac{1}{\sqrt{t}} e^{-i\langle \xi + 2\pi T^{-1} d_i, d_p \rangle} \\ &= \mu_{f_{l,p}}(\xi) D_{p,i}(\xi), \end{aligned}$$

by the $T^{-1}\mathbb{Z}^2$ periodicity of $\mu_{f_{l,p}}(\xi)$, where

$$D_{j,k}(\xi) = e^{i\langle \xi, d_j \rangle} \frac{1}{\sqrt{t}} e^{-i\langle 2\pi T^{-1} d_k, d_j \rangle}.$$

Note that $D = (D_{j,k})$ is a unitary valued matrix and

$$m^i_{f_l} = \sum_{p=0}^{t-1} m^i_{f_{l,p}} = \sum_{p=0}^{t-1} \mu^i_{f_{l,p}} D_{p,i} = (U'_{\mathcal{F}} D)_{l,i}.$$

Thus

$$U_{\mathcal{F}} = U'_{\mathcal{F}} D,$$

where $U'_{\mathcal{F}}$, which we call the *coset filter matrix* for $\mathcal{F}$ and T, is defined by

$$U'_{\mathcal{F}}(\xi) = \big(\mu_{f_{l,p}}(\xi)\big)_{l=1,p=0}^{t,t-1}.$$

4.2 Latin Square Wavelets

Let $A = \left[\begin{smallmatrix} 3 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, $B = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 3 \end{smallmatrix}\right]$ be (weak) dilation matrices providing a commuting dilation pair (A, B) and unitary dilation operators D_A, D_B on $L^2(\mathbb{R}^2)$. Let ϕ be the characteristic function of the unit square $[0,1]^2$ in $\mathbb{R}^2$. Then ϕ and (A, B) generate a BMRA $(\phi, \mathfrak{V})$ in $L^2(\mathbb{R}^2)$. In fact $\mathcal{V}$ is simply the tensor product BMRA of two copies of the triadic Haar wavelet MRA. We have

$$\phi\big((AB)^{-1}x\big) = \sum_{i=0}^{2} \sum_{j=0}^{2} \phi\big(x - (i, j)\big).$$

The distinctiveness of this scaling relation is that $\phi\big((AB)^{-1}x\big)$ is simply a linear combination of translates of ϕ by a set of digits for $T = AB$. This property, as we shall see, persists in the wavelets that we construct for $(\phi, \mathfrak{V})$ by the filter matrix completion method of Theorem 3.4. We have

$$\phi(A^{-1}x) = \phi(x) + \phi(x_1 - 1, x_2) + \phi(x_1 - 2, x_2)$$

and so

$$m^A_\phi(\xi_1, \xi_2) = \frac{1}{3}(1 + e^{-i\xi_1} + e^{-i2\xi_1}).$$

Let $(0,0),(1,0),(2,0)$ be the natural set of digits for A. Then the three coset functions for $m_\phi^A(\xi)$ are simply the pure frequency functions $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}e^{-i\xi_1}, \frac{1}{\sqrt{3}}e^{-i2\xi_1}$. To complete the row

$$\frac{1}{\sqrt{3}}\begin{bmatrix}1 & e^{-i\xi_1} & e^{-i2\xi_1}\end{bmatrix}$$

to a unitary matrix valued function of coset functions we first complete

$$\frac{1}{\sqrt{3}}\begin{bmatrix}1 & 1 & 1\end{bmatrix}$$

to a 3×3 unitary scalar matrix. One such completion is given by

$$U_A'(\xi_1,\xi_2)=\begin{bmatrix}\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}}\\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0\end{bmatrix}M_A(\xi),$$

where $M_A(\xi) = \operatorname{diag}(1, e^{-i\xi_1}e^{-i2\xi_1})$. A similar completion matrix $U_B'(\xi_1,\xi_2)$, with $M_B(\xi)=\operatorname{diag}(1,e^{-i\xi_2}e^{-i2\xi_2})$, is associated with

$$m_\phi^B(\xi_1,\xi_2)=\frac{1}{3}(1+e^{-i\xi_1}+e^{-i2\xi_1})$$

and the digits $(0,0),(0,1),(0,2)$.

Rows $2,3$ of the completions above provide functions $\psi_1^A,\psi_2^A,\psi_1^B,\psi_2^B$ such that the five functions $\{\phi,\psi_1^A,\psi_2^A,\psi_1^B,\psi_2^B\}$ are part of a wavelet family in the sense of Definition 3.3. Moreover, for this orthonormal set the coset functions for digits for $T=AB$ provide a partial isometry

$$\begin{bmatrix}
\frac13 & \frac13 & \frac13 & \frac13 & \frac13 & \frac13 & \frac13 & \frac13 & \frac13\\
\frac16\sqrt2 & \frac16\sqrt2 & \frac16\sqrt2 & \frac16\sqrt2 & \frac16\sqrt2 & \frac16\sqrt2 & -\frac13\sqrt2 & -\frac13\sqrt2 & -\frac13\sqrt2\\
\frac16\sqrt6 & \frac16\sqrt6 & \frac16\sqrt6 & -\frac16\sqrt6 & -\frac16\sqrt6 & -\frac16\sqrt6 & 0 & 0 & 0\\
\frac16\sqrt2 & \frac16\sqrt2 & -\frac13\sqrt2 & \frac16\sqrt2 & \frac16\sqrt2 & -\frac13\sqrt2 & \frac16\sqrt2 & \frac16\sqrt2 & -\frac13\sqrt2\\
\frac16\sqrt6 & -\frac16\sqrt6 & 0 & \frac16\sqrt6 & -\frac16\sqrt6 & 0 & \frac16\sqrt6 & -\frac16\sqrt6 & 0
\end{bmatrix}$$

where we have ordered the columns according to the digit order

$$(0,0),(0,1),(0,2),(1,0),(1,1),(1,2),(2,0),(2,1),(2,2).$$

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We may now complete to a 9×9 unitary matrix,

$$\begin{bmatrix}
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{\sqrt{2}}{6} & \frac{\sqrt{2}}{6} & \frac{\sqrt{2}}{6} & \frac{\sqrt{2}}{6} & \frac{\sqrt{2}}{6} & \frac{\sqrt{2}}{6} & -\frac{\sqrt{2}}{3} & -\frac{\sqrt{2}}{3} & -\frac{\sqrt{2}}{3} \\
\frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{6} & -\frac{\sqrt{6}}{6} & -\frac{\sqrt{6}}{6} & -\frac{\sqrt{6}}{6} & 0 & 0 & 0 \\
\frac{\sqrt{2}}{6} & \frac{\sqrt{2}}{6} & -\frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{6} & \frac{\sqrt{2}}{6} & -\frac{\sqrt{2}}{3} & \frac{\sqrt{2}}{6} & \frac{\sqrt{2}}{6} & -\frac{\sqrt{2}}{3} \\
\frac{\sqrt{6}}{6} & -\frac{\sqrt{6}}{6} & 0 & \frac{\sqrt{6}}{6} & -\frac{\sqrt{6}}{6} & 0 & \frac{\sqrt{6}}{6} & -\frac{\sqrt{6}}{6} & 0 \\
\frac{\sqrt{10}}{6} & -\frac{\sqrt{10}}{30} & -\frac{2\sqrt{10}}{15} & -\frac{2\sqrt{10}}{15} & -\frac{\sqrt{10}}{30} & \frac{\sqrt{10}}{6} & -\frac{\sqrt{10}}{30} & \frac{\sqrt{10}}{15} & -\frac{\sqrt{10}}{30} \\
0 & -\frac{\sqrt{15}}{30} & \frac{\sqrt{15}}{30} & -\frac{2\sqrt{15}}{15} & \frac{2\sqrt{15}}{15} & 0 & \frac{2\sqrt{15}}{15} & -\frac{\sqrt{15}}{10} & -\frac{\sqrt{15}}{30} \\
\frac{\sqrt{15}}{15} & -\frac{\sqrt{15}}{6} & \frac{\sqrt{15}}{10} & 0 & \frac{\sqrt{15}}{15} & -\frac{\sqrt{15}}{15} & -\frac{\sqrt{15}}{15} & \frac{\sqrt{15}}{10} & -\frac{\sqrt{15}}{30} \\
\frac{\sqrt{10}}{10} & 0 & -\frac{\sqrt{10}}{10} & 0 & \frac{\sqrt{10}}{10} & -\frac{\sqrt{10}}{10} & -\frac{\sqrt{10}}{10} & -\frac{\sqrt{10}}{10} & \frac{\sqrt{10}}{5}
\end{bmatrix}.$$

Just as the dilated scaling function $\phi(A^{-1}B^{-1}\xi)$ is a linear combination of digit translates of ϕ, so too are the wavelets, $\psi_1, \psi_2, \psi_3, \psi_4$ which are determined by the last four rows of the completion. We can confirm and understand the orthogonality of these wavelets by arranging the coefficients of $\frac{30}{\sqrt{10}}\psi_1$, $\frac{30}{\sqrt{15}}\psi_2$, $\frac{30}{\sqrt{15}}\psi_3$, $\frac{10}{\sqrt{10}}\psi_4$ as in the diagram of Figure 1. One readily sees that the construction creates in this way a quadruple of latin squares which are pairwise orthogonal, that is, have vanishing inner products. Such constructs are natural to study in their own right, and indeed may be used to provide wavelets which, as here, are entirely natural variants of Haar wavelets.

