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A C^0-COLLOCATION-LIKE METHOD FOR ELLIPTIC EQUATIONS ON RECTANGULAR REGIONS

ZBIGNIEW LEYK[1]

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Abstract

We describe a C^0-collocation-like method for solving two-dimensional elliptic Dirichlet problems on rectangular regions, using tensor products of continuous piecewise polynomials. Nodes of the Lobatto quadrature formula are taken as the points of collocation. We show that the method is stable and convergent with order $h^r (r \geq 1)$ in the H^1–norm and $h^{r+1} (r \geq 2)$ in the L^2–norm, if the collocation solution is a piecewise polynomial of degree not greater than r with respect to each variable. The method has an advantage over the Galerkin procedure for the same space in that no integrals need be evaluated or approximated.

1. Introduction

In this paper we define and analyze a C^0-collocation-like method for elliptic Dirichlet problems with variable coefficients on a rectangular domain. The method uses tensor products of continuous piecewise-polynomial spaces. The collocation points are the nodes of the Lobatto quadrature formula.

The idea of a collocation method at Gaussian points was introduced and analyzed for two-point boundary value problems by de Boor and Swartz [4]. A C^1 finite-element collocation method was studied by Prenter and Russell [13] and Percell and Wheeler [12]. The well-known advantage of collocation finite-element methods over Galerkin finite-element methods is that the formation of the coefficients in the resulting system of equations is very fast since no integrals need be evaluated or approximated. But the smoothness of the approximate solution required by C^1- and C^2-collocation methods is higher than that required by the finite-element method for the same problems. In order to weaken the smoothness C^0-collocation methods have been considered, where the approximate solution is only continuous (see [5, 6, 7, 10, 11, 15]) and C^0-H^{-1} methods where the approximate solution is discontinuous (see [7]). The methods presented

[1]SMS, Australian National University, Canberra, ACT 0200, Australia

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in [5, 6, 7, 15] are combinations of the collocation methods with the finite-element method and in these methods one also has to evaluate or approximate integrals but fewer than is required in the finite-element method. The methods described in [10, 11] need not do any evaluation or approximation of integrals. The method presented in this paper is a nontrivial extension of the C^0-collocation-like method introduced in [10] to two-dimensional elliptic problems.

The outline of this paper is as follows. In Section 2 we formulate an elliptic Dirichlet problem in a weak form and describe a partition of a rectangle. The C^0-collocation-like method is defined in Section 3. In Section 4 the equivalent variational formulation of the method is introduced. The stability of the method and some auxiliary results are asserted in Section 5. The optimal order error estimates in the H^1 and L^2 norms are obtained in Section 6. The idea of the analysis used in Section 5 and 6 is taken from [2], but the analysis is substantially different from that used in [2] since the second partial derivatives must also be taken into consideration (not only the first partial derivatives as in the standard finite-element method). Therefore most of the auxiliary results are also new.

2. The problem and notation

The basic notation used in the paper is adopted from [2]. For any bounded open set Ω in two-dimensional space, the Sobolev space $W^{m,q}(\Omega)$ consists of functions f such that $\partial^\alpha f \in L^q(\Omega)$, $|\alpha| \le m$, with the norm

$$\|f\|_{m,q,\Omega} = \left(\sum_{|\alpha| \le m} \|\partial^\alpha f\|_{0,q,\Omega}^q \right)^{1/q},$$

where $\alpha = (\alpha_1, \alpha_2)$ are multi-integers, $\partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2}$, $\partial_1 = \partial/\partial x$, $\partial_2 = \partial/\partial y$ and $\|f\|_{0,q,\Omega}^q = \int_\Omega |f|^q \, dxdy$. We shall also use the seminorms $|\cdot|_{m,q,\Omega}$. If $q = 2$, we shall write $H^m(\Omega)$, $\|\cdot\|_{m,\Omega}$ and $|\cdot|_{m,\Omega}$ instead of $W^{m,2}(\Omega)$, $\|\cdot\|_{m,2,\Omega}$ and $|\cdot|_{m,2,\Omega}$, respectively. The Sobolev space $H^m(\Omega)$ is equipped with the inner product $(\cdot,\cdot)_{m,\Omega}$. $H_0^1(\Omega)$ is defined as the closure of the space $C_0^\infty(\Omega)$ in the sense of the norm $\|\cdot\|_{1,\Omega}$. The restriction of a function v to the set K is denoted by $v_{|K}$. Throughout the paper C (also with subscripts) will denote a generic constant with possibly different values in different contexts. The space of all polynomials in x (in y) of degree at most r restricted to the interval K_x (K_y) is denoted by $P_r^x(K_x)$ (by $P_r^y(K_y)$) and $R_r(K_x \times K_y) = P_r^x(K_x) \otimes P_r^y(K_y)$, $r \ge 0$. The space of all polynomials in x, y on $K \subset \mathbb{R}^2$ of degree at most r is denoted by $P_r(K)$.

We consider the Dirichlet problem

$$\begin{aligned} Lu &= f \quad \text{on } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

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where $\Omega = (0, l_1) \times (0, l_2)$ and

$$\begin{aligned} Lu(x, y) = &- \sum_{m,l=1}^{2} \partial_m \left(a_{ml}(x, y)\partial_l u(x, y)\right) \\ &+ \sum_{l=1}^{2} b_l(x, y)\partial_l u(x, y) + c(x, y)u(x, y). \end{aligned} \tag{2.2}$$

We can rewrite the problem (2.1) in a weak form.

Find $u \in H_0^1(\Omega)$ such that

$$a(u, v) = l(v), \quad \forall v \in H_0^1(\Omega), \tag{2.3}$$

where

$$a(u, v) = \int_\Omega \left(\sum_{m,l=1}^{2} a_{ml}\partial_l u \partial_m v + \sum_{l=1}^{2} b_l \partial_l u v + cuv \right) dxdy$$

and

$$l(v) = \int_\Omega f v \, dxdy.$$

We shall also consider the problem dual to (2.3).

Find $w \in H_0^1(\Omega)$ such that

$$a(v, w) = l(v), \quad \forall v \in H_0^1(\Omega). \tag{2.4}$$

Assume that the following conditions are satisfied.

Z1: $a_{ml}, b_l \in C^1(\Omega)$ for $m, l = 1, 2$ and $c, f \in C(\Omega)$.
Z2: There exists a constant $a_0 > 0$ such that for each $(\xi_1, \xi_2, \eta) \in \mathbb{R}^3$ and $(x, y) \in \Omega$

$$\sum_{m,l=1}^{2} a_{ml}(x, y)\xi_m \xi_l + \sum_{l=1}^{2} b_l(x, y)\xi_l \eta + c(x, y)\eta^2 \geq a_0 \sum_{l=1}^{2} \xi_l^2.$$

From Z2 it follows that the coefficient matrix $\{a_{ml}(x, y)\}$ must be positive definite and $c(x, y) \geq 0$ for any $(x, y) \in \Omega$.

Under the conditions Z1 and Z2 there exist unique solutions to the problems (2.3) and (2.4) (see [2]). Moreover , the solutions u and w belong to $H^2(\Omega) \cap H_0^1(\Omega)$ and satisfy

$$\|u\|_{2,\Omega} \leq C_1 \|f\|_{0,\Omega}, \qquad \|w\|_{2,\Omega} \leq C_2 \|f\|_{0,\Omega}, \tag{2.5}$$

where C_1, C_2 are constants independent of u, w, f (see [9]).

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We divide the interval $[0, l_1]$ into subintervals $e_i^x = [x_i, x_{i+1}]$, $i = 0, 1, \dots, N_1 - 1$, where

$$0 = x_0 < x_1 < \cdots < x_{N_1-1} < x_{N_1} = l_1.$$

The interval $[0, l_2]$ is divided analogously into $e_j^y = [y_j, y_{j+1}]$. Write $e_{ij} = e_i^x \times e_j^y$.

