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Affine Lines on Affine Surfaces and the Makar–Limanov Invariant

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Abstract. A smooth affine surface X defined over the complex field $\mathbb{C}$ is an ML_0 surface if the Makar–Limanov invariant $\mathrm{ML}(X)$ is trivial. In this paper we study the topology and geometry of ML_0 surfaces. Of particular interest is the question: Is every curve C in X which is isomorphic to the affine line a fiber component of an $\mathbb{A}^1$-fibration on X? We shall show that the answer is affirmative if the Picard number $\rho(X) = 0$, but negative in case $\rho(X) \geq 1$. We shall also study the ascent and descent of the ML_0 property under proper maps.

1 Preliminaries

Let X be a smooth affine surface defined over the complex field $\mathbb{C}$, which we can replace in most arguments by an algebraically closed field of characteristic zero. The Makar–Limanov invariant $\mathrm{ML}(X)$ is defined as $\bigcap_\sigma A^\sigma$, where σ runs over all algebraic actions of the additive group G_a and A^σ is the invariant subring of the coordinate ring A of X under the G_a-action σ. We call X an ML_i surface ($i = 0, 1, 2$) if the transcendence degree of the quotient field of $\mathrm{ML}(X)$ over $\mathbb{C}$ is equal to i. Thus, ML_2 surfaces have no nontrivial G_a-actions. In the present article, we are mostly interested in ML_0 surfaces and the geometry of curves on such surfaces.

The simplest example of an ML_0 surface is the affine plane $\mathbb{A}^2$, and we have the well-known theorem of Abhyankar, Moh, and Suzuki (AMS theorem, for future use) which states that any curve in $\mathbb{A}^2$ isomorphic to the affine line $\mathbb{A}^1$ is a fiber of an $\mathbb{A}^1$-fibration. This easily implies that there exists a system of coordinates $\{x, y\}$ on $\mathbb{A}^2$ such that the given curve is defined by $x = 0$. On the other hand, the affine plane $\mathbb{A}^2$ is characterized as an ML_0 surface with $\mathrm{Pic}(X) = 0$. We define $\rho(X)$ to be the rank of $\mathrm{Pic}(X)_\mathbb{Q}$ and $\gamma(X)$ to be the rank of $\Gamma(X, \mathcal{O})^*/\mathbb{C}^*$, which is a finitely generated free abelian group.

We are interested in the following.

Question Let X be an ML_0 surface and let C be a curve on X isomorphic to the affine line $\mathbb{A}^1$. Does there exist an $\mathbb{A}^1$-fibration $f : X \to B$ such that C is a fiber-component of f and $B \cong \mathbb{A}^1$?

An affirmative answer to this question would have been a good generalization of the AMS theorem. But we will see that the answer is negative if $\rho(X) \geq 1$ and affirmative if $\rho(X) = 0$.

The main results in this paper are Theorems 2.1, 3.11, 4.2, 4.3 and 5.1. We have also given several examples where the analogue of the AMS theorem is false for ML_0

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surfaces with $\rho > 0$ and for ML_1 surfaces (Theorem 2.3, Claims 3.7, 3.8). In addition, an example of a simply-connected ML_0 surface with $\rho > 0$ which does not contain $\mathbb{C}^2$ as a Zariski-open subset is given (§6).

We will deal only with complex algebraic (and analytic) varieties. We let $\mathbb{C}^*$ denote the affine curve $\mathbb{P}^1 - \{\text{two points}\}$.

- A $\mathbb{P}^1$-fibration on a smooth algebraic surface S is a morphism $f \colon S \to B$ onto a smooth curve such that a general fiber of f is isomorphic to $\mathbb{P}^1$. An $\mathbb{A}^1$-fibration and a $\mathbb{C}^*$-fibration on a smooth surface are defined similarly.
- An (n)-curve on a smooth surface S is a smooth projective rational curve C with $(C^2) = n$.
- A linear chain $C_1 + \cdots + C_n$ of smooth rational curves is called *admissible* if $({C_i}^2) \leq -2$ for $1 \leq i \leq n$. For related definitions and relevant results we refer to [20].

In the following, we denote by $e(S)$ the topological Euler–Poincaré characteristic of a topological space S. The following results by Suzuki [28] and Zaidenberg [30] (see also [10]) will play an important role.