In summary, the arguments above have led to the following theorem, where $\chi_{ij}(x, y)$ denotes the characteristic function of the set $[0, 1/3]^2 + (i/3, j/3)$.

Theorem 4.3 *Let $\psi_1, \dots, \psi_4$ be the functions on $\mathbb{R}^2$ given by*

$$\psi_1 = \frac{\sqrt{10}}{30}(5\chi_{00} - \chi_{01} - 4\chi_{02} - 4\chi_{10} - \chi_{11} + 5\chi_{12} - \chi_{20} + 2\chi_{21} - \chi_{22}),$$

$$\psi_2 = \frac{\sqrt{15}}{30}(-\chi_{01} + \chi_{02} - 4\chi_{10} + 4\chi_{11} + 4\chi_{20} - 3\chi_{21} - \chi_{22}),$$

$$\psi_3 = \frac{\sqrt{10}}{30}(2\chi_{00} - 5\chi_{01} + 3\chi_{02} + 2\chi_{11} - 2\chi_{12} - 2\chi_{20} + 3\chi_{21} - \chi_{22}),$$

$$\psi_4 = \frac{\sqrt{10}}{10}(\chi_{00} - \chi_{02} + \chi_{11} - \chi_{12} - \chi_{20} - \chi_{21} + 2\chi_{22}).$$

Then the set $\{\psi_1, \dots, \psi_4\}$ is a bidyadic wavelet set. That is, the set

$$\{3^{(n+m)/2}\psi_i(3^m x + k_1, 3^n y + k_2) : (m, n), (k_1, k_2) \in \mathbb{Z}^2, 1 \le i \le 4\}$$

is an orthonormal basis in $L^2(\mathbb{R}^2)$.

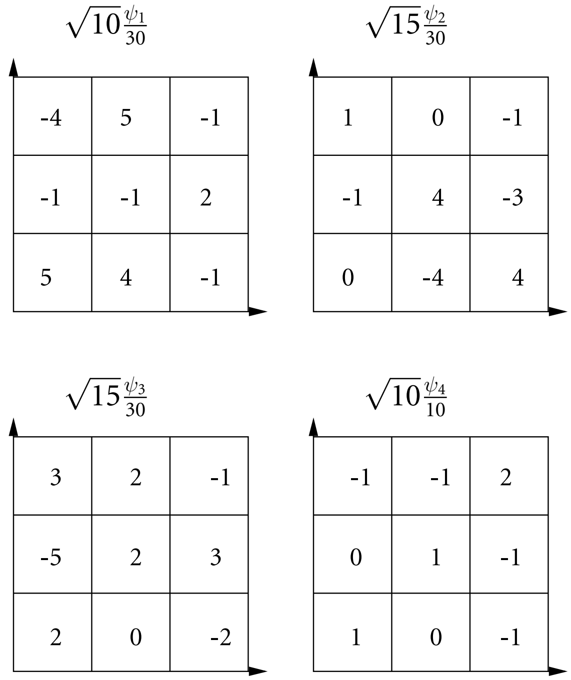

Figure 1: Orthogonal Latin squares for the biscaled wavelets $\psi_1, \dots, \psi_4$.

5 Dyadic Biscaled Wavelets and Filter Formulae

In this section we derive the filter formula, as given in the introduction, that corresponds to the commuting projection lattice property of a BMRA.

We start by reproving the well-known fact that an MRA $(\phi, \{V_i\})$ in $L^2(\mathbb{R})$ for the dyadic dilation matrix $A = [2]$ has an essentially unique wavelet ψ_0. It is determined by the necessary and sufficient condition that

$$\hat{\psi}_0(\xi) = m_{\psi_0}(\xi/2)\hat{\phi}(\xi/2)$$

where the filter m_{ψ_0} for ψ_0 for the dilation $T = A$ is given by

$$m_{\psi_0} = \nu(\xi)e^{-i\xi}\overline{m_\phi(\xi + \pi)}$$

where $\nu(\xi)$ is an arbitrary 2π-periodic unimodular function in $L^\infty(\mathbb{R})$.

To see this note that the row matrix function

$$[m_\phi(\xi) \quad m_\phi(\xi + \pi)]$$

determined by the scaling function ϕ has a 2×2 unitary matrix completion of the form

$$\begin{bmatrix} m_\phi(\xi) & m_\phi(\xi + \pi) \\ m_{\psi_0}(\xi) & m_{\psi_0}(\xi + \pi) \end{bmatrix}$$

for every function $\nu(\xi)$ as above. Furthermore, every unitary completion in $M_2\big(L^2(2\pi\mathbb{T})\big)$ necessarily has this form. It follows from Lemma 2.7 that each such function ψ_0 has orthonormal translates forming a basis for the difference space $V_1 \ominus V_0$, and hence that ψ_0 is a wavelet. Conversely, if ψ' is a wavelet, then by Lemma 2.7 $\{\phi, \psi'\}$ necessarily has unitary filter matrix for $T = A$ and so ψ' necessarily is of the same form as ψ_0.

Consider now a dyadic biscaled wavelet ψ by which we mean a wavelet for a BMRA $(\phi, \mathfrak{B})$ in $L^2(\mathbb{R})$ for the dilation matrices

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

obtained from Theorem 3.4. More precisely, since $(\det A - 1)(\det B - 1) = 1$ the proof of Theorem 3.4 shows that there exists a wavelet set which is a singleton ψ whose filter function $m_\psi^{AB}(\xi)$ for $T = AB$ is associated with a filter bank $\{\phi, \psi^A, \psi^B, \psi\}$. This filter, $m_\psi^{AB}(\xi)$, arises from the step-wise unitary completion of the row matrix function

$$\Big[m_\phi^{AB}(\xi) \quad m_\phi^{AB}\big(\xi + \pi(1,0)\big) \quad m_\phi^{AB}\big(\xi + \pi(0,1)\big) \quad m_\phi^{AB}\big(\xi + \pi(1,1)\big)\Big].$$

By our previous remarks we can make explicit how $\psi^A(x)$ and $\psi^B(x)$ may be defined. Let us do this and recap the construction process for ψ.

Define a function ψ^A by specifying its filter $m_{\psi^A}^A(\xi)$ for $T = A$ to have the form

$$m_{\psi^A}^A(\xi) = e^{-i\xi_1}\overline{m_\phi^A\big(\xi + \pi(1,0)\big)},$$

and likewise define ψ^B via its filter function for $T = B$ given by

$$m_{\psi_B}^B(\xi) = e^{-i\xi_2}\overline{m_\phi^B\big(\xi + \pi(0,1)\big)}.$$

As we have already observed above, the 2×2-translate filter matrices for $\{\phi, \psi^A\}$ and $T = A$, and for $\{\phi, \psi^B\}$ and $T = B$, are unitary almost everywhere. It follows from Lemma 2.7 that ψ^A (resp. ψ^B) has orthonormal translates spanning $V_{1,0} \ominus V_{0,0}$ (resp. $V_{0,1} \ominus V_{0,0}$). Since these difference spaces are orthogonal, by the commuting lattice property of a BMRA subspace grid, we obtain, via Lemma 2.7, a 3×4 partial isometry valued filter matrix function $U_{\mathcal{F}}(\xi)$ for $T = AB$. It suffices to complete this to a 4×4 unitary-valued filter matrix in order to obtain an explicit filter function m_ψ^{AB} which then determines the desired wavelet ψ.

5.1 The Commuting Lattice Filter Relation

We now examine more directly the orthogonality of the rows of the 3×4 filter matrix for $T = AB$ and $\{\phi, \psi^A, \psi^B\}$.

For convenience we assume that the support of $\hat{\phi}$ contains $[-\pi, \pi]^2$. We are now regarding ψ^A as a function in $V_{1,1}$. This will necessarily have a filter function $m_{\psi^A}^{AB}(\xi)$ for $T = AB$ satisfying

$$\hat{\psi}^A(AB\xi) = m_{\psi^A}^{AB}(\xi)\hat{\phi}(\xi).$$

We have

$$\hat{\psi}^A\big(A(B\xi)\big) = m^A_{\psi^A}(B\xi)\hat{\phi}(B\xi)$$
$$= m^A_{\psi^A}(B\xi) m^B_\phi(\xi)\hat{\phi}(\xi),$$

and so, almost everywhere on the support of $\hat{\phi}$, we have

$$m^{AB}_{\psi^A}(\xi) = m^A_{\psi^A}(B\xi) m^B_\phi(\xi)$$
$$= e^{-i\xi_1}\overline{m^A_\phi(\xi_1+\pi, 2\xi_2)} m^B_\phi(\xi_1, \xi_2).$$

By the support assumption and the $2\pi\mathbb{Z}^2$ periodicity of the filters the identitity holds almost everywhere. Similarly,

$$m^{AB}_{\psi^B}(\xi) = e^{-i\xi_2}\overline{m^B_\phi(2\xi_1, \xi_2+\pi)} m^A_\phi(\xi_1, \xi_2).$$