Let $h_{1,i} = (x_{i+1} - x_i)$, $h_{2,j} = (y_{j+1} - y_j)$ and

$$h = \max_{k=1,2}\left(\max_{0\le i\le N_k-1} h_{k,i}\right), \qquad h_{\min} = \min_{k=1,2}\left(\min_{0\le i\le N_k-1} h_{k,i}\right).$$

We assume that the family of partitions is regular, that is, there exists a constant $\sigma > 0$ such that $h/h_{\min} \le \sigma$, if $h \to 0$.

For $f \in W^{m,q}(e_{ij})$, we adopt the notation

$$|||f|||_{m,q} = \left(\sum_{i=0}^{N_1-1}\sum_{j=0}^{N_2-1} \|f\|_{m,q,e_{ij}}^q\right)^{1/q}.$$

If $q = 2$, we shall write $|||f|||_m$ instead of $|||f|||_{m,2}$.

Let $V_r = V_r^x \otimes V_r^y$, where V_r^x and V_r^y are finite-element spaces of the form

$$V_r^x = \{v \in C([0, l_1]) : v_{|e_i^x} \in P_r^x(e_i^x),\ i = 0, \dots, N_1 - 1,\ v(0) = v(l_1) = 0\}$$

and

$$V_r^y = \{v \in C([0, l_2]) : v_{|e_j^y} \in P_r^y(e_j^y),\ j = 0, \dots, N_2 - 1,\ v(0) = v(l_2) = 0\}.$$

3. The C^0-collocation-like method

The Lobatto points $\{t_i\}_{i=1}^{r-1}$ on $(0, 1)$ are the roots of the orthogonal Jacobi polynomial $J_{r-1}^{(1,1)}(t) = C_r P_r'(t)$, $t \in [0, 1]$, where P_r is the Legendre polynomial of degree r, $P_r : [0, 1] \longrightarrow \mathbb{R}$ and C_r is a constant. Let $t_0 = 0$, $t_r = 1$.

Introduce the affine mappings of the form

$$F_i^x(t) = x_i + h_{1,i}t, \qquad F_j^y(t) = y_j + h_{2,j}t, \quad t \in [0, 1].$$

Denote by $(x_{ik}, y_{jl}) = (F_i^x(t_k), F_j^y(t_l))$, $k, l = 0, \dots, r$, the Lobatto points on the element e_{ij}. Note that $x_{i-1,r} = x_{i0} = x_i$, $y_{j-1,r} = y_{j0} = y_j$. Further, by

$$[g]_x(x, y) = g(x-, y) - g(x+, y)$$

and

$$[g]_y(x, y) = g(x, y-) - g(x, y+)$$

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we denote the discontinuity jump of g at (x, y) in the x and y axes respectively. The weighted averages of g at a point (x, y) are denoted by

$$\{g\}_x(\alpha)(x, y) = \alpha g(x-, y) + (1-\alpha) g(x+, y)$$

and

$$\{g\}_y(\beta)(x, y) = \beta g(x, y-) + (1-\beta) g(x, y+),$$

where $\alpha, \beta \in [0, 1]$. Set

$$D_m g(x, y) = \sum_{l=1}^{2} a_{ml}(x, y)\partial_l g(x, y), \quad m = 1, 2, \ (x, y) \in \Omega,$$

the conormal derivative of g associated with the operator L.

The C^0-*collocation-like method* is defined as follows.

Find $U \in V_r$ such that

$$LU(x_{ik}, y_{jl}) = f(x_{ik}, y_{jl}), \tag{3.1}$$

$$w\hbar_{1,i}^{-1}[\![D_1 U]\!]_x(x_i, y_{jl}) + \{LU\}_x(\alpha_i)(x_i, y_{jl}) = f(x_i, y_{jl}), \tag{3.2}$$

$$w\hbar_{2,j}^{-1}[\![D_2 U]\!]_y(x_{ik}, y_j) + \{LU\}_y(\beta_j)(x_{ik}, y_j) = f(x_{ik}, y_j), \tag{3.3}$$

$$\begin{aligned} &w\hbar_{1,i}^{-1}\{[\![D_1 U]\!]_x\}_y(\beta_j)(x_i, y_j) + w\hbar_{2,j}^{-1}\{[\![D_2 U]\!]_y\}_x(\alpha_i)(x_i, y_j) \\ &\quad + \{\{LU\}_x(\alpha_i)\}_y(\beta_j)(x_i, y_j) = f(x_i, y_j), \end{aligned} \tag{3.4}$$

where $1 \le k, l \le r-1$ and $i = 0, \dots, N_1 - 1$, $j = 0, \dots, N_2 - 1$ for (3.1)–(3.3) and $i = 1, \dots, N_1 - 1$, $j = 1, \dots, N_2 - 1$ for (3.4). Moreover, $w = r^2 + r$, $\hbar_{1,i} = h_{1,i-1} + h_{1,i}$, $\hbar_{2,j} = h_{2,j-1} + h_{2,j}$, $\alpha_i = h_{1,i-1}/\hbar_{1,i}$, $\beta_j = h_{2,j-1}/\hbar_{2,j}$.

REMARK. If $r = 1$, then Equations (3.1)–(3.3) vanish. If $h_{1,i} = h_1$ and $h_{2,j} = h_2$ for any i and j then $\hbar_{1,i} = 2h_1$ and $\hbar_{2,j} = 2h_2$ and $\alpha_i = \beta_j = 0.5$.

In the Appendix we present another form of the system (3.1)–(3.4).

4. The variational form

We want to give an equivalent formulation of the method (3.1)–(3.4) in a variational form in order to simplify its investigation. The variational formulation uses a discrete inner product based on the Lobatto points. In this way we are able to prove existence and uniqueness as well as obtain optimal rates of convergence for the method (3.1)–(3.4). The variational formulation is not good for practical computations. Instead, the system (3.1)–(3.4) should be used.

Let $I(g) = \int_0^1 g(t)\,dt$ and denote by $Q(g)$ the Lobatto quadrature formula on $[0, 1]$ (see [3]), that is, $Q(g) = \sum_{j=0}^{r} w_j g(t_j)$. Note that $w_0 = w_r = (r^2 + r)^{-1}$. The following relation is satisfied (see [3]):

$$Q(g) = I(g), \qquad \forall g \in P_{2r-1}([0, 1]). \tag{4.1}$$

Let $T(g) = g(1-) - g(0+)$ and $I_i^x(g) = h_{1,i} I(\tilde{g})$, $Q_i^x(g) = h_{1,i} Q(\tilde{g})$, $T_i^x(g) = T(\tilde{g})$, where $\tilde{g}(t) = g(F_i^x(t))$, $t \in [0, 1]$. Analogously we define $I_j^y(g)$, $Q_j^y(g)$, $T_j^y(g)$. Moreover, we define the tensor product of any two of the above functionals as follows

$$\left(G_1^x \otimes G_2^y\right)(f) = G_1^x(G_2^y(f)),$$

where $f = f(x, y)$ and G_1 and G_2 stands for I, Q or T with indices i or j.

We transform the integrals $\int_{e_{ij}} D_k w \partial_k v \, dx dy$ using Green's formula.

