BULL. AUSTRAL. MATH. SOC. MOS 46H05

VOL. 5 (1971), 387-390.

The weak density of the non-invertible elements of a commutative algebra

L. Narici, G. Bachman and E. Beckenstein

Let X be a commutative locally convex Hausdorff topological algebra with identity over a non-trivially valued field F. Let M_c denote the continuous nontrivial homomorphisms of X into F and M the set of all maximal ideals of X. If the spectrum of each element x in X is the set of scalars $\{f(x) \mid f \in M_c\}$, it is shown that the singular elements of X are weakly dense in X if and only if M is an infinite set.

In this paper we extend the above quoted result, proved for commutative Banach algebras with identity by Graham in [4, p. 179], to a category of topological algebras over arbitrary nontrivially valued fields. Included among these are a number of nonarchimedean locally multiplicatively convex algebras over nonarchimedean valued fields ([3]) such as algebras $F(T)$ of continuous functions from an infinite totally disconnected space T into a complete nonarchimedean valued field F with compact-open topology.

LEMMA 1. *If X is a topological algebra over a nontrivially valued field F such that $\cup M_c = \cup M$, then M_c is finite if and only if M is finite.*

Proof. Since $M_c \subset M$, finiteness of M clearly implies finiteness of M_c. Conversely we show that if M_c is finite, then $M_c = M$. To do this let us suppose that $M \in M$ and $M \notin M_c$. Letting

Received 7 July 1971.

$M_c = \{M_1, \ldots, M_n\}$, then, by hypothesis, $M = \bigcup_{i=1}^{n} (M \cap M_i)$. With no loss of generality we may assume that the above union is irredundant. Thus for each i, $1 \le i \le n$, there must be an element $x_i \in M \cap M_i$ such that $x_i \notin M \cap M_j$ for $j \ne i$. Consider $y = x_i + \prod_{j \ne i} x_j$. Since $y \in M$, $y \in M \cap M_k$ for some k, $1 \le k \le n$. If $k \ne i$, we are led to the contradictory result: $x_i = y - \prod_{j \ne i} x_j \in M \cap M_k$. If $k = i$ we find that $\prod_{j \ne i} x_j \in M \cap M_i$. Since M_i is a maximal - hence prime - ideal, this too is contradictory. Thus there can be no $M \in \mathcal{M}$ such that $M \notin \mathcal{M}_c$, and the proof is complete.

Clearly $\cup\mathcal{M} = \cup\mathcal{M}_c$ in any topological algebra X where the spectrum $\sigma(x)$ of each $x \in X$ satisfies the relationship $\sigma(x) = \{f(x) \mid f \in \mathcal{M}_c\}$. We call such algebras *Arens algebras*. As a result of [1, p. 462] the following algebras are Arens algebras:

(a) any complex commutative complete Hausdorff locally m-convex algebra ([5, p. 18]);

(b) by (a), any complex commutative Banach algebra with identity;

(c) any complete nonarchimedean locally multiplicatively convex algebra over a complete nonarchimedean valued field whose factor algebras are Gelfand algebras ([3], [6, p. 432]).

Taking the radical of X to be $R = \cap\mathcal{M}$, we note that by [8, p. 33], $R = \cap\mathcal{M}_c$ in any Arens algebra. Before proceeding to the main result, the following lemma is needed.

LEMMA 2. *If X is an Arens algebra, then $\mathcal{M}$ is infinite if and only if X/R is infinite-dimensional.*

Proof. Consider the Gelfand map ([2])

$$\psi : X \to F(\mathcal{M}_c)$$

$$x \to \hat{x}$$

https://doi.org/10.1017/S0004972700047377 Published online by Cambridge University Press

where for any $M \in M_c$, $\hat{x}(M) = x + M$. Since X is an Arens algebra, we observe that $\ker\psi = R$. Thus X/R is isomorphic to $\psi(X)$. If X/R is infinite-dimensional, surely M_c is an infinite set. Thus M is infinite. Conversely, by Lemma 1, if M is infinite, then M_c must be infinite. Now $\psi(X)$ is a subalgebra of $F(M_c)$ which separates points of M_c so, by elementary considerations, $\psi(X)$ must be infinite-dimensional. Thus X/R is infinite-dimensional.

THEOREM. *If* X *is a locally convex Hausdorff Arens algebra over the reals, the complexes, or any spherically complete valued field, the singular elements of* X *are weakly dense in* X *if and only if* M *is infinite.*

Proof. We refer to Graham's proof ([4, p. 180]) making note of the fact that the validity of his proof in this setting depends on the infinite-dimensionality of X/R which has been shown to be true in Lemma 2, the fact that $R = \cap M_c$ since X is an Arens algebra, and the availability of the Hahn-Banach Theorem in our setting ([7, p. 78]).

References

[1] Richard Arens, "A generalization of normed rings", *Pacific J. Math.* 2 (1952), 455-471.

[2] E. Beckenstein, G. Bachman and L. Narici, "Maximal ideals in topological algebras", *J. Analyse Math.* 25 (1971), (to appear).

[3] E. Beckenstein, G. Bachman and L. Narici, "Function algebras over valued fields", (to appear).

[4] Colin C. Graham, "The weak density of the non-invertible elements of a commutative Banach algebra", *Bull. Austral. Math. Soc.* 4 (1971), 179-182.

[5] Ernest A. Michael, "Locally multiplicatively-convex topological algebras", *Mem. Amer. Math. Soc.* 11 (Amer. Math. Soc., Providence, Rhode Island, 1952).

[6] Lawrence Narici, "On nonarchimedean Banach algebras", *Arch. Math.* 19 (1968), 428-435.

[7] Lawrence Narici, Edward Beckenstein and George Bachman, *Functional analysis and valuation theory* (Marcel Dekker, New York, 1971).

[8] W. Żelazko, "Metric generalizations of Banach algebras", *Rozprawy Mat.* 47 (Polsk. Akad. Nauk, Inst. Mat., Warszawa, 1965).

St John's University,
Jamaica,
New York, USA,

Polytechnic Institute of Brooklyn,
Brooklyn,
New York, USA

and

Polytechnic Institute of Brooklyn,
Brooklyn,
New York, USA.