To compactify notation we suppress ξ_1 and ξ_2 and set

$$A^{0,0}_{1,1} = m^A_\phi(\xi_1, \xi_2), \quad A^{\pi,0}_{1,1} = m^A_\phi(\xi_1+\pi, \xi_2),$$
$$A^{0,0}_{2,1} = m^A_\phi(2\xi_1, \xi_2), \quad A^{\pi,0}_{2,1} = m^A_\phi\big(2(\xi_1+\pi), \xi_2\big) = A^{0,0}_{2,1},$$
$$\overline{A}^{0,\pi}_{2,1} = \overline{m^A_\phi(2\xi_1, \xi_2+\pi)},$$

and so on, with $B^{0,0}_{1,1}, B^{\pi,0}_{1,1}, \dots$ similarly associated with $m^B_\phi(\xi_1, \xi_2)$. The 3×4 filter matrix for $\{\phi, \psi^A, \psi^B\}$ is the product of the diagonal matrix $D = \operatorname{diag}\{1, e^{-i\xi_1}, e^{-i\xi_2}\}$ and the 3×4 matrix

$$U = \begin{bmatrix} A^{0,0}_{1,2}B^{0,0}_{1,1} & A^{\pi,0}_{1,2}B^{\pi,0}_{1,1} & A^{0,0}_{1,2}B^{0,\pi}_{1,1} & A^{\pi,0}_{1,2}B^{\pi,pi}_{1,1} \\ \overline{A}^{\pi,0}_{1,2}B^{0,0}_{1,1} & -\overline{A}^{0,0}_{1,2}B^{\pi,0}_{1,1} & \overline{A}^{\pi,0}_{1,2}B^{0,\pi}_{1,1} & -\overline{A}^{0,0}_{1,2}B^{\pi,\pi}_{1,1} \\ A^{0,0}_{1,1}\overline{B}^{0,\pi}_{2,1} & A^{\pi,0}_{1,1}\overline{B}^{0,\pi}_{2,1} & -A^{0,\pi}_{1,1}\overline{B}^{0,0}_{2,1} & -A^{\pi,\pi}_{1,1}\overline{B}^{0,0}_{2,1} \end{bmatrix}.$$

The first row of U can be written in the alternate form

$$\begin{bmatrix} A^{0,0}_{1,1}B^{0,0}_{2,1} & A^{\pi,0}_{1,1}B^{0,0}_{2,1} & A^{0,\pi}_{1,1}B^{0,\pi}_{2,1} & A^{\pi,\pi}_{1,1}B^{0,\pi}_{2,1} \end{bmatrix}$$

in view of the intertwining relations

$$m^{AB}_\phi(\xi_1, \xi_2) = A^{0,0}_{1,2}B^{0,0}_{1,1,} = A^{0,0}_{1,1}B^{0,0}_{2,1}, \quad \textit{etc.}$$

We note that the unitarity almost everywhere of the filter matrices for $\{\phi, \psi^A\}$ and $\{\phi, \psi^B\}$ implies that almost everywhere

$$|A^{0,0}_{1,1}|^2 + |A^{\pi,0}_{1,1}|^2 = 1, \quad |B^{0,0}_{1,1}|^2 + |B^{0,\pi}_{1,1}|^2 = 1.$$

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Thus, the inner product of rows 1 and 2 of U is

$$A^{0,0}_{1,2}A^{\pi,0}_{1,2}|B^{0,0}_{1,1}|^2 - A^{\pi,0}_{1,2}A^{0,0}_{1,2}|B^{\pi,0}_{1,1}|^2 + A^{0,0}_{1,2}A^{\pi,0}_{1,2}|B^{0,\pi}_{1,1}|^2 - A^{\pi,0}_{1,2}A^{0,0}_{1,2}|B^{\pi,\pi}_{1,1}|^2$$
$$= A^{0,0}_{1,2}A^{\pi,0}_{1,2}(|B^{0,0}_{1,1}|^2 + |B^{0,\pi}_{1,1}|^2) - A^{\pi,0}_{1,2}A^{0,0}_{1,2}(|B^{\pi,0}_{1,1}|^2 + |B^{\pi,\pi}_{1,1}|^2) = 0.$$

Likewise, using the alternative form for row 1, the rows 1 and 3 of U are orthogonal.

To recap, we have a 3×4 partial isometry translate filter matrix arising from the $T = AB$ filters for ϕ, ψ^A, ψ^B, and

$$m^{AB}_{\psi^A}(\xi) = e^{-i\xi_1}\overline{m^A_\phi(\xi_1+\pi, 2\xi_2)}m^B_\phi(\xi_1,\xi_2) = e^{-i\xi_1}\overline{A}^{\pi,0}_{1,2}B^{0,0}_{1,1},$$
$$m^{AB}_{\psi^B}(\xi) = e^{-i\xi_2}m^A_\phi(\xi_1,\xi_2)\overline{m^B_\phi(2\xi_1,\xi_2+\pi)} = e^{-i\xi_2}A^{0,0}_{1,1}\overline{B}^{0,\pi}_{2,1}.$$

As we have remarked, ψ^A and ψ^B lie in $V_{1,0} \ominus V_{0,0}$ and $V_{0,1} \ominus V_{0,0}$ respectively. This is a consequence of the commuting lattice property of a BMRA and it is for this reason that rows 3 and 4 are orthogonal almost everywhere. Conversely this necessary condition (together with the previous row orthogonality) is sufficient for the commuting lattice property. Indeed, the orthogonality of $V_{1,0} \ominus V_{0,0}$ and $V_{0,1} \ominus V_{0,0}$ follows from this (via Lemma 2.7 yet again) and the commuting projection lattice property follows. The row orthogonality is the following formula, which is written in expanded form in the introduction.

$$A^{\pi,0}_{1,2}A^{0,0}_{1,1}\overline{B^{0,0}_{1,1}B^{0,\pi}_{2,1}} - A^{0,0}_{1,2}A^{\pi,0}_{1,1}\overline{B^{\pi,0}_{1,1}B^{0,\pi}_{2,1}}$$
$$- A^{\pi,0}_{1,2}A^{0,\pi}_{1,1}\overline{B^{0,\pi}_{1,1}B^{0,0}_{2,1}} + A^{0,0}_{1,2}A^{\pi,\pi}_{1,1}\overline{B^{\pi,\pi}_{1,1}B^{0,0}_{2,1}} = 0.$$

Remark 5.2 We have shown that for a dyadic BMRA scaling function ϕ there are two necessary conditions on the "marginal" filters $m^A_\phi(\xi)$ and $m^B_\phi(\xi)$, namely the intertwining condition and the commuting lattice (or orthogonality) condition above. In the final section we shall construct a function ϕ which satisfies these requirements and which defines a nonseparable BMRA. Evidently this nonseparable scaling function construction, *ab initio*, is considerably more complicated than that of constructing nonseparable wavelets, such as the Latin square wavelets, from a given (possibly separable) BMRA.

Remark 5.3 It would be interesting to pursue a "Riesz theory" of general not-necessarily-commuting lattices associated with scaling functions with, perhaps, Riesz basis translates. Going somewhat in this direction, we remark that the following is true (and the proof is rather delicate). For a "noncommuting-BMRA", that is a pair $(\phi, \mathcal{V})$ satisfying all the axioms for a BMRA except the commuting lattice axiom (v), the subspace grid is necessarily a lattice.

6 Impossibility of Compact Support for Nonseparable BMRA Scaling Functions

In this section we show that if the scaling function of a BMRA $(\phi, \mathfrak{V})$ in $L^2(\mathbb{R})$ is compactly supported then $\phi(x, y)$ is separable.

Lemma 6.1 *Let $a(\xi_1, \xi_2), b(\xi_1, \xi_2)$ be non-zero trigonometric polynomials with frequencies in $\mathbb{Z}^2$. Suppose that $\alpha, \beta \geq 2$ are integers and for all ξ_1, ξ_2,*

$$a(\xi_1, \beta\xi_2)b(\xi_1, \xi_2) = a(\xi_1, \xi_2)b(\alpha\xi_1, \xi_2). \tag{19}$$

Then $a(\xi_1, \xi_2) = a(\xi_1)$ and $b(\xi_1, \xi_2) = b(\xi_2)$ for some single variable trigonometric polynomials $a(\xi_1), b(\xi_2)$.

Proof Write

$$a(\xi_1, \xi_2) = \sum_{j=L_1^a}^{M_1^a} \sum_{k=L_2^a}^{M_2^a} a_{j,k} e^{i(j\xi_1 + k\xi_2)} \tag{20}$$

and

$$b(\xi_1, \xi_2) = \sum_{j=L_1^b}^{M_1^b} \sum_{k=L_2^b}^{M_2^b} b_{j,k} e^{i(j\xi_1 + k\xi_2)}, \tag{21}$$

where $Q_a := [L_1^a, M_1^a] \times [L_2^a, M_2^a]$ and $Q_b := [L_1^b, M_1^b] \times [L_2^b, M_2^b]$ are the minimal rectangles containing the support of the coefficients $a_{j,k}, b_{j,k}$ respectively. Also define $a_{j,k}, b_{j,k}$ to be zero outside their respective rectangles. For given p, q the (p, q)-th term of $a(\xi_1, \beta\xi_2)b(\xi_1, \xi_2)$ is

$$\sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} a_{p-j,k} b_{j,q-\beta k}, \tag{22}$$

while the (p, q)-th term of $a(\xi_1, \xi_2)b(\alpha\xi_1, \xi_2)$ is

$$\sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} a_{p-\alpha j,k} b_{j,q-k}. \tag{23}$$

Consider the $(p, \beta M_2^a + M_2^b)$-th coefficient of $a(\xi_1, \beta\xi_2)b(\xi_1, \xi_2)$. Since

$$a_{p-j,k} b_{j,q-\beta k} = a_{p-j,k} b_{j,\beta(M_2^a - k) + M_2^b}$$

this term is nonzero only if $k \leq M_2^a$ and $\beta(M_2^a - k) \leq 0$, that is, only if $k = M_2^a$. Thus the Fourier coefficient is simply

$$\sum_{j=-\infty}^{\infty} a_{p-j,M_2^a} b_{j,M_2^b}. \tag{24}$$

On the other hand, the $(p, \beta M_2^a + M_2^b)$-th term of $a(\xi_1, \xi_2)b(\alpha\xi_1, \xi_2)$ is

$$\sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} a_{p-\alpha j,k} b_{j,\beta M_2^a + M_2^b - k}. \tag{25}$$

There are nonzero terms in this sum only if $k \le M_2^a$ and $\beta M_2^a + M_2^b - k \le M_2^b$. Thus all terms, and the Fourier coefficient, are zero if $M_2^a \neq 0$.