For $k = 1$

$$\begin{aligned}\int_{e_{ij}} D_1 w \partial_1 v \, dx dy &= \int_{e_j^y} ((D_1 w v)(x_{i+1}-, t) - (D_1 w v)(x_i+, t))\, dt - \int_{e_{ij}} \partial_1 D_1 w v \, dx dy \\ &= (T_i^x \otimes I_j^y)(D_1 w v) - (I_i^x \otimes I_j^y)(\partial_1 D_1 w v)\end{aligned}$$

and for $k = 2$

$$\int_{e_{ij}} D_2 w \partial_2 v \, dx dy = (I_i^x \otimes T_j^y)(D_2 w v) - (I_i^x \otimes I_j^y)(\partial_2 D_2 w v).$$

We recall that $D_k w$, $k = 1, 2$, is the co-normal derivative of w associated with L. We can now rewrite the bilinear form $a(w, v)$ as

$$a(w, v) = \sum_{i=0}^{N_1-1} \sum_{j=0}^{N_2-1} (T_i^x \otimes I_j^y)(D_1 w v) + (I_i^x \otimes T_j^y)(D_2 w v) + (I_i^x \otimes I_j^y)(L w v). \tag{4.2}$$

Substituting in (4.2) the quadratures Q_i^x and Q_j^y for the integrals I_i^x and I_j^y respectively, we obtain the bilinear form $a_h(w, v)$, that is,

$$a_h(w, v) = \sum_{i=0}^{N_1-1} \sum_{j=0}^{N_2-1} (T_i^x \otimes Q_j^y)(D_1 w v) + (Q_i^x \otimes T_j^y)(D_2 w v) + (Q_i^x \otimes Q_j^y)(L w v). \tag{4.3}$$

Similarly

$$l_h(v) = \sum_{i=0}^{N_1-1} \sum_{j=0}^{N_2-1} (Q_i^x \otimes Q_j^y)(f v). \tag{4.4}$$

The problem

find $U \in V_r$ such that

$$a_h(U, V) = l_h(V), \quad \forall V \in V_r, \tag{4.5}$$

is equivalent to (3.1)–(3.4). To show this, we choose $\{\phi^x_{ik}\phi^y_{jl}\}$ as the basis functions of V_r satisfying

$$\phi^x_{ik} \in V^x_r, \quad \phi^y_{jl} \in V^y_r, \tag{4.6}$$

$$\phi^x_{ik}(x_{mn}) = \delta_{im}\delta_{kn}, \tag{4.7}$$

$$\phi^y_{jl}(y_{mn}) = \delta_{jm}\delta_{ln}, \tag{4.8}$$

where x_{mn} and y_{mn} are grid nodes on x and y axis respectively and δ_{im} denotes the Kronecker delta. To get (3.1)–(3.4) one should substitute the functions $\phi^x_{ik}\phi^y_{jl}$ for V in (4.5), use (4.7), (4.8) and reduce these expressions.

5. Stability of the method

In this section we prove that the method is stable. The main result of this section is that the bilinear form $a_h(W, V)$ is uniformly V_r-elliptic for sufficiently small h.

THEOREM 1. *Let $a_{ml} \in W^{2,\infty}(e_{ij})$, $b_l, c \in C(e_{ij})$ for $i = 0, 1, \dots, N_1 - 1$, $j = 0, 1, \dots, N_2 - 1$, where $l, m = 1, 2$, and let Condition Z2 be satisfied. Then there exist constants $\gamma \geq 0$ and $h_0 \geq 0$ such that*

$$\gamma \|V\|^2_{1,\Omega} \leq a_h(V, V), \quad \forall V \in V_r, \tag{5.1}$$

for any $h \leq h_0$.

Stability, and hence existence and uniqueness of the solution to (3.1)–(3.4), is a consequence of Theorem 1.

COROLLARY 1. *Let $f \in C(e_{ij})$ for $i = 0, \dots, N_1 - 1$, $j = 0, \dots, N_2 - 1$, and the assumptions of Theorem 1 be satisfied. Then for any $h \leq h_0$*

$$\|U\|_{1,\Omega} \leq C\|f\|_{0,\infty,\Omega},$$

where U is the unique solution to (3.1)–(3.4) and h_0 is given in Theorem 1.

Let $\tilde{K} = [0, 1] \times [0, 1]$. We introduce the quadrature error functionals on e_{ij}

$$E_{ij}(f) = I^x_i \otimes I^y_j(f) - Q^x_i \otimes Q^y_j(f),$$

where $i = 0, \dots, N_1 - 1$, $j = 0, \dots, N_2 - 1$. To prove Theorem 1 we need several auxiliary lemmas. Here s, k and q are some parameters and n is a degree of a polynomial p.

LEMMA 1. *Let $2/q < s \leq 2r - n$, $q \in [1, \infty]$, $0 \leq k \leq n$. Assume that $f \in W^{s+k,q}(e_{ij})$. Then for any $p \in R_n(e_{ij})$*

$$|(Q_i^x - I_i^x) \otimes T_j^y(fp)| \leq Ch^{s+k-(2/q)} \|f\|_{s+k,q,e_{ij}} \|p\|_{k,e_{ij}}, \tag{5.2}$$

$$|(Q_i^x - I_i^x) \otimes S_j^y(fp)| \leq Ch^{s+k+1-(2/q)} \|f\|_{s+k,q,e_{ij}} \|p\|_{k,e_{ij}}, \tag{5.3}$$

$$|T_i^x \otimes (Q_j^y - I_j^y)(fp)| \leq Ch^{s+k-(2/q)} \|f\|_{s+k,q,e_{ij}} \|p\|_{k,e_{ij}}, \tag{5.4}$$

$$|S_i^x \otimes (Q_j^y - I_j^y)(fp)| \leq Ch^{s+k+1-(2/q)} \|f\|_{s+k,q,e_{ij}} \|p\|_{k,e_{ij}}, \tag{5.5}$$

$$|E_{ij}(fp)| \leq Ch^{s+k+1-(2/q)} \|f\|_{s+k,q,e_{ij}} \|p\|_{k,e_{ij}}, \tag{5.6}$$

where S stands for Q or I and if $k = 0$, then the norm $\|f\|_{s+k,q,e_{ij}}$ is replaced by the seminorm $|f|_{s+k,q,e_{ij}}$.

PROOF. First we shall prove inequality (5.2). We have

$$(Q_i^x - I_i^x) \otimes T_j^y(fp) = h_{1,i}(Q - I) \otimes T(\tilde{f}\tilde{p}), \tag{5.7}$$

where $\tilde{f}(\xi_1, \xi_2) = f(F_i^x(\xi_1), F_j^y(\xi_2))$ and $\tilde{p}(\xi_1, \xi_2) = p(F_i^x(\xi_1), F_j^y(\xi_2))$ for $(\xi_1, \xi_2) \in \tilde{K}$.

I. We prove (5.2) for $k = 0$. Let $G(\tilde{f}) = (Q - I) \otimes T(\tilde{f}\tilde{p})$. The functional G is linear with respect to $\tilde{f} \in W^{s,q}(\tilde{K})$ and continuous for $s > 2/q$, ($q \in [1, \infty]$) since

$$|G(\tilde{f})| \leq C\|\tilde{f}\|_{0,\infty,\tilde{K}} \|\tilde{p}\|_{0,\infty,\tilde{K}} \leq C\|\tilde{f}\|_{s,q,\tilde{K}} \|\tilde{p}\|_{0,\tilde{K}}.$$

In the above, the equivalence of norms in the finite-dimensional space $R_n(\tilde{K})$ and the inclusion $W^{s,q}(\tilde{K}) \subset L^\infty(\tilde{K})$ ($s > 2/q$) have been used (see [2]).

Since $(I - Q) \otimes T(g) = 0$ for $g \in R_{s+n-1}(\tilde{K}) \subset R_{2r-1}(\tilde{K})$ (see (4.1)), the functional G vanishes over the space $P_{s-1}(\tilde{K}) \subset R_{s-1}(\tilde{K})$. By the Bramble-Hilbert Lemma (see [1])

$$|G(\tilde{f})| \leq C|\tilde{f}|_{s,q,\tilde{K}} \|\tilde{p}\|_{0,\tilde{K}}. \tag{5.8}$$

Hence

$$\begin{aligned} |(Q_i^x - I_i^x) \otimes T_j^y(fp)| = h_{1,i}|(Q - I) \otimes T(\tilde{f}\tilde{p})| &\leq Ch|\tilde{f}|_{s,q,\tilde{K}} \|\tilde{p}\|_{0,\tilde{K}} \\ &\leq Ch^{s-(2/q)}|f|_{s,q,e_{ij}} \|p\|_{0,e_{ij}}, \end{aligned}$$

where $2/q < s \leq 2r - n$, $q \in [1, \infty]$, $0 \leq n$.