Lemma 1.1 *Let X be a smooth affine surface with a morphism $f \colon X \to B$ with connected general fiber, where B is a smooth curve. Let F be a general fiber of f and let F_i be the singular fibers in the scheme-theoretic sense for $1 \leq i \leq s$. Then we have*

$$e(X) = e(B)e(F) + \sum_{i=1}^{s} (e(F_i) - e(F)).$$

Further, $e(F_i) \geq e(F)$ for all i. If the equality holds for some i, then F is isomorphic to either $\mathbb{A}^1$ or $\mathbb{C}^*$ and F_i is isomorphic to F for all i if taken with reduced structure.

Let S be a smooth irreducible quasi-projective surface. A projective completion $S \subset W$ is called a *normal completion* of S if W is smooth and $D := W - S$ is a divisor with simple normal crossings. Furthermore, if any (-1)-curve in D intersects at least three other irreducible components of D, then we say that this completion is a *minimal normal* completion of S.

If S is an ML_0 surface, then it is trivial to see that S is rational and the boundary divisor $W - S$ in any normal completion of S is a tree of $\mathbb{P}^1$'s. In particular, the boundary divisor is simply connected.

We have the following result which characterizes an ML_0 surface in terms of the boundary graph [5,8,9,13]. The study of ML_0 surfaces was begun by M. H. Gizatullin in the influential papers [8, 9]

Lemma 1.2 *Let X be a smooth affine surface and let V be a minimal normal completion of X. Then X is an ML_0 surface if and only if $\gamma(X) = 0$ and the dual graph of the boundary divisor $D := V - X$ is a linear chain.*

M. H. Gizatullin has described properties of a singular fiber of a $\mathbb{P}^1$-fibration on a smooth projective surface [18, Ch. II, §2]. We will use this result implicitly in some of the proofs later on.

We begin with some easy results.

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Lemma 1.3 *Let C be an irreducible curve on a smooth affine surface X and let $X' = X - C$. Then we have $\gamma(X') - \rho(X') = \gamma(X) - \rho(X) + 1$.*

Proof In the case where mC is a principal divisor for some $m > 0$, write $mC = (u)$. Then $\gamma(X') = \gamma(X) + 1$ and $\rho(X') = \rho(X)$. In the case where the class $[C]$ is not zero in $\mathrm{Pic}(X)_{\mathbb{Q}}$, we have $\rho(X') = \rho(X) - 1$ and $\gamma(X') = \gamma(X)$. In both cases the result follows. ∎

Lemma 1.4 *Let X be an ML_0 surface. Then the following assertions hold.*

(i) $\gamma(X) = 0$.

(ii) *The torsion part $\mathrm{Pic}(X)_{\mathrm{tor}}$ is isomorphic to $\pi_1(X)$ and is a finite cyclic group, while $H_2(X)$ is the free part of $\mathrm{Pic}(X)$.*

(iii) *If $B \cong \mathbb{P}^1$ (resp. $B \cong \mathbb{A}^1$), any $\mathbb{A}^1$-fibration $f\colon X \to B$ has at most two (resp. one) multiple fibers.*

Proof (i) Since there are two independent G_a-actions on X, there is a dominant morphism $\varphi\colon \mathbb{A}^2 \to X$. Hence, for any non-constant unit u on X, $\varphi^*(u)$ is a non-constant unit on $\mathbb{A}^2$. But this is a contradiction. Hence $\gamma(X) = 0$.

(ii) Let $X \hookrightarrow V$ be a normal completion. Let $D := V - X$. Then we have an exact sequence of cohomology groups with $\mathbb{Z}$-coefficients

$$H^1(D) \to H^2(V, D) \to H^2(V) \to H^2(D) \to H^3(V, D) \to H^3(V),$$

where $H^1(D) = (0)$ since D is simply-connected, $H^2(V) \cong \mathrm{Pic}(V)$, $H^3(V, D) \cong H_1(X)$ and $H^3(V) \cong H_1(V) = (0)$. On the other hand, we have an exact sequence

$$0 \to H_2(D) \to H_2(V) \to \mathrm{Pic}(X) \to 0.$$

Taking the dual of the last exact sequence, we have an exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}(\mathrm{Pic}(X), \mathbb{Z}) \to \mathrm{Hom}(H_2(V), \mathbb{Z}) \to \mathrm{Hom}(H_2(D), \mathbb{Z}) \\ \to \mathrm{Ext}^1(\mathrm{Pic}(X), \mathbb{Z}) \to \mathrm{Ext}^1(H_2(V), \mathbb{Z}), \end{aligned}$$