Assume, by way of contradiction that this is the case, so that, by the assumed identity $a(\xi_1, \xi_2)b(\alpha\xi_1, \xi_2) = a(\xi_1, \beta\xi_2)b(\xi_1, \xi_2)$ we have

$$\sum_{j=\infty}^{\infty} a_{p-j,M_2^a} b_{j,M_2^b} = 0. \tag{26}$$

For the case $p = M_1^a + M_1^b$, equation (26) implies

$$a_{M_1^a,M_2^a} b_{M_1^b,M_2^b} = 0, \tag{27}$$

hence at least one of $a_{M_1^a,M_2^a}$, $b_{M_1^b,M_2^b}$ is zero. Define

$$z_a = \max_{j\in[L_1^a,M_1^a]} \{ j : a_{j,M_2^a} \neq 0 \}, \tag{28}$$

and

$$z_b = \max_{j\in[L_1^b,M_1^b]} \{ j : b_{j,M_2^b} \neq 0 \}. \tag{29}$$

Let $p = z_a + z_b$. We may now rewrite (26) as

$$\sum_{j=-\infty}^{z_b-1} a_{p-j,M_2^a} b_{j,M_2^b} + a_{z_a,M_1^a} b_{z_b,M_1^b} + \sum_{j=z_b+1}^{\infty} a_{p-j,M_2^a} b_{j,M_2^b} = 0. \tag{30}$$

For $j > z_b$, $b_{j,M_2^b} = 0$ by the definition of z_b. For $j < z_b$, $p - j > z_a$ so by the definition of z_a, $a_{p-j,M_2^a} = 0$. It then follows from (30) either a_{z_a,M_2^a} or b_{z_b,M_2^b} is zero, which is a contradiction, and so we must have $M_2^a = 0$.

An analogous argument to the one just given, beginning with consideration of $q = \beta L_2^a + L_2^b$, gives $L_2 = 0$ and so

$$a(\xi_1, \xi_2) = \sum_{j=L_1^a}^{M_1^a} a_{j,0} e^{ij\xi_1}. \tag{31}$$

Exchanging roles of the variables it follows that $b(\xi_1, \xi_2)$ is independent of ξ_1, as required. ■

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Theorem 6.2 *Let $(\phi, \mathfrak{B})$ be a BMRA with respect to dilation pair (A, B) with scaling function $\phi \in L^2(\mathbb{R}^2)$ of compact support. Then ϕ is separable.*

Proof Recall that for a BMRA $(\phi, \mathfrak{B})$ with respect to dilation pair (A, B), for (ξ_1, ξ_2) we have the fundamental intertwining relation

$$m_\phi^A(\xi_1, \beta\xi_2) m_\phi^B(\xi_1, \xi_2) = m_\phi^A(\xi_1, \xi_2) m_\phi^B(\alpha\xi_1, \xi_2).$$

If ϕ has compact support, then $\hat{\phi}(\xi)$ is non-vanishing almost everywhere. Furthermore $\phi(\alpha\xi_1, \xi_2)$ and $\phi(\xi_1, \beta\xi_2)$ are finite linear combinations of $\mathbb{Z}^2$-translates of ϕ and so the filters m_ϕ^A, m_ϕ^B are trigonometric polynomials. By Lemma 6.1, $m_\phi^A(\xi_1, \xi_2) = f(\xi_1)$ and $m_\phi^B(\xi_1, \xi_2) = g(\xi_2)$, where f and g are trigonometric polynomials in one variable. It is routine to check that $\phi_1(\xi_1) := \phi(\xi_1, 0)$, with filter $f(\xi_1)$ gives rise to a rank 1 univariate MRA with respect to dilation by α. Likewise $\phi_2(\xi_2) := \phi(0, \xi_2)$ with filter $g(\xi_2)$ gives rise to an MRA with respect to dilation by β.

Using the filter relation (14) $N + 1$ times gives

$$\hat{\phi}(\xi_1, \xi_2) = \Big(\prod_{n=0}^{N} f(\alpha^{-n}\xi_1) \Big) \hat{\phi}(\alpha^{-n}\xi_1, \xi_2).$$

As ϕ is compactly supported, $\hat{\phi}$ is continuous. Furthermore f is a trigonometric polynomial, hence Lipschitz, and so the product $\prod_{n=0}^{N} f(\alpha^{-n}\xi_1)$ converges almost uniformly, to $F(\xi_1)$, say. Hence

$$\hat{\phi}(\xi_1, \xi_2) = F(\xi_1)\hat{\phi}(0, \xi_2). \tag{32}$$

Similarly

$$\hat{\phi}(\xi_1, \xi_2) = G(\xi_2)\hat{\phi}(\xi_1, 0), \tag{33}$$

where $G(\xi_2) = \lim_{N\to\infty} \prod_{n=0}^{N} g(\beta^{-n}\xi_2)$. Hence we have $F(\xi_1) = \hat{\phi}(\xi_1, 0)$, $G(\xi_2) = \hat{\phi}(0, \xi_2)$ almost everywhere and so $\hat{\phi}(\xi_1, \xi_2) = \hat{\phi}_1(\xi_1)\hat{\phi}(\xi_2)$ almost everywhere, as required. ∎

7 Higher Rank Meyer BMRAs

In this section we construct a family of bidyadic BMRAs which include purely non-separable examples. The construction is a higher rank version of the well-known method used by Meyer to construct wavelets belonging to the Schwartz class. In particular, the Fourier transform of the scaling function and the resulting wavelet have compact support. In fact, the separable BMRAs obtained from the tensor product of two rank-1 dyadic Meyer type MRAs are included here as a special case. For our construction the scaling function and wavelet have discontinuous Fourier transforms; thus our wavelets do not lie in the Schwartz class, and it is not immediately obvious to what extent the decay may be improved.

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It is natural, by way of motivation and orientation, to recall the construction of Meyer wavelets, which we now do.

Suppose that $\phi \in L^2(\mathbb{R})$ is a unit vector with orthonormal translates which satisfies the scaling relation

$$\phi(x/2) = \sum_{k\in\mathbb{Z}} a_k \phi(x-k),$$

with convergence in $L^2(\mathbb{R})$, and is such that $\hat{\phi}(\xi)$ is continuous at 0, with $\hat{\phi}(0) \neq 0$. Then from ϕ and the scaling unitary for $A = [2]$ one obtains an MRA $(\phi, \mathfrak{V})$. This fact is well known [17, Theorem 2.13].

A scaling function ϕ of this type may be constructed by specifying its Fourier transform $\theta(\xi) = \hat{\phi}(\xi)$ to have the following three properties:

(i) $\sum_{l\in\mathbb{Z}} |\theta(\xi + 2\pi l)|^2 = \frac{1}{2\pi}$, almost everywhere. This condition is equivalent to the orthonormality of translates (see Lemma 2.6).

(ii) $\theta(2\xi) = \psi(\xi)\theta(\xi)$, for some 2π-periodic function $\psi(\xi)$. This is equivalent to the scaling relation above.

(iii) $\theta(\xi)$ is continuous at 0 with $\theta(0) \neq 0$.

To construct such a function $\theta(\xi)$ one may take the following route of Meyer and construct first a nonnegative function θ on $\mathbb{R}$ which is symmetric on $[-2\pi, 2\pi]$, with

$$\theta(\xi)^2 + \theta(\xi - 2\pi)^2 = \frac{1}{2\pi} \quad \text{on } [0, 2\pi],$$

$$\theta(\xi) = \frac{1}{\sqrt{2\pi}} \quad \text{for } |\xi| < \frac{2\pi}{3},$$

$$\theta(\xi) = 0 \quad \text{for } |\xi| > \frac{4\pi}{3}.$$

Thus (i) holds, with at most two nonzero summands for each ξ. Let $f(\xi)$ be the 2π-periodic extension of $\sqrt{2\pi}\theta(2\xi)$ for $\xi \in [-\pi, \pi]$. Then it follows that the scaling relation (ii) holds. If in addition $\theta(\xi)$ is continuous at 0 with $\theta(0) \neq 0$, then the construction is complete.

It is completely elementary to construct a function θ on $\mathbb{R}$ with the properties above. The main point in the construction is that, firstly, since $\theta(\xi) = \frac{1}{\sqrt{2\pi}}$, for $|\xi| \leq \frac{2\pi}{3}$, we have the scaling relation

$$f(\xi)\theta(\xi) = \sqrt{2\pi}\theta(2\xi)\frac{1}{\sqrt{2\pi}} = \theta(2\xi),$$

which holds in fact for the bigger range $|\xi| \leq \pi$ since $\theta(2\xi)$, and hence $f(\xi)$, are zero in the range $\frac{2\pi}{3} \leq |\xi| \leq \pi$. Thus, there is no obstacle to periodically extending $f(\xi)$ to a function on $\mathbb{R}$ and maintaining the scaling relation (ii).