II. We prove (5.2) for $k \geq 1$ using the method of mathematical induction with respect to k. Note that the lemma is valid for $k = 0$ (see Part I of the proof). Let the lemma be true for $k = l \geq 0$. We prove that it is true for $k = l + 1$. Let $p_{ij} \in P_1(e_{ij})$, if $l = 0$, and $p_{ij} \in P_l(e_{ij})$, if $l \geq 1$, be an interpolatory polynomial of p, that is,

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$p_{ij}(z_{ijm}) = p(z_{ijm})$, where z_{ijm} are different points from e_{ij} with $m = 1, 2, 3$, if $l = 0$, and with $m = 1, \dots, l(l+1)/2$, if $l \geq 1$. We have

$$(Q_i^x - I_i^x) \otimes T_j^y(fp) = (Q_i^x - I_i^x) \otimes T_j^y(fp_{ij}) + (Q_i^x - I_i^x) \otimes T_j^y(f(p - p_{ij})). \quad (5.9)$$

We now estimate the first term of the right-hand side of (5.9) using (5.2) with $k = l$, $n = l$. For $2/q < s \leq 2r - l$, $q \in [1, \infty]$

$$|(Q_i^x - I_i^x) \otimes T_j^y(fp_{ij})| \leq Ch^{s+l-(2/q)} \|f\|_{s+l,q,e_{ij}} \|p\|_{l,e_{ij}}. \quad (5.10)$$

We have used the following inequalities: for $l = 0, 1$

$$\|p_{ij}\|_{l,e_{ij}} \leq \|p - p_{ij}\|_{l,e_{ij}} + \|p\|_{l,e_{ij}} \leq Ch^{2-l}|p|_{2,e_{ij}} + \|p\|_{l,e_{ij}} \leq C\|p\|_{l,e_{ij}},$$

since $|p|_{2,e_{ij}} \leq Ch^{l-2}|p|_{l,e_{ij}}$ (see [2]); for $l \geq 2$

$$\|p_{ij}\|_{l,e_{ij}} \leq \|p - p_{ij}\|_{l,e_{ij}} + \|p\|_{l,e_{ij}} \leq C\|p\|_{l,e_{ij}}.$$

Replacing s by $s + 1$ in (5.10), we get

$$|(Q_i^x - I_i^x) \otimes T_j^y(fp_{ij})| \leq Ch^{s+l+1-(2/q)} \|f\|_{s+l+1,q,e_{ij}} \|p\|_{l,e_{ij}} \quad (5.11)$$

for $-1 + 2/q < s \leq 2r - l - 1$. Note that (5.11) is also valid for $2/q < s \leq 2r - n$, if $l + 1 \leq n < 2r - 2/q$. We estimate the second term of the right-hand side of (5.9) using (5.2) with $k = l$, $2/q < s \leq 2r - n$, $q \in [1, \infty]$, $l \leq n$. We have

$$\begin{aligned} |(Q_i^x - I_i^x) \otimes T_j^y(f(p - p_{ij}))| &\leq Ch^{s+l-(2/q)} \|f\|_{s+l,q,e_{ij}} \|p - p_{ij}\|_{l,e_{ij}} \\ &\leq Ch^{s+l+1-(2/q)} \|f\|_{s+l,q,e_{ij}} |p|_{l+1,e_{ij}}. \end{aligned} \quad (5.12)$$

We have used the inequalities

$$\|p - p_{ij}\|_{0,e_{ij}} \leq Ch^2|p|_{2,e_{ij}}$$

and

$$|p|_{2,e_{ij}} \leq Ch^{-1}|p|_{1,e_{ij}},$$

for $l = 0$ (see [2]). Combining (5.9) with (5.11) and (5.12) yields (5.2) with $k = l + 1$. Hence (5.2) is valid for any $0 \leq k < 2r - 2/q$. Analogously we estimate (5.4).

Similarly we get bounds on (5.3) and (5.5). Note that now we have one h more (compare with (5.7)), since

$$S^x \otimes (Q^y - I^y)(fp) = h_{1,i}h_{2,j}S \otimes (Q - I)(\tilde{f}\tilde{p})$$

and

$$(Q_i^x - I_i^x) \otimes S_j^y(fp) = h_{1,i}h_{2,j}(Q - I) \otimes S(\tilde{f}\tilde{p}),$$

where S stands for Q or I. Combining (5.3) and (5.5) with the inequality

$$|E_{ij}(fp)| \le |Q_i^x \otimes (Q_j^y - I_j^y)(fp)| + |(Q_i^x - I_i^x) \otimes I_j^y)(fp)|$$

gives (5.6).

The following lemma will be useful in showing the equivalence of norms in V_r.

LEMMA 2. *There exists a constant $C_0 > 0$ independent of h such that for any $V \in V_r$*

$$\left(\sum_{i=0}^{N_1-1}\sum_{j=0}^{N_2-1} Q_i^x \otimes Q_j^y \left(\sum_{k=1}^{2}(\partial_k V)^2\right)\right)^{1/2} \ge C_0 \|V\|_{1,\Omega}.$$

PROOF. We have

$$\begin{aligned} Q_i^x \otimes Q_j^y \left(\sum_{k=1}^{2}(\partial_k V)^2\right) &= h_{1,i}h_{2,j} Q \otimes Q \left(h_{1,i}^{-2}(\partial_1 \tilde{V})^2 + h_{2,j}^{-2}(\partial_2 \tilde{V})^2\right) \\ &\ge \sigma^{-1} Q \otimes Q \left(\sum_{k=1}^{2}(\partial_k \tilde{V})^2\right), \end{aligned}$$

where $\tilde{V}(\xi_1, \xi_2) = V(F_i^x(\xi_1), F_j^y(\xi_2))$ for $(\xi_1, \xi_2) \in \tilde{K}$.

The mapping $\tilde{V} \to \left(Q \otimes Q \left(\sum_{k=1}^{2}(\partial_k \tilde{V})^2\right)\right)^{1/2}$ defines a norm in the quotient space $R_r(\tilde{K})/P_0(\tilde{K})$. Since the mapping $\tilde{V} \to |\tilde{V}|_{1,\tilde{K}}$ is also a norm in this finite-dimensional space, there exists a positive constant C_1 such that

$$\left(Q \otimes Q\left(\sum_{k=1}^{2}(\partial_k \tilde{V})^2\right)\right)^{1/2} \ge C_1 |\tilde{V}|_{1,\tilde{K}}, \quad \forall \tilde{V} \in R_r(\tilde{K}).$$

Hence, for any $V \in V_r$

$$Q_i^x \otimes Q_j^y \left(\sum_{k=1}^{2}(\partial_k V)^2\right) \ge \sigma^{-1} C_1^2 |\tilde{V}|_{1,\tilde{K}}^2 \ge C|V|_{1,e_{ij}}^2$$

and

$$\sum_{i=0}^{N_1-1}\sum_{j=0}^{N_2-1} Q_i^x \otimes Q_j^y \left(\sum_{k=1}^{2}(\partial_k V)^2\right) \ge C|V|_{1,\Omega}^2.$$

To complete the proof, notice that the seminorm $|\cdot|_{1,\Omega}$ is a norm in the space $H_0^1(\Omega)$ equivalent to the norm $\|\cdot\|_{1,\Omega}$.