where $\mathrm{Hom}(\mathrm{Pic}(X), \mathbb{Z})$ is the dual of the free part of $\mathrm{Pic}(X)$, $\mathrm{Hom}(H_2(V), \mathbb{Z}) \cong H^2(V)$ by the universal coefficient theorem and the fact that $H_1(V) = (0)$, $\mathrm{Hom}(H_2(D), \mathbb{Z}) \cong H^2(D)$, $\mathrm{Ext}^1(\mathrm{Pic}(X), \mathbb{Z}) \cong \mathrm{Pic}(X)_{\mathrm{tor}}$ and $\mathrm{Ext}^1(H_2(V), \mathbb{Z}) = (0)$ because $H_2(V)$ is free. Hence we obtain an exact sequence

$$0 \to \mathrm{Hom}(\mathrm{Pic}(X), \mathbb{Z}) \to H^2(V) \to H^2(D) \to \mathrm{Pic}(X)_{\mathrm{tor}} \to 0.$$

Hence it follows that $H_1(X) \cong \mathrm{Pic}(X)_{\mathrm{tor}}$ and $H_2(X)$ is the free part of $\mathrm{Pic}(X)$.

(iii) Let $f\colon X \to B$ be an $\mathbb{A}^1$-fibration. Let $X \hookrightarrow V$ be a smooth normal completion. We may assume that the $\mathbb{A}^1$-fibration f extends to a $\mathbb{P}^1$-fibration $\overline{f}\colon V \to \overline{B}$, where $\overline{B} \cong \mathbb{P}^1$. Let M be the cross-section of $\overline{f}$ lying in the boundary divisor $D := V - X$. Suppose that there exist two or more multiple fibers, say $m_1F_1, \ldots, m_rF_r$

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in the fibration f. We include the fiber of $\overline{f}$ over the point $\overline{B}-B$ as m_rF_r (even though it is reduced) if $B \cong \mathbb{A}^1$. The corresponding fibers $\Phi_1, \ldots, \Phi_r$ of $\overline{f}$ have respectively the components $B_1, \ldots, B_r$ in D meeting the section M. Then M is a branching component of D, *i.e.*, a component meeting three or more other components of D. If we make the boundary D minimal by successively contracting all (-1) curves in D which lie in fibers of $\overline{f}$ so that the resulting boundary divisor still has normal crossings, then the resulting divisor must be a linear chain by Lemma 1.2. But this is impossible as long as $r \geq 3$. So, f has at most two (resp. one) multiple fibers if $B \cong \mathbb{P}^1$ (resp. $B \cong \mathbb{A}^1$). ■

With the above notations, let Φ be a fiber of $\overline{f}$ which cuts out a singular fiber of f. Then Φ consists of a boundary part $C_1 + C_2 + \cdots + C_n$ with C_1 meeting M and several components $A_1, \ldots, A_m$ such that $A_i \cap X$ is isomorphic to $\mathbb{A}^1$ for $1 \leq i \leq m$ (see Lemma 1.5). We sometimes call any of $A_1 \cap X, \ldots, A_m \cap X$ a *feather* of the $\mathbb{A}^1$-fibration f. If there is no fear of confusion, we also call any of $A_1, \ldots, A_m$ a feather of X.

For later use, we shall look into a more systematic construction of ML_0 surfaces. The following result is well known and easy [20].

Lemma 1.5 *Let $U \subset X$ be smooth affine surfaces such that U is an open set of X and U has an $\mathbb{A}^1$-fibration. Then $X - U$ is a disjoint union of irreducible curves which are isomorphic to $\mathbb{A}^1$.*

Lemma 1.6 *Let X be an ML_0 surface and $f\colon X \to \mathbb{A}^1$ an $\mathbb{A}^1$-fibration. Then there exists a smooth normal completion $\overline{X} = \overline{X}_{f,n}$ of X such that the boundary divisor $D := \overline{X} - X$ consists of a linear chain $D = \ell - M - A$, where ℓ is a (0)-curve, M is an (n)-curve and A is an admissible linear chain. The fibration f extends to a $\mathbb{P}^1$-fibration $\overline{f}\colon \overline{X}_{f,n} \to \mathbb{P}^1$ such that $\ell = \overline{f}^{-1}(\infty)$ $(:= F_\infty)$, M is a cross-section of $\overline{f}$ and $F_0 := \overline{f}^{-1}(0) = A + A_1 + \cdots + A_r$, where each A_i is isomorphic to $\mathbb{P}^1$ and meets $\overline{X} - X$ normally in a component A_i' of A (hence A_i is a feather of X).*

We then write X as $X_{f,n}$.