We are going to follow a similar procedure to construct a bidyadic scaling function in $L^2(\mathbb{R}^2)$ which determines a multiresolution for $A = \left[\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, $B = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right]$. However, while one can readily construct a nonnegative function $\theta(\xi, \xi_2)$ on $[-2\pi, 2\pi]^2$ with

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the properties

$$\theta(\xi) = \frac{1}{2\pi} \quad \text{for } \xi \in \Big(-\frac{2\pi}{3}, \frac{2\pi}{3}\Big)^2$$
$$\theta(\xi) = 0 \quad \text{for } \xi \notin \Big[-\frac{4\pi}{3}, \frac{4\pi}{3}\Big]^2$$

and

$$\theta(\xi)^2 + \theta\big(\xi - 2\pi(1,0)\big)^2 + \theta\big(\xi - 2\pi(0,1)\big)^2 + \theta\big(\xi - 2\pi(1,1)\big)^2 = \frac{1}{4\pi^2},$$

there is no guarantee that one can periodically extend the functions $\psi^A(\xi), \psi^B(\xi)$ defined on the support of θ by

$$\psi^A(\xi) = \frac{\theta(2\xi_1, \xi_2)}{\theta(\xi_1, \xi_2)}, \quad \psi^B(\xi) = \frac{\theta(\xi_1, 2\xi_2)}{\theta(\xi_1, \xi_2)}.$$

Our first main task is to construct θ with extra structure so that this will be possible. This step is necessary because as we have seen in the general theory, if ϕ and A, B provide a BMRA then, by virtue of the subspace inclusions, ϕ will have periodic filter functions $m_\phi^A(\xi)$ and $m_\phi^B(\xi)$ satisfying the intertwining relation. In fact, we are arguing here in the reverse direction. We construct periodic extensions $\psi^A(\xi), \psi^B(\xi)$. These will be the filters for ϕ, since

$$\hat\phi(A\xi) = \theta(2\xi_1, \xi_2) = \psi^A(\xi)\theta(\xi) = \psi^A(\xi)\hat\phi(\xi),$$
$$\hat\phi(B\xi) = \theta(\xi_1, 2\xi_2) = \psi^B(\xi)\theta(\xi) = \psi^B(\xi)\hat\phi(\xi).$$

The intertwining condition follows from the equalities

$$\begin{aligned}\psi^A(\xi_1, 2\xi_2)\psi^B(\xi_1, \xi_2) &= \Big(\frac{\theta(2\xi_1, 2\xi_2)}{\theta(\xi_1, 2\xi_2)}\Big)\Big(\frac{\theta(\xi_1, 2\xi_2)}{\theta(\xi_1, \xi_2)}\Big)\\ &= \frac{\theta(2\xi)}{\theta(\xi)} = \Big(\frac{\theta(2\xi_1, \xi_2)}{\theta(\xi_1, \xi_2)}\Big)\Big(\frac{\theta(2\xi_1, 2\xi_2)}{\theta(2\xi_1, \xi_2)}\Big)\\ &= \psi^A(\xi_1, \xi_2)\psi^B(2\xi_1, \xi_2).\end{aligned}$$

However, such a condition does not yet guarantee the orthogonality structure of the commuting lattice property and we must construct θ with further structure to ensure this. We do this in Theorem 7.2; we consider a function $\phi \in L^2(\mathbb{R}^2)$ and identify sufficient conditions on its Fourier transform $\hat\phi$ that ensure that ϕ, A, B determine a BMRA.

Note first the following lemma. This follows from the rank one case (which is a basic fact; see [10], [17]), since every BMRA $(\phi, \mathfrak{B})$ contains the MRA $\{V_{i,i} : i \in \mathbb{Z}\}$.

Lemma 7.1 *Let $\phi \in L^2(\mathbb{R}^d)$ be such that $\{\phi(x-k) : k \in \mathbb{Z}^d\}$ is an orthonormal set in $L^2(\mathbb{R}^d)$ spanning the closed subspace $V_{0,0}$. Let $V_{i,j} = D_A^i D_B^j V_{0,0}$. Then $\bigcap_{(i,j)\in\mathbb{Z}^2} V_{i,j} = \{0\}$. If, moreover, $\hat\phi(0) \neq 0$ and $\hat\phi(\xi)$ is continuous at 0, then $\bigcup_{(i,j)\in\mathbb{Z}^2} V_{i,j}$ is dense in $L^2(\mathbb{R}^d)$.*

We introduce notation for four families of rectangles that lie in the big square $(-\frac{4\pi}{3}, \frac{4\pi}{3})^2$.

For $i, j, k \in \{0, 1\}$, let

$$I_{(i,j,k)} = \Big((-1)^i 2^{i+k} \frac{\pi}{3}, (-1)^i 2^{(1-i)+k} \frac{\pi}{3} \Big) \\ \times \Big((-1)^j 2^{j+(1-k)} \frac{\pi}{3}, (-1)^j 2^{(1-j)+(1-k)} \frac{\pi}{3} \Big),$$

and

$$J_{(i,j)} = \Big(-\frac{\pi}{3} + i(-1)^j \pi, \frac{\pi}{3} + i(-1)^j \pi \Big) \\ \times \Big(-\frac{\pi}{3} + (1-i)(-1)^j \pi, \frac{\pi}{3} + (1-i)(-1)^j \pi \Big).$$

These rectangles lie between the big square and the small square $(-\frac{2\pi}{3}, \frac{2\pi}{3})^2$. In particular $I_{0,0,0}$ is the rectangle which is the image of the north-east square $(\frac{2\pi}{3}, \frac{4\pi}{3})^2$ under A^{-1} and the rectangles $I_{i,j,k}$ have similar determinations. Note also that $J_{0,0}$ is the closure of the union of the disjoint rectangles $A^{-k}\big((\frac{2\pi}{3}, \frac{4\pi}{3})^2\big)$, $k = 1, 2, \dots,$.

Also for $i, j = 0, 1$ let

$$K_{i,j} = \Big(\frac{-4\pi}{3} + i(2\pi), \frac{-2\pi}{3} + i(2\pi) \Big) \times \Big(\frac{-4\pi}{3} + j(2\pi), \frac{-2\pi}{3} + j(2\pi) \Big),$$

$$L_{i,j} = \Big(\frac{-2\pi}{3} + i(\pi), \frac{-\pi}{3} + i(\pi) \Big) \times \Big(\frac{-2\pi}{3} + j(\pi), \frac{-\pi}{3} + j(\pi) \Big).$$

Thus $L_{0,0}$ lies in the south west corner of the small square.

In the following theorem, $\hat{\phi}$ is assumed to be supported on the big square. Condition (d) is a condition on the restriction of $\hat{\phi}$ to the four corner squares (translates of $(\frac{2\pi}{3}, \frac{4\pi}{3})^2$. Conditions (e) and (f) show how $\hat{\phi}$ is determined on the border rectangles by the values of $\hat{\phi}$ on pairs of corner squares. Condition (g) is an additional condition on the restrictions to corner squares.

Despite the detail in conditions (a)–(g) the construction of examples of such functions $\hat{\phi}$ is quite elementary. (Also there is further flexibility to arrange ϕ to be real-valued.) Indeed one may define $\hat{\phi}$ on the four corner squares to comply with (d) and (g), and then $\hat{\phi}$ is constructed (and uniquely determined) by the conditions (a), (b), (f). For example, in Figure 3 we show the regions of constancy of a function $\hat{\phi}$ which takes constant values on the triangular subsets of the corner squares. Condition (d) is elementary and (g) holds trivially with both products in (g) identically zero. It is evident, from the triangularity of support in the corners, that $\hat{\phi}$ and hence ϕ are not separable.

Theorem 7.2 *Let $\phi \in L^2(\mathbb{R}^2)$ satisfy the following properties:*

(a) *For $(\xi_1, \xi_2) \in \hat{\mathbb{R}}^2$ we have $0 \le \hat{\phi}(\xi_1, \xi_2) \le \frac{1}{2\pi}$.*

(b) *For $(\xi_1, \xi_2) \in (-\frac{2\pi}{3}, \frac{2\pi}{3})^2$ we have $\hat{\phi}(\xi_1, \xi_2) = \frac{1}{2\pi}$.*

(c) *For $(\xi_1, \xi_2) \notin (-\frac{4\pi}{3}, \frac{4\pi}{3})^2$ we have $\hat{\phi}(\xi_1, \xi_2) = 0$.*