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LEMMA 3. *Let* $a_{ml} \in W^{2,\infty}(e_{ij})$ *for* $i = 0, 1, \ldots, N_1 - 1$, $j = 0, 1, \ldots, N_2 - 1$, *where* $m, l = 1, 2$. *Then for any* $V \in V_r$

$$\left| \sum_{i=0}^{N_1-1} \sum_{j=0}^{N_2-1} (I_i^x - Q_i^x) \otimes Q_j^y \left(\partial_1 (D_1 V V)\right) \right| \le C_1 h \|V\|_{1,\Omega}^2, \tag{5.13}$$

and

$$\left| \sum_{i=0}^{N_1-1} \sum_{j=0}^{N_2-1} Q_i^x \otimes (I_j^y - Q_j^y) \left(\partial_2 (D_2 V V)\right) \right| \le C_2 h \|V\|_{1,\Omega}^2. \tag{5.14}$$

PROOF. We first prove (5.13). Since $W^{2,\infty}(e_{ij}) \subset C^1(e_{ij})$, we can define $a_{1l}^0 = a_{1l}(x_i, y_j)$, $l = 1, 2$, for fixed i and j. Furthermore,

$$\|a_{1l} - a_{1l}^0\|_{0,\infty,e_{ij}} \le Ch|a_{1l}|_{1,\infty,e_{ij}} \tag{5.15}$$

(see [2]). By virtue of (4.1) it follows that

$$(I_i^x - Q_i^x) \otimes Q_j^y (a_{1l}^0 \partial_1 W V) = 0,$$

for any $W, V \in R_r(e_{ij})$. Thus, we can write

$$(I_i^x - Q_i^x) \otimes Q_j^y \big(\partial_1 (a_{1l} \partial_l V V)\big) = J_1 + J_2 + J_3, \tag{5.16}$$

where

$$\begin{aligned} J_1 &= (I_i^x - Q_i^x) \otimes Q_j^y (\partial_1 a_{1l} \partial_l V V), \\ J_2 &= (I_i^x - Q_i^x) \otimes Q_j^y \left((a_{1l} - a_{1l}^0) \partial_1 \partial_l V V\right), \\ J_3 &= (I_i^x - Q_i^x) \otimes Q_j^y \left((a_{1l} - a_{1l}^0) \partial_l V \partial_1 V\right). \end{aligned}$$

In the sequel, we shall frequently use the result

$$|V|_{m,q,e_{ij}} \le Ch^{l-m} h^{(2/q)-1} |V|_{l,e_{ij}}, \tag{5.17}$$

where $V \in R_n(e_{ij})$, $q \in [1, \infty]$, $0 \le l \le m$, $n \ge 0$ and the constant C is independent of h, see [2, p. 140]. Also we make use of the inequality

$$|fw|_{m,q,e_{ij}} \le C \sum_{l=0}^{m} |f|_{l,\infty,e_{ij}} |w|_{m-l,q,e_{ij}}, \tag{5.18}$$

where $f \in W^{m,\infty}(e_{ij})$, $w \in W^{m,q}(e_{ij})$, $m \ge 0$, $q \in [1, \infty]$ and C is independent of h (see [2, p. 192]).

Using (5.17), (5.18) and (5.3) with $f = (\partial_1 a_{1l})V$, $p = \partial_l V$, $s = 1$, $k = 0$, $n = r$, $q = \infty$, we estimate the first term of the right-hand side of (5.16) as

$$|J_1| \le Ch^2 |\partial_1 a_{1l} V|_{1,\infty,e_{ij}} \|\partial_l V\|_{0,e_{ij}} \le Ch\|a_{1l}\|_{2,\infty,e_{ij}} \|V\|^2_{1,e_{ij}}. \tag{5.19}$$

We now get a bound on the second term of the right-hand side of (5.16) using (5.15), (5.17), (5.18) and (5.3) with $f = (a_{1l} - a^0_{1l})V$, $p = \partial_1\partial_l V$, $s = 2$, $k = 0$, $q = \infty$, where $n = r$, $r \ge 2$, if $l = 1$, or $n = r - 1$, $r \ge 1$, if $l = 2$. This yields

$$\begin{aligned}
|J_2| &\le Ch^3 |(a_{1l} - {a_{1l}}^0)V|_{2,\infty,e_{ij}} \|\partial_1\partial_l V\|_{0,e_{ij}} \\
&\le Ch^3 \left(|a_{1l} - a^0_{1l}|_{0,\infty,e_{ij}} |V|_{2,\infty,e_{ij}} + |a_{1l}|_{1,\infty,e_{ij}} |V|_{1,\infty,e_{ij}} + |a_{1l}|_{2,\infty,e_{ij}} |V|_{0,\infty,e_{ij}}\right) |V|_{2,e_{ij}} \\
&\le Ch^3 \left(Ch|a_{1l}|_{1,\infty,e_{ij}} Ch^{-1}|V|_{1,\infty,e_{ij}} + \|a_{1l}\|_{2,\infty,e_{ij}} \|V\|_{1,\infty,e_{ij}}\right) Ch^{-1}|V|_{1,e_{ij}} \\
&\le Ch^2 \|a_{1l}\|_{2,\infty,e_{ij}} \|V\|^2_{1,e_{ij}}.
\end{aligned} \tag{5.20}$$

Note that if $r = 1$, $l = 1$, the inequality (5.20) is valid, since in this case $\partial_1^2 V = 0$ and hence $J_2 = 0$.

Finally, we estimate the third term of the right-hand side of (5.16) using (5.15), (5.18), (5.17) and (5.3) with $f = (a_{1l} - a^0_{1l})\partial_l V$, $p = \partial_1 V$, $s = 1$, $k = 0$, $n = r$ and $q = \infty$. The result is

$$\begin{aligned}
|J_3| &\le Ch^2 \left(|a_{1l} - {a_{1l}}^0|_{0,\infty,e_{ij}} \|\partial_l V\|_{1,\infty,e_{ij}} + |a_{1l}|_{1,\infty,e_{ij}} |\partial_l V|_{0,\infty,e_{ij}}\right) \|\partial_1 V\|_{0,e_{ij}} \\
&\le Ch^2 \left(Ch|a_{1l}|_{1,\infty,e_{ij}} Ch^{-1}|V|_{1,\infty,e_{ij}} + |a_{1l}|_{1,\infty,e_{ij}} |V|_{1,\infty,e_{ij}}\right) |V|_{1,e_{ij}} \\
&\le Ch^2 |a_{1l}|_{1,\infty,e_{ij}} |V|^2_{1,e_{ij}}.
\end{aligned} \tag{5.21}$$

Using the definition of D_1 and combining (5.16) with (5.19), (5.20) and (5.21), we derive (5.13) with C_1 independent of h (but dependent on a_{1l}, $l = 1, 2$). Analogously we prove (5.14).

We are now in a position to prove Theorem 1.

PROOF OF THEOREM 1. We shall use the equalities $T^x_i(f) = I^x_i(\partial_1 f)$, $T^y_j(f) = I^y_j(\partial_2 f)$ and $\partial_m(D_m V)V = \partial_m((D_m V)V) - D_m V \partial_m V$ for $m = 1, 2$.

The bilinear form $a_h(V, V)$ (see (4.3)) can be rewritten as

$$\begin{aligned}
a_h(V, V) = \sum_{i=0}^{N_1-1}\sum_{j=0}^{N_2-1} \Bigg(& I^x_i \otimes Q^y_j(\partial_1(D_1 V V)) \\
& + Q^x_i \otimes I^y_j(\partial_2(D_2 V V)) - Q^x_i \otimes Q^y_j(\partial_1(D_1 V V)) - Q^x_i \otimes Q^y_j(\partial_2(D_2 V V)) \\
& + Q^x_i \otimes Q^y_j(D_1 V \partial_1 V + D_2 V \partial_2 V) + Q^x_i \otimes Q^y_j\left(\sum_{l=1}^{2} b_l \partial_l V V + cV^2\right)\Bigg) \\
= J_1 + J_2 + J_3, &
\end{aligned} \tag{5.22}$$

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where

$$J_1 = \sum_{i=0}^{N_1-1} \sum_{j=0}^{N_2-1} (I_i^x - Q_i^x) \otimes Q_j^y (\partial_1 (D_1 V V)),$$

$$J_2 = \sum_{i=0}^{N_1-1} \sum_{j=0}^{N_2-1} Q_i^x \otimes (I_j^y - Q_j^y)(\partial_2 (D_2 V V)),$$

$$J_3 = \sum_{i=0}^{N_1-1} \sum_{j=0}^{N_2-1} Q_i^x \otimes Q_j^y \left(\sum_{m,l=1}^{2} a_{ml} \partial_l V \partial_m V + \sum_{l=1}^{2} b_l \partial_l V V + c V^2 \right).$$