Furthermore, the following assertions hold.

(i) *Given an ML_0 surface $X = X_{f,n}$, an ML_0 surface $X_{f,n'}$ with an arbitrary n' is obtained by elementary transformations on $\overline{f}^{-1}(\infty)$ of $\overline{X}_{f,n}$, where ℓ (resp. M) is replaced by a (0)-curve (resp. (n')-curve) ℓ' (resp. M') and A, X are not touched in the process.*

(ii) *We have $r \geq 1$ and $\rho(X) = r - 1$. Let m_i be the multiplicity of A_i in $\overline{f}^*(0)$. Then $\mathrm{Pic}(X)$ is generated by the classes of $A_1, \ldots, A_r$ which are subject to the relation $\sum_{i=1}^{r} m_iA_i = 0$. Hence $\mathrm{Pic}_{\mathrm{tor}}(X)$ is a finite cyclic group of order $d = \gcd(m_1, \ldots, m_r)$. (*cf. *Lemma* 1.4*).*

(iii) *At least one A_i is a (-1)-curve.*

We call a chain $D = \ell - M - A$ as above a *standard chain* and the completion $\overline{X}_{f,n}$ of X attaching D as the boundary divisor a *standard completion* of X. Note that for any standard chain on a smooth projective rational surface $\overline{X}$, the curve ℓ induces an $\mathbb{A}^1$-fibration on $X := \overline{X} - D$. Then X is an ML_0 surface precisely when X is affine.

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We shall explain a method of attaching feathers to construct any ML_0 surface. Let Σ_s be the Hirzebruch surface with a $\mathbb{P}^1$-fibration $p\colon \Sigma_s \to \mathbb{P}^1$. Let $\ell = \ell_\infty$ and ℓ_0 be two fibers of p and let M be a cross-section with $(M^2) = n$. We apply the following two operations.

(A) Blow up above a point on ℓ_0 not on M so that the proper transform ℓ_0' of ℓ_0 together with the exceptional curves form a linear chain $B^{(1)}$. Suppose $B^{(1)}$ has length ≥ 2. Let $B_0^{(1)}$ be the tip of $B^{(1)}$ not meeting M and $B_1^{(1)}$ the component meeting $B_0^{(1)}$. Blow up above a point on $B_1^{(1)}$ not on any other component of $B^{(1)}$ to produce a chain $B^{(2)}$ attached to $B_1^{(1)}$. Continuing this way, we produce a linear chain

$$A' = (B^{(1)} \setminus B_0^{(1)}) - (B^{(2)} \setminus B_0^{(2)}) - \cdots - (B^{(s-1)} \setminus B_0^{(s-1)}) - B^{(s)}.$$

We allow $B^{(s)}$ to have length 1 or to be empty.

(B) Choose irreducible components $A_1', \ldots, A_t'$ of A', including all (-1)-components, and blow up n_i points of A_i' not on any other component of A' to produce exceptional curves A_{ij}. Let A be the same chain as A' with $(A_i'^2)$ replaced by $(A_i'^2) - n_i$. Relabel $A_{11}, \ldots, A_{tn_t}$ as $A_1, \ldots, A_{r'}$ and $B_0^{(1)}, \ldots, B_0^{(s-1)}$ as $A_{r'+1}, \ldots, A_r$.

We produce a projective surface $\overline{X}$ with a $\mathbb{P}^1$-fibration $\overline{f} : \overline{X} \to \mathbb{P}^1$ induced by p such that $\overline{f}^{-1}(\infty) = \ell$, M is a cross-section of $\overline{f}$ and $\overline{f}^{-1}(0) = A + A_1 + \cdots + A_r$. If we assume $r \geq 2$ in case $s = 1$ and $B^{(1)}$ has length 1 (*i.e.*, $A = \ell_0$), A is an admissible chain.

Lemma 1.7 *Let $\overline{X}$ be as constructed above and let $D = \ell + M + A$. Then $X := \overline{X} - D$ is an* ML_0 *surface and $\overline{f}$ induces an $\mathbb{A