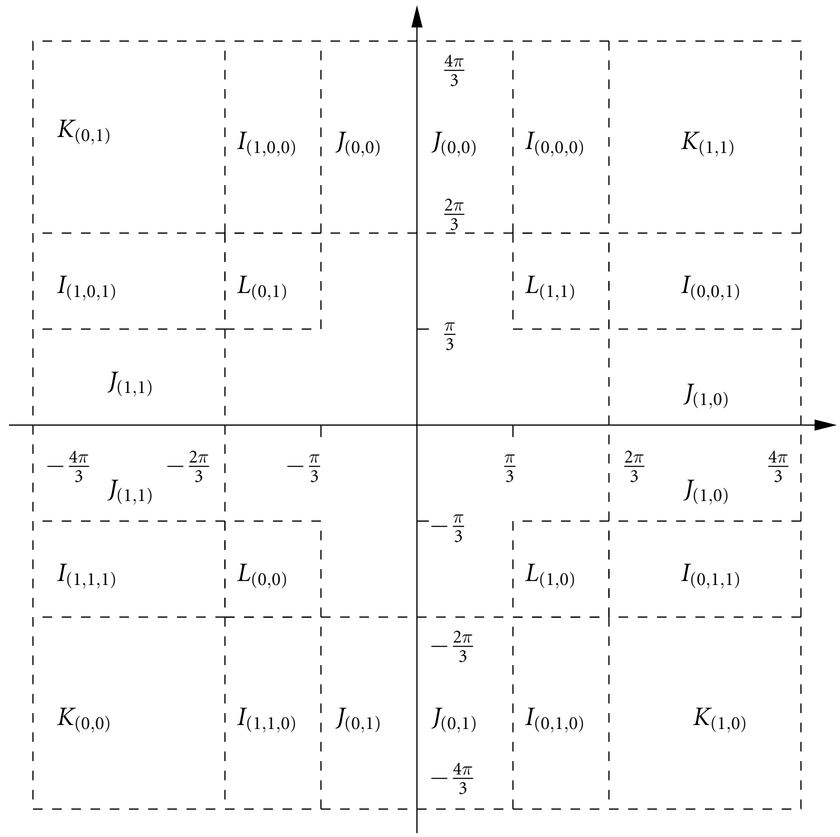

Figure 2: Border rectangles in the support of $\hat{\phi}$

(d) *For* $(\xi_1, \xi_2) \in (\frac{2\pi}{3}, \frac{4\pi}{3})^2$,

$$\sum_{i,j \in \{0,1\}} |\hat{\phi}(\xi_1 - 2\pi i, \xi_2 - 2\pi j)|^2 = \frac{1}{4\pi^2},$$

and the terms of this sum are nonzero.

(e) *For* $(\xi_1, \xi_2) \in I_{(i,j,k)}$ *and* $i, j, k \in \{0, 1\}$,

$$\hat{\phi}(\xi_1, \xi_2)^2 = \Big(\frac{1}{2\pi}\Big)^2 \frac{\theta_1(\xi_1, \xi_2)^2}{\theta_1(\xi_1, \xi_2)^2 + \theta_2(\xi_1, \xi_2)^2},$$

where

$$\theta_1(\xi_1, \xi_2) = \hat{\phi}(2^{(1-k)}\xi_1, 2^k \xi_2),$$
$$\theta_2(\xi_1, \xi_2) = \hat{\phi}\big(2^{(1-k)}\xi_1 - 2\pi(-1)^i k, 2^k \xi_2 - 2\pi(-1)^j (1 - k)\big)$$

(f) *For* $(\xi_1, \xi_2) \in J_{(i,j)}$ *and* $i, j \in \{0, 1\}$,

$$\hat{\phi}(\xi_1, \xi_2) = \hat{\phi}(2^{(1-i)}\xi_1, 2^i\xi_2).$$

(g) *For* $(\xi_1, \xi_2) \in (\frac{2\pi}{3}, \frac{4\pi}{3})^2$,

$$\hat{\phi}(\xi_1, \xi_2)\hat{\phi}(\xi_1 - 2\pi, \xi_2 - 2\pi)^2 - \hat{\phi}(\xi_1 - 2\pi, \xi_2)\hat{\phi}(\xi_1, \xi_2 - 2\pi)^2 = 0.$$

Let $V_{0,0}$ *be the closed subspace spanned by* $\{\phi(\xi_1 - k_1, \xi_2 - k_2) : k_1, k_2 \in \mathbb{Z}\}$, *let* $V_{i,j} = D_A^i D_B^j V_{0,0}$ *and let* $\mathfrak{B} = \{V_{i,j} : i, j \in \mathbb{Z}\}$. *Then* $(\phi, \mathfrak{B})$ *is a BMRA with respect to the dilation pair* A, B.

Proof We show that the conditions of Definition 2.3 hold. Let $V_{i,j}$ be as above. Then part (ii) of Definition 2.3 is automatically satisfied. By Lemma 7.1, parts (iii) and (iv) of Definition 2.3 will follow from (vi). Let

$$S(\xi_1, \xi_2) = \sum_{(k_1,k_2)\in\mathbb{Z}^2} |\hat{\phi}(\xi_1 - 2\pi k_1, \xi_2 - 2\pi k_2)|^2,$$

then (iv) will follow from Lemma 2.6 if we show that $S(\xi_1, \xi_2) = \frac{1}{4\pi^2}$ almost everywhere. It is immediate from (b) and (c) of the theorem that this is the case for $(\xi_1, \xi_2) \in (\frac{-2\pi}{3}, \frac{2\pi}{3})^2 + 2\pi(k_1, k_2)$. Likewise from (d) it follows for $(\xi_1, \xi_2) \in (\frac{2\pi}{3}, \frac{4\pi}{3})^2 + 2\pi(k_1, k_2)$. For $(\xi_1, \xi_2) \in I_{(0,0,0)} = (\frac{\pi}{3}, \frac{2\pi}{3}) \times (\frac{2\pi}{3}, \frac{4\pi}{3})$ we have, from (e),

$$\begin{aligned} S(\xi_1, \xi_2) &= |\hat{\phi}(\xi_1, \xi_2)|^2 + |\hat{\phi}(\xi_2, \xi_2 - 2\pi)|^2 \\ &= \frac{1}{4\pi^2} \frac{\hat{\phi}(2\xi_1, \xi_2)^2}{\hat{\phi}(2\xi_1, \xi_2)^2 + \hat{\phi}(2\xi_1, \xi_2 - 2\pi)^2} \\ &\quad + \frac{1}{4\pi^2} \frac{\hat{\phi}(2\xi_1, \xi_2 - 2\pi)^2}{\hat{\phi}(\xi_1, \xi_2)^2 + \hat{\phi}(\xi_1, \xi_2 - 2\pi)^2} \\ &= \frac{1}{4\pi^2}, \end{aligned}$$

as required. It is straightforward to carry out the preceding calculation for general $I_{(i,j,k)}$, $i, j, k \in \{0, 1\}$, and so we obtain

$$S(\xi_1, \xi_2) = \frac{1}{4\pi^2} \quad \text{for all } (\xi_1, \xi_2) \in \left\{ I_{(i,j,k)} + 2\mathbb{Z}^2 : i, j, k \in \{0, 1\} \right\}.$$

Now let $(\xi_1, \xi_2) \in (0, \frac{\pi}{3}) \times (\frac{2\pi}{3}, \frac{4\pi}{3})$. Then there exists $r \in \mathbb{N}$ such that $\frac{\pi}{3} \le 2^r\xi_1 \le \frac{2\pi}{3}$, and

$$S(\xi_1, \xi_2) = \hat{\phi}(2^r\xi_1, \xi_2)^2 + \hat{\phi}(2^r\xi_1, \xi_2 - 2\pi)^2 = \frac{1}{4\pi^2}.$$

Again, this calculation may be repeated to show that $S(\xi_1, \xi_2) = \frac{1}{4\pi^2}$ for all $(\xi_1, \xi_2) \in \{J_{(i,j)} + 2\pi\mathbb{Z}^2 : i, j \in \mathbb{Z}\}$.

Hence we have shown that $S(\xi_1,\xi_2) = \frac{1}{4\pi^2}$ almost everywhere, and so $(\phi, \mathfrak{B})$ satisfies (iii), (iv), and (vi) of Definition 2.3.

Next we show the BMRA inclusion property (i). From the definition of $V_{i,j}$ it suffices to show $V_{-1,0} \subset V_{0,0}$ and $V_{0,-1} \subset V_{0,0}$. We show the former. By Proposition 2.4 this is equivalent to the equation

$$\hat{\phi}(2\xi_1,\xi_2) = g(\xi_1,\xi_2)\hat{\phi}(\xi_1,\xi_2)$$

for some $2\pi\mathbb{Z}^2$ periodic function g with square summable restriction to $2\pi\mathbb{T}^2$. We show such a function exists. The equality holds trivially regardless of the value taken by g for $(\xi_1,\xi_2) \notin (\frac{-4\pi}{3}, \frac{4\pi}{3})^2$, hence if g may be constructed to be $2\pi\mathbb{Z}^2$ periodic on this square then its periodic extension will satisfy the equation everywhere.

For (ξ_1,ξ_2) such that $\frac{2\pi}{3} < |\xi_1| < \frac{4\pi}{3}$, we have $\hat{\phi}(2\xi_1,\xi_2) = 0$ but $\hat{\phi}(\xi_1,\xi_2) \neq 0$, so that we must take $g(\xi_1,\xi_2) = 0$; also $g(\xi_1,\xi_2) = g(\xi_1 - 2\pi,\xi_2) = 0$ for (ξ_1,ξ_2) such that $\frac{2\pi}{3} < \xi_1 < \frac{4\pi}{3}$, so periodicity is preserved. For $(\xi_1,\xi_2) \in (-\frac{2\pi}{3}, \frac{2\pi}{3})^2$ we have $\hat{\phi}(\xi_1,\xi_2) = \frac{1}{2\pi}$, so we set $g(\xi_1,\xi_2) = \hat{\phi}(2\xi_1,\xi_2)$. As

$$\left(\left(-\frac{2\pi}{3}, \frac{2\pi}{3}\right)^2 + 2\pi(k_1,k_2)\right) \cap \left(-\frac{4\pi}{3}, \frac{4\pi}{3}\right)^2 = \varnothing$$

if $(k_1,k_2) \neq (0,0)$, periodicity is maintained.