Using condition Z2 and Lemma 2 gives

$$J_3 \geq \sum_{i=0}^{N_1-1} \sum_{j=0}^{N_2-1} a_0 Q_i^x \otimes Q_j^y \left(\sum_{l=1}^{2} (\partial_l V)^2 \right) \geq a_0 C_0 \| V \|_{1,\Omega}^2 = C_3 \| V \|_{1,\Omega}^2. \quad (5.23)$$

To obtain (5.1), it suffices to use the equality (5.22) and the inequalities (5.13), (5.14) and (5.23). Indeed, taking $h_0 = 0.5 C_3 (C_1 + C_2)^{-1}$, we get for $h \leq h_0$

$$\begin{aligned} a_h(V, V) &\geq -C_1 h \| V \|_{1,\Omega}^2 - C_2 h \| V \|_{1,\Omega}^2 + C_3 \| V \|_{1,\Omega}^2 \\ &\geq (-(C_1 + C_2) h_0 + C_3) \, \| V \|_{1,\Omega}^2 \geq 0.5 C_3 \| V \|_{1,\Omega}^2. \end{aligned}$$

REMARK. If $a_{1l}(x, y)$ and $a_{2l}(x, y)$, $l = 1, 2$, are functions independent of x and y respectively, then (5.1) is valid for any $h \geq 0$, since J_1 and J_2 vanish (as a consequence of (4.1)).

6. Error estimates

In this section we prove that the C^0-collocation method has the optimal rate of convergence in $H^1(\Omega)$ and $L^2(\Omega)$ norms for sufficiently small h.

THEOREM 2. *Assume that $f \in W^{s,q}(e_{ij})$, $u \in H^{s+2}(e_{ij}) \cap H^2(\Omega) \cap H_0^1(\Omega)$ for $i = 0, \ldots, N_1 - 1$, $j = 0, \ldots, N_2 - 1$, where $2/q < s \leq r$, $r \geq 1$, $q \geq 2$. Let the assumptions of Theorem 1 be satisfied.*

Then for any $h \leq h_0$

$$\| u - U \|_{1,\Omega} \leq C h^s \left(|\!|\!| u |\!|\!|_{s+2} + |\!|\!| f |\!|\!|_{s,q} \right),$$

where u and U are the solutions to (2.3) and (3.1)–(3.4) respectively, f is as in (2.1) and h_0 is given in Theorem 1.

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To get an error estimate in the L^2-norm, we apply the inequality

$$\|u-U\|_{0,\Omega} \le \sup_{g\in L^2(\Omega)} \|g\|_{0,\Omega}^{-1}\Big(\inf_{V\in V_r} \big(C\|u-U\|_{1,\Omega}\|w-V\|_{1,\Omega} + |a(U,V)-a_h(U,V)| + |l(V)-l_h(V)|\big)\Big) \tag{6.1}$$

(see [2, p. 203]), where w is the solution to (2.4) with $l(w) = \int_\Omega gv\,dx\,dy$, and u and U are the solutions to (2.3) and (3.1)–(3.4) respectively.

THEOREM 3. *Assume that* $f \in W^{s+1,q}(e_{ij})$, $u \in H^{\tilde{s}}(e_{ij}) \cap H^2(\Omega) \cap H_0^1(\Omega)$ *for* $i = 0,\dots,N_1-1$, $j = 0,\dots,N_2-1$, *where* $2/q < s \le r$, $\tilde{s} = \max(s+2,4)$, $r \ge 2$, $q \ge 2$. *Let the assumptions of Theorem 1 be satisfied.*

Then for any $h \le h_0$

$$\|u-U\|_{0,\Omega} \le Ch^{s+1}\left(\|u\|_{\tilde{s}} + \|f\|_{s+1,q}\right),$$

where u, U *are the solutions to (2.3) and (3.1)–(3.4) respectively.*

To prove Theorems 2 and 3 some lemmas are required.

LEMMA 4. *Let* $a_{ml} \in W^{s+k,\infty}(e_{ij})$, $b_l, c \in W^{s+k-1,\infty}(e_{ij})$ *for* $i = 0,\dots,N_1-1$, $j = 0,\dots,N_2-1$, *where* $m, l = 1,2$ *and* $1 \le s,k \le r$. *Then for any* $W, V \in V_r$

$$|a(W,V)-a_h(W,V)| \le Ch^{s+k-1}\|W\|_{s+k+1}\|V\|_k, \tag{6.2}$$

$$|a(W,V)-a_h(W,V)| \le Ch^{s+k-1}\|V\|_{s+k}\|W\|_{k+1}. \tag{6.3}$$

PROOF. We can write (see (4.2) and (4.3))

$$a(W,V)-a_h(W,V) = \sum_{i=0}^{N_1-1}\sum_{j=0}^{N_2-1}\left(\sum_{l=1}^{2}\left(J_{1,l}^{(i,j)}+J_{2,l}^{(i,j)}\right) + \sum_{m,l=1}^{2}\left(J_{3,m,l}^{(i,j)}+J_{4,m,l}^{(i,j)}\right) + \sum_{l=1}^{2} J_{5,l}^{(i,j)} + J_6^{(i,j)}\right), \tag{6.4}$$

where $J_{1,l}^{(i,j)} = T_i^x \otimes (I_j^y - Q_j^y)(a_{1l}\partial_1 WV)$, $J_{2,l}^{(i,j)} = (I_i^x - Q_i^x)\otimes T_j^y(a_{2l}\partial_l WV)$, $J_3^{(i,j)} = E_{ij}(-\partial_m a_{ml}\partial_l WV)$, $J_{4,m,l}^{(i,j)} = E_{ij}(-a_{ml}\partial_m\partial_l WV)$, $J_{5,l}^{(i,j)} = E_{ij}(b_l\partial_l WV)$, $J_6^{(i,j)} = E_{ij}(cWV)$. We estimate each term of the above equality for fixed i, j using (5.17), (5.18) and Lemma 1 with $n = r$, $q = \infty$. Thus, from (5.4) it follows that

$$|J_{1,l}^{(i,j)}| \le Ch^{s+k}\|a_{1l}\partial_l W\|_{s+k,\infty,e_{ij}}\|V\|_{k,e_{ij}} \le Ch^{s+k-1}\|a_{1l}\|_{s+k,\infty,e_{ij}}\|W\|_{s+k+1,e_{ij}}\|V\|_{k,e_{ij}}$$

for $l = 1, 2$, and

$$|J_{1,l}^{(i,j)}| \leq Ch^{s+k} \|a_{1l}V\|_{s+k,\infty,e_{ij}} \|\partial_l W\|_{k,e_{ij}} \leq Ch^{s+k-1} \|a_{1l}\|_{s+k,\infty,e_{ij}} \|V\|_{s+k} \|W\|_{k+1,e_{ij}}$$

for $l = 1, 2$, where $1 \leq s \leq r$, $r \geq k$.

Similarly we estimate $J_{2,l}^{(i,j)}$. Further on, using (5.17), (5.18) and (5.6) with $n = r$, $q = \infty$ and $k - 1$ instead of k we obtain

$$\begin{aligned} |J_{3,m,l}^{(i,j)}| &\leq Ch^{s+k} \|\partial_m a_{1l} \partial_l W\|_{s+k-1,\infty,e_{ij}} \|V\|_{k-1,e_{ij}} \\ &\leq Ch^{s+k-1} \|a_{ml}\|_{s+k,\infty,e_{ij}} \|W\|_{s+k,e_{ij}} \|V\|_{k-1,e_{ij}}, \quad m, l = 1, 2 \end{aligned}$$

and

$$\begin{aligned} |J_{3,m,l}^{(i,j)}| &\leq Ch^{s+k} \|\partial_m a_{ml} V\|_{s+k-1,\infty,e_{ij}} \|\partial_l W\|_{k-1,e_{ij}} \\ &\leq Ch^{s+k-1} \|a_{ml}\|_{s+k,\infty,e_{ij}} \|V\|_{s+k-1,e_{ij}} \|W\|_{k,e_{ij}}, \quad m, l = 1, 2, \end{aligned}$$

where $1 \leq s \leq r$, $r \geq k - 1 \geq 0$.

Similarly we estimate $J_{4,m,l}^{(i,j)}$, $J_{5,l}^{(i,j)}$ and $J_6^{(i,j)}$, $m, l = 1, 2$. To complete the proof of (6.2) combine (6.4) with the above estimates using the Cauchy-Schwarz inequality.

Analogously we prove (6.3).

LEMMA 5. *Let* $u \in H^{\tilde{s}}(e_{ij}) \cap H^2(\Omega) \cap H_0^1(\Omega)$ *for* $i = 0, \ldots, N_1 - 1$, $j = 0, \ldots, N_2 - 1$, *where* $0 \leq s \leq r + 3$ *and* $\tilde{s} = \max(s, 2)$. *Then*

$$|||U_I|||_s \leq C |||u|||_{\tilde{s}},$$

where $U_I \in V_r$ *is a* V_r*-interpolant of the function* u.

PROOF. Observe that $H^2(\Omega) \subset C(\Omega)$, so U_I is well defined. We first prove the inequality ($s = r + l$)

$$|U_I|_{r+l,e_{ij}} \leq c|u|_{r+l,e_{ij}} \tag{6.5}$$

for fixed i, j, where C is a constant independent of h and $l = 2, 3$. Since $U_{I|e_{ij}} \in R_r(e_{ij})$, then

$$\begin{aligned} |U_I|^2_{r+l,e_{ij}} &= \int_{e_{ij}} \sum_{k=l}^{r} (\partial_1^k \partial_2^{r+l-k} U_I)^2 \, dx dy \\ &\leq 2 \int_{e_{ij}} \sum_{k=l}^{r} (\partial_1^k \partial_2^{r+l-k} (u - U_I))^2 \, dx dy + 2 \int_{e_{ij}} \sum_{k=l}^{r} (\partial_1^k \partial_2^{r+l-k} u)^2 \, dx dy. \end{aligned} \tag{6.6}$$

Let $G_k(u) = \int_{e_{ij}} \partial_1^k \partial_2^{r+l-k}(u - U_l) f_k \, dx dy$, where $f_k \in L^2(e_{ij})$, $k = l, \dots, r$, and $\tilde{G}_k(\tilde{u}) = \int_{\tilde{K}} \partial_1^k \partial_2^{r+l-k}(\tilde{u} - \tilde{U}_l) \tilde{f}_k \, d\xi_1 d\xi_2$, where $\tilde{w}(\xi_1, \xi_2) = w(F_i^x(\xi_1), F_j^y(\xi_2))$. The symbol w stands for u, U_l or f_k, $k = l, \dots, r$.

We have the estimate $|G_k(u)| \le Ch^{-r-l+2}|\tilde{G}_k(\tilde{u})|$. The functional $\tilde{G}_k$ is linear with respect to $\tilde{u} \in H^{r+l}(\tilde{K})$ and continuous, since

$$|\tilde{G}_k(\tilde{u})| \le (\|\tilde{u}\|_{r+l,\tilde{K}} + C\|\tilde{u}\|_{0,\infty,\tilde{K}})\|\tilde{f}\|_{0,\tilde{K}} \le C_k \|\tilde{u}\|_{r+l,\tilde{K}} \|\tilde{f}_k\|_{0,\tilde{K}}.$$

Above we have used the inequality $\|\tilde{U}_l\|_{r+l,\tilde{K}} \le C\|\tilde{u}\|_{0,\infty,\tilde{K}} \le C\|\tilde{u}\|_{r+l,\tilde{K}}$ and the inclusion $H^{r+l}(\tilde{K}) \subset C(\tilde{K})$.

Now we show that the functional $\tilde{G}_k$ vanishes over the space $P_{r+l-1}(\tilde{K})$. If $\tilde{u}(\xi_1, \xi_2) = \xi_1^m \xi_2^n$ and $0 \le m, n \le r$, then $\tilde{U}_l(\xi_1, \xi_2) = \xi_1^m \xi_2^n$ and thus $\tilde{G}_k(\tilde{u}) = 0$.

If $u(\xi_1, \xi_2) = \xi_1^{r+l-1-n} \xi_2^n$, $n = 0, \dots, l-2$, then $\tilde{U}_l(\xi_1, \xi_2) = p_r^{(1,l,n)}(\xi_1) + p_r^{(2,l,n)}(\xi_1)\xi_2^n$, where $p_r^{(m,l,n)} \in P_r([0,1])$, $m = 1, 2$. Hence $\partial_1^k \partial_2^{r+l-k}(\tilde{u} - \tilde{U}_l) = 0$ for $k = l, \dots, r$ and $\tilde{G}_k(\tilde{u}) = 0$.

In a similar way we obtain that $\tilde{G}_k(\tilde{u}) = 0$, if $\tilde{u}(\xi_1, \xi_2) = \xi_2^{r+l-1-n}\xi_1^n$, $n = 0, \dots, l-2$. From the above equalities it follows that $\tilde{G}_k(\tilde{u}) = 0$ for $\tilde{u} \in P_{r+l-1}(\tilde{K})$. By the Bramble-Hilbert Lemma (see [1])

$$|\tilde{G}_k(\tilde{u})| \le C_k |\tilde{u}|_{r+l,\tilde{K}} \|\tilde{f}_k\|_{0,\tilde{K}}, \quad k = l, \dots, r$$

and hence

$$|G_k(u)| \le Ch^{-r-l+2}|\tilde{u}|_{r+l,\tilde{K}} \|\tilde{f}_k\|_{0,\tilde{K}} \le C|u|_{r+l,e_{ij}} \|f_k\|_{0,e_{ij}}.$$

Taking $f_k = \partial_1^k \partial_2^{r+l-k}(u - U_l) \in L^2(e_{ij})$, $k = l, \dots, r$, yields

$$\|f_k\|_{0,e_{ij}}^2 \le C|u|_{r+l,e_{ij}} \|f_k\|_{0,e_{ij}},$$

that is,

$$\|\partial_1^k \partial_2^{r+l-k}(u - U_l)\|_{0,e_{ij}} \le C|u|_{r+l,e_{ij}}.$$

Combining (6.6) with the above inequalities produces (6.5).

For $2 \le s \le r+1$ we have

$$|U_l|_{s,e_{ij}} \le |u - U_l|_{s,e_{ij}} + |u|_{s,e_{ij}} \le C|u|_{s,e_{ij}}, \tag{6.7}$$

and for $s = 0, 1$

$$|U_l|_{s,e_{ij}} \le |u - U_l|_{s,e_{ij}} + |u|_{s,e_{ij}} \le C\|u\|_{2,e_{ij}}, \tag{6.8}$$

(see [2, pp. 122–124]).

The conclusion follows from the estimates (6.5), (6.7), (6.8) and the Cauchy-Schwarz inequality.