For the remaining $(\xi_1,\xi_2) \in (\frac{-4\pi}{3}, \frac{4\pi}{3})^2$, both $\hat{\phi}(\xi_1,\xi_2)$, and $\hat{\phi}(2\xi_1,\xi_2)$ are nonzero and we may define, as in the rank 1 case described earlier,

$$g(\xi_1,\xi_2) = \frac{\hat{\phi}(2\xi_1,\xi_2)}{\hat{\phi}(\xi_1,\xi_2)}.$$

Then for $(\xi_1,\xi_2) \in I_{(0,0,0)}$ we have

$$\begin{aligned}
g(\xi_1,\xi_2) &= \hat{\phi}(2\xi_1,\xi_2)\sqrt{\frac{\hat{\phi}(2\xi_1,\xi_2)^2 + \hat{\phi}(2\xi_1,\xi_2 - 2\pi)^2}{\hat{\phi}(2\xi_1,\xi_2)^2}} \\
&= \sqrt{\hat{\phi}(2\xi_1,\xi_2)^2 + \hat{\phi}(2\xi_1,\xi_2 - 2\pi)^2} \\
&= \hat{\phi}(2\xi_1,\xi_2 - 2\pi)\sqrt{\frac{\left(\hat{\phi}\big(2\xi_1,(\xi_2 - 2\pi) + 2\pi\big)\right)^2 + \big(\hat{\phi}(2\xi_1,\xi_2 - 2\pi)\big)^2}{\big(\hat{\phi}(2\xi_1,\xi_2 - 2\pi)\big)^2}} \\
&= \frac{\hat{\phi}(2\xi_1,\xi_2 - 2\pi)}{\hat{\phi}(\xi_1,\xi_2 - 2\pi)} = g(\xi_1,\xi_2 - 2\pi)
\end{aligned}$$

so that g is periodic on $I_{(0,0,0)} \cup I_{(0,1,0)}$. Again repeating a variation of this calculation, or by appealing to symmetry, we obtain periodicity for points in all $I_{(i,j,k)}$.

It remains to consider $(\xi_1,\xi_2) \in J_{(0,0)} \cup J_{(0,1)}$. Periodicity of g for such (ξ_1,ξ_2) follows from the recursive definition of $\hat{\phi}(\xi_1,\xi_2)$ on this interval and the periodicity of g on $I_{i,j,k}$. Hence g is periodic and $V_{-1,0} \subset V_{0,0}$. By symmetry it follows that $V_{0,-1} \subset V_{0,0}$, and the other inclusions in Definition 2.3 (i) are satisfied.

Finally we must show that the spaces $V_{i,j}$ form a commuting lattice. It is sufficient to show that $(V_{0,1} \ominus V_{0,0}) \perp (V_{1,0} \ominus V_{0,0})$. As we have discussed in Section 5.1, we require

$$e^{-i(\xi_1+\xi_2)} f(\xi_1, \xi_2) = 0, \tag{34}$$

for almost every $(\xi_1, \xi_2) \in \hat{\mathbb{R}}^2$, where

$$\begin{aligned} f(\xi_1, \xi_2) = {} & A^{\pi,0}_{1,2} B^{0,0}_{1,1} B^{0,\pi}_{2,1} A^{0,0}_{1,1} - A^{0,0}_{1,2} B^{\pi,0}_{1,1} B^{0,\pi}_{2,1} A^{\pi,0}_{1,1} \\ & + A^{\pi,0}_{1,2} B^{0,\pi}_{1,1} B^{0,0}_{2,1} A^{0,\pi}_{1,1} + A^{0,0}_{1,2} B^{\pi,\pi}_{1,1} B^{0,0}_{2,1} A^{\pi,\pi}_{1,1}. \end{aligned}$$

The factors here arise from the filters m^A_ϕ, m^B_ϕ of ϕ. Note that we have determined these filters as ratios, for example, m^A_ϕ is given explicitly by the function $g(\xi_1, \xi_2)$ above.

Our aim is to show that the unwieldy expression above is equivalent to condition (g) for a function ϕ satisfying conditions (a)–(f). To that end we first observe that if on some rectangle at least one term in each of the products present in f vanishes, then f vanishes identically in that rectangle.

For $(\xi_1, \xi_2) \in \{K_{i,j}\}_{i,j\in\{0,1\}}$, $A^{\pi,0}_{1,2} = A^{0,0}_{1,2} = B^{0,\pi}_{2,1} = B^{0,0}_{2,1} = 0$, hence at least one term in each of the products in $f(\xi_1, \xi_2)$ is zero and so $f(\xi_1, \xi_2) = 0$ for almost every $(\xi_1, \xi_2) \in \{K_{i,j}\}_{i,j=0,1}$. For (ξ_1, ξ_2) such that $\frac{2\pi}{3} < |\xi_2| < \frac{4\pi}{3}$, $A^{\pi,0}_{1,2} = A^{0,0}_{1,2} = 0$, and so again for almost all such (ξ_1, ξ_2) we have $f(\xi_1, \xi_2) = 0$. Likewise, for (ξ_1, ξ_2) such that $\frac{2\pi}{3} < |\xi_1| < \frac{4\pi}{3}$, $B^{0,\pi}_{2,1} = B^{0,0}_{2,1} = 0$ and so $f(\xi_1, \xi_2) = 0$ for such (ξ_1, ξ_2). Hence we may restrict our attention to $(\xi_1, \xi_2) \in (-\frac{2\pi}{3}, \frac{2\pi}{3})^2$. For (ξ_1, ξ_2) in the central square $(-\frac{\pi}{3}, \frac{\pi}{3})^2$, $A^{\pi,0}_{1,1} = A^{\pi,\pi}_{1,1} = B^{0,\pi}_{1,1} = B^{\pi,\pi}_{1,1} = A^{\pi,0}_{1,2} = B^{0,\pi}_{2,1} = 0$, and hence $f(\xi_1, \xi_2) = 0$. For (ξ_1, ξ_2) such that $\frac{\pi}{3} < |\xi_2| < \frac{2\pi}{3}$ and $|\xi_1| < \frac{\pi}{3}$, $A^{\pi,0}_{1,2} = A^{\pi,0}_{1,1} = A^{\pi,\pi}_{1,1} = 0$. Likewise for (ξ_1, ξ_2) with $\frac{\pi}{3} < |\xi_1| < \frac{2\pi}{3}$ and $|\xi_2| < \frac{\pi}{3}$, $B^{0,\pi}_{1,1} = B^{\pi,\pi}_{1,1} = B^{0,\pi}_{2,1} = 0$. Hence f vanishes for almost every (ξ_1, ξ_2) outside $L_{i,j\in\{0,1\}}$.

Before proceeding, we observe that, for almost every $(\xi_1, \xi_2) \in \mathbb{R}^2$

$$\begin{aligned} f(\xi_1 + \pi, \xi_2) &= A^{2\pi,0}_{1,2} B^{\pi,0}_{1,1} B^{\pi,\pi}_{2,1} A^{\pi,0}_{1,1} - A^{\pi,0}_{1,2} B^{2\pi,0}_{1,1} B^{\pi,\pi}_{2,1} A^{2\pi,0}_{1,1} \\ &\quad + A^{2\pi,0}_{1,2} B^{\pi,\pi}_{1,1} B^{\pi,0}_{2,1} A^{\pi,\pi}_{1,1} + A^{\pi,0}_{1,2} B^{2\pi,\pi}_{1,1} B^{\pi,0}_{2,1} A^{2\pi,\pi}_{1,1} \\ &= A^{0,0}_{1,2} B^{\pi,0}_{1,1} B^{0,\pi}_{2,1} A^{\pi,0}_{1,1} - A^{\pi,0}_{1,2} B^{0,0}_{1,1} B^{0,\pi}_{2,1} A^{0,0}_{1,1} \\ &\quad + A^{0,0}_{1,2} B^{\pi,\pi}_{1,1} B^{0,0}_{2,1} A^{\pi,\pi}_{1,1} + A^{\pi,0}_{1,2} B^{0,\pi}_{1,1} B^{0,0}_{2,1} A^{0,\pi}_{1,1} \\ &= -f(\xi_1, \xi_2), \end{aligned}$$

where we have used the $2\pi\mathbb{Z}^2$ periodicity of the filters (implicitly in $A^{0,\pi}_{1,2} = A^{0,0}_{1,2}$ and $B^{\pi,0}_{2,1} = B^{0,0}_{2,1}$). Similar calculations show that $f(\xi_1, \xi_2 + \pi) = -f(\xi - \pi, \xi_2)$ and $f(\xi_1, \xi_2) = f(\xi_1 + \pi, \xi_2 + \pi)$, that is, $e^{-(\xi_1+\xi_2)} f(\xi_1, \xi_2)$ is a $\pi\mathbb{Z}^2$-periodic function, and so it suffices to check that equation (34) holds on $L_{0,0}$ for it to hold on all of the $L_{i,j}$.

https://doi.org/10.4153/CJM-2011-012-1 Published online by Cambridge University Press