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LEMMA 6. *Let* $f \in W^{s+k,q}(e_{ij})$ *for* $i = 0, \dots, N_1 - 1$, $j = 0, \dots, N_2 - 1$, *where* $2/q < s \le r$, $q \ge 2$, $r \ge k \ge 0$. *Then for any* $V \in V_r$

$$|l(V) - l_h(V)| \le Ch^{s+k} |||f|||_{s+k,q} |||V|||_k.$$

PROOF. By (2.3) and (4.4)

$$l(V) - l_h(V) = \sum_{i=0}^{N_1-1} \sum_{j=0}^{N_2-1} E_{ij}(fv).$$

Using (5.6) with $n = r$ and the Hölder inequality (see [2, p. 200]) we have for any $V \in V_r$ and $q \ge 2$

$$\begin{aligned}
|l(V) - l_h(V)| &\le \sum_{i=0}^{N_1-1} \sum_{j=0}^{N_2-1} Ch^{s+k+1-(2/q)} \|f\|_{s+k,q,e_{ij}} \|V\|_{k,e_{ij}} \\
&\le Ch^{s+k} \left(\sum_{i=0}^{N_1-1} \sum_{j=0}^{N_2-1} h^{1-(2/q)} \|f\|_{s+k,q,e_{ij}} \|V\|_{k,e_{ij}} \right) \\
&\le Ch^{s+k} \left(\sum_{i=0}^{N_1-1} \sum_{j=0}^{N_2-1} h^2 \right)^{(1/2-(1/q))} \left(\sum_{i=0}^{N_1-1} \sum_{j=0}^{N_2-1} \|f\|^q_{s+k,q,e_{ij}} \right)^{1/q} \left(\sum_{i=0}^{N_1-1} \sum_{j=0}^{N_2-1} \|V\|^2_{k,e_{ij}} \right)^{1/2} \\
&\le Ch^{s+k} |||f|||_{s+k,q} |||V|||_k.
\end{aligned}$$

We can now prove Theorem 2.

PROOF OF THEOREM 2. Let U_I be a V_r-interpolant of the function u. Applying the interpolation error estimates (see [2, 14]) we note that

$$\|u - U_I\|_{1,\Omega} \le Ch^s |||u|||_{s+1}, \tag{6.9}$$

where $2/q < s \le r$. Using the first Strang Lemma (see [2, p. 186]), (6.2) with $k = 1$, Lemma 5, Lemma 6 with $k = 0$ and the estimate (6.9) we obtain

$$\begin{aligned}
\|u - U\|_{1,\Omega} &\le C\|u - U_I\|_{1,\Omega} + C \sup_{V \in V_r} \frac{|a(U_I, V) - a_h(U_I, V)|}{\|V\|_{1,\Omega}} + C \sup_{V \in V_r} \frac{|l(V) - l_h(V)|}{\|V\|_{1,\Omega}} \\
&\le Ch^s |||u|||_{s+1} + Ch^s |||U_I|||_{s+2} + Ch^s |||f|||_{s,q} \\
&\le Ch^s \left(|||u|||_{s+2} + |||f|||_{s,q} \right),
\end{aligned}$$

which completes the proof.

Finally, we prove Theorem 3.

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Proof of Theorem 3. Let $W_I \in V_1$ and $U_I \in V_r$ be interpolants of w and u respectively. From the interpolation error estimate and (2.5) it follows that

$$\|w - W_I\|_{1,\Omega} \le Ch\|w\|_{2,\Omega} \le Ch\|g\|_{0,\Omega} \tag{6.10}$$

and

$$\|W_I\|_{1,\Omega} \le \|W_I - w\|_{2,\Omega} + \|w\|_{2,\Omega} \le C\|w\|_{2,\Omega} \le C\|g\|_{0,\Omega}. \tag{6.11}$$

We now estimate $|||U|||_3$ for $r \ge 2$. By the triangle inequality

$$|||U|||_3 \le |||U - U_I|||_3 + |||U_I|||_3. \tag{6.12}$$

By (5.17)

$$|||U - U_I|||_3 \le Ch^{-2}\|U - U_I\|_{1,\Omega}. \tag{6.13}$$

The triangle inequality yields

$$\|U - U_I\|_{1,\Omega} \le \|U - u\|_{1,\Omega} + \|u - U_I\|_{1,\Omega}. \tag{6.14}$$

Theorem 2 and (6.9) ($s = 2$) imply

$$\|U - U_I\|_{1,\Omega} \le Ch^2(|||u|||_4 + |||f|||_{2,q}). \tag{6.15}$$

Note that (see Lemma 5)

$$|||U_I|||_3 \le C|||u|||_3. \tag{6.16}$$

Combining (6.12) with (6.13), (6.15) and (6.16) yields

$$|||U|||_3 \le C(|||u|||_4 + |||f|||_{2,q}). \tag{6.17}$$

By (6.3) with $k = 2$, $1 \le s \le r$, $r \ge 2$,

$$|a(U, W_I)| - a_h(U, W_I)| \le Ch^{s+1}|||W_I|||_2|||U|||_3, \tag{6.18}$$

since $W_I \in V_1$ and $|||W_I|||_{s+2} = |||W_I|||_2$. Setting W_I instead of V in (6.1) and using (6.1), Theorem 2, (6.10), (6.18), (6.11), Lemma 6 with $k = 1$ and (6.17) produces

$$\begin{aligned}\|u - U\|_{0,\Omega} \le \sup_{g\in L^2(\Omega)} \|g\|_{0,\Omega}^{-1}\big(&Ch^s\left(|||u|||_{s+2} + |||f|||_{s,q}\right) Ch\|g\|_{0,\Omega} \\ &+ Ch^{s+1}|||U|||_3\|g\|_{0,\Omega} + Ch^{s+1}|||f|||_{s+1,q}\|g\|_{0,\Omega}\big) \\ \le Ch^{s+1}&(|||u|||_{\tilde{s}} + |||f|||_{s+1,q}),\end{aligned}$$

where $2/q < s \le r$, $r \ge 2$, $q \ge 2$, $\tilde{s} = \max(s + 2, 4)$.

Appendix

We present another form of the system (3.1)–(3.4). The idea is similar to that in the finite-element method. First we generate local matrices k_{ij} and local right-hand sides g_{ij} on elements e_{ij}, $i = 0, \dots, N_1 - 1$, $j = 0, \dots, N_2 - 1$. Then we extend them to the whole region Ω, obtaining the matrices $\tilde{k}_{ij}$ and the vectors $\tilde{g}_{ij}$. Next we assemble the matrices to create a global matrix K and the vectors to create a global right-hand side G, that is,

$$K = \sum_{i=0}^{N_1-1} \sum_{j=0}^{N_2-1} \tilde{k}_{ij},$$

$$G = \sum_{i=0}^{N_1-1} \sum_{j=0}^{N_2-1} \tilde{g}_{ij}.$$

Let the solution U of the system (3.1)–(3.4) be represented as

$$U = \sum_{i=0}^{N_1-1} \sum_{j=0}^{N_2-1} \sum_{k,l=0}^{r} U_{ik,jl} \phi_{ik}^x \phi_{jl}^y,$$

where ϕ_{ik}^x and ϕ_{jl}^y are defined in (4.6)–(4.8). Denote the vector $\bar{U} = \{U_{ik,jl}\}$. Then $\bar{U}$ is the solution of the system of equations

$$K\bar{U} = G.$$

The local system of equations on any element e_{ij} is as follows:

$$\begin{gathered} LU(x_{ik}, y_{jl}) = f(x_{ik}, y_{jl}), \\ \theta_i^x(x) w D_1 U(x, y_{jl}) + h_{1,i} LU(x, y_{jl}) = h_{1,i} f(x, y_{jl}), \\ \theta_j^y(y) w D_2 U(x_{ik}, y) + h_{2,j} LU(x_{ik}, y) = h_{2,j} f(x_{ik}, y), \\ \theta_i^x(x) h_{2,j} w D_1 U(x, y) + \theta_j^y(y) h_{1,i} w D_2 U(x, y) + h_{1,i} h_{2,j} LU(x, y) = h_{1,i} h_{2,j} f(x, y), \end{gathered}$$

where $1 \le k$, $l \le r - 1$, $x = x_{i+}$ or $x = x_{i+1}-$, $y = y_{j+}$ or $y = y_{j+1}-$ and $\theta_i^x(x_{i+}) = -1$, $\theta_i^x(x_{i+1}-) = 1$, $\theta_j^y(y_{j+}) = -1$, $\theta_j^y(y_{j+1}-) = 1$. The above system defines the local matrix k_{ij} and local right-hand side g_{ij}.

Another approach is to use the tensor product of one-dimensional operators (see [8]).

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