For $(\xi_1,\xi_2)\in L_{0,0}$, we have, by the definition of $A^{0,0}_{1,1}$;

$$A^{0,0}_{1,1}=\frac{\hat\phi(2\xi_1,\xi_2)}{\hat\phi(\xi_1,\xi_2)}=2\pi\hat\phi(2\xi_1,\xi_2) \tag{35}$$

$$=\frac{\hat\phi(2\xi_1,2\xi_2)}{\sqrt{\hat\phi(2\xi_1,2\xi_2)^2+\hat\phi(2\xi_1+2\pi,2\xi_2)^2}}. \tag{36}$$

At this point we introduce the notation

$$\Phi^{a,b}=\hat\phi(2\xi_1+a,2\xi_2+b)^2, \tag{37}$$

for $a,b=0,2\pi$ so that

$$A^{0,0}_{1,1}=\sqrt{\frac{\Phi^{0,0}}{\Phi^{0,0}+\Phi^{2\pi,0}}}. \tag{38}$$

The other scaled filters also simplify. For example

$$\begin{aligned}A^{\pi,0}_{1,2}=m_A(\xi_1+\pi,2\xi_2)=\frac{\hat\phi(2\xi_1+2\pi,2\xi_2)}{\hat\phi(\xi_1+\pi,2\xi_2)}&=\sqrt{\Phi^{2\pi,0}}\sqrt{\frac{\Phi^{2\pi,0}+\Phi^{2\pi,2\pi}}{\Phi^{2\pi,0}}}\\&=\sqrt{\Phi^{0,0}+\Phi^{2\pi,2\pi}}.\end{aligned}$$

In fact on $L_{0,0}$ all the scaled filters in the formula for $f(\xi_1,\xi_2)$ are expressible in terms of $\hat\phi$ evaluated at the four points $(2\xi_1,2\xi_2)+(i2\pi,j2\pi)$, $i,j\in\{0,1\}$. With the other filters similarly expressed this leads to the equality

$$\begin{aligned}f(\xi_1,\xi_2)=\Phi^{0,0}&\sqrt{\frac{(\Phi^{2\pi,0}+\Phi^{2\pi,2\pi})(\Phi^{0,2\pi}+\Phi^{2\pi,2\pi})}{(\Phi^{0,0}+\Phi^{2\pi,0})(\Phi^{0,0}+\Phi^{0,2\pi})}}\\
&-\Phi^{2\pi,0}\sqrt{\frac{(\Phi^{0,0}+\Phi^{0,2\pi})(\Phi^{0,2\pi}+\Phi^{2\pi,2\pi})}{(\Phi^{0,0}+\Phi^{2\pi,0})(\Phi^{2\pi,0}+\Phi^{2\pi,2\pi})}}\\
&-\Phi^{0,2\pi}\sqrt{\frac{(\Phi^{2\pi,0}+\Phi^{2\pi,2\pi})(\Phi^{0,0}+\Phi^{2\pi,0})}{(\Phi^{0,2\pi}+\Phi^{2\pi,2\pi})(\Phi^{0,0}+\Phi^{0,2\pi})}}\\
&+\Phi^{2\pi,2\pi}\sqrt{\frac{(\Phi^{0,0}+\Phi^{0,2\pi})(\Phi^{0,0}+\Phi^{2\pi,0})}{(\Phi^{0,2\pi}+\Phi^{2\pi,2\pi})(\Phi^{2\pi,0}+\Phi^{2\pi,2\pi})}}.\end{aligned}$$

We simplify the expression by taking out the factor

$$g(\xi_1,\xi_2):=\Big(\sqrt{(\Phi^{0,0}+\Phi^{0,2\pi})(\Phi^{0,0}+\Phi^{2\pi,0})(\Phi^{0,2\pi}+\Phi^{2\pi,2\pi})(\Phi^{2\pi,0}+\Phi^{2\pi,2\pi})}\Big)^{-1}.$$

https://doi.org/10.4153/CJM-2011-012-1 Published online by Cambridge University Press

By the definition of ϕ, the function g is nonvanishing almost every on $L_{0,0}$ and so condition (34) holds if and only if

$$0 = \Phi^{0,0}(\Phi^{2\pi,0} + \Phi^{2\pi,2\pi})(\Phi^{0,2\pi} + \Phi^{2\pi,2\pi}) - \Phi^{2\pi,0}(\Phi^{0,0} + \Phi^{0,2\pi})(\Phi^{0,2\pi} + \Phi^{2\pi,2\pi})$$
$$- \Phi^{0,2\pi}(\Phi^{2\pi,0} + \Phi^{2\pi,2\pi})(\Phi^{0,0} + \Phi^{2\pi,0}) + \Phi^{2\pi,2\pi}(\Phi^{0,0} + \Phi^{0,2\pi})(\Phi^{0,0} + \Phi^{2\pi,0}),$$

which, after some routine algebra, simplifies to

$$(\Phi^{0,0}\Phi^{2\pi,2\pi} - \Phi^{2\pi,0}\Phi^{0,2\pi})(\Phi^{0,0} + \Phi^{2\pi,0} + \Phi^{0,2\pi} + \Phi^{2\pi,2\pi}) = 0.$$

We observe that

$$(\Phi^{0,0} + \Phi^{2\pi,0} + \Phi^{0,2\pi} + \Phi^{2\pi,2\pi}) = \frac{1}{4\pi^2},$$

since on rewriting in terms of ϕ it coincides with condition (d). Thus (34) holds almost everywhere if and only if, for almost every $(\xi_1, \xi_2) \in K_{0,0}$,

$$\big(\hat{\phi}(\xi_1, \xi_2)\hat{\phi}(\xi_1 - 2\pi, \xi_2 - 2\pi)\big)^2 - \big(\hat{\phi}(\xi_1 - 2\pi, \xi_2)\hat{\phi}(\xi_1, \xi_2 - 2\pi)\big)^2 = 0,$$

as required. ■

In summary, we have the following theorem.

Theorem 7.3 *Let* $A = \left[\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, $B = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right]$. *Then, with respect to dilation by* (A, B), *there exists a nonseparable real valued wavelet* ψ, *associated with a nonseparable BMRA* $(\phi, \mathfrak{B})$ *in* $L^2(\mathbb{R}^2)$, *for which* $\hat{\psi}$ *has compact support.*

Example 7.4 We finish with some further examples. Let $\{\Xi_i\}_{i\in\{0,1\}}$ be a partition of $\{K_{i,j}\}_{i,j\in\{0,1\}}$ such that

(i) $\{\Xi_i\}_{i\in\{0,1\}}$ is invariant under the group $\mathcal{G}$ of translation by elements of $2\pi\mathbb{Z}^2$, when $\{K_{i,j}\}_{i,j\in\{0,1\}}$ is viewed as a subset of $\mathbb{R}^2/4\pi\mathbb{Z}^2$,

(ii) if $(\xi_1, \xi_2) \in \Xi_0$, then $(-\xi_1, -\xi_2) \in \Xi_1$.

Observe that if $\hat{\phi}$ is chosen to satisfy conditions (d) and (g) of Theorem 6.2 on Ξ_0, then the function defined by $\hat{\phi}(\xi_1, \xi_2) = \hat{\phi}(-\xi_1, -\xi_2)$ for $(\xi_1, \xi_2) \in \Xi_1$ satisfies (d) and (g) on the whole of $K_{0,0}$. Consider the simplest case, with $\hat{\phi}$ constant on each set $K_{i,j} \cap \Xi_0$ for $i, j \in \{0, 1\}$. Thus, set $\hat{\phi}(\xi_1, \xi_2) = \alpha$ for $(\xi_1, \xi_2) \in (K_{0,0} \cap \Xi_0)$, and $\hat{\phi}(\xi_1, \xi_2) = \delta$ for $(\xi_1, \xi_2) \in (K_{1,1} \cap \Xi_0)$ for some constants $\alpha, \delta > 0$ satisfying $\alpha^2 + \delta^2 < \frac{1}{8\pi^2}$. Define $\hat{\phi}(\xi_1, \xi_2) = \beta$ for $(\xi_1, \xi_2) \in (K_{1,0} \cap \Xi_0)$, where

$$\beta = \frac{1}{2}\Big(\frac{1}{4\pi^2} - \alpha^2 - \delta^2\Big) + \frac{1}{2}\sqrt{\Big(\frac{1}{4\pi^2} - \alpha - \delta\Big)^2 - 4\alpha\delta},$$

and $\hat{\phi}(\xi_1, \xi_2) = \gamma$ for $(\xi_1, \xi_2) \in (K_{0,1} \cap \Xi_0)$, where

$$\gamma = \frac{1}{4\pi^2} - \alpha - \beta - \delta.$$

We have now defined $\hat{\phi}$ on Ξ_0 and we define $\hat{\phi}(\xi_1, \xi_2) = \hat{\phi}(-\xi_1, -\xi_2)$ for $(\xi_1, \xi_2) \in \Xi_1$, as suggested above. Thus we have defined $\hat{\phi}$ to satisfy (d), (a) and (g) of Theorem 7.2, and so, as before, $\hat{\phi}$, and ϕ, are determined and yield a BMRA. Note that we have $\hat{\phi}(\xi_1, \xi_2) = \hat{\phi}(-\xi_1, -\xi_2)$, and so the scaling function is real valued.

Notice that $\hat{\phi}$ takes at most 4 values on each set $I_{(i,j,k)}$ and these values, together with the values α, β, γ, δ and $\frac{1}{2\pi}$ are the only values taken by $\hat{\phi}$. In particular, $\hat{\phi}$ is a finite linear combination of characteristic functions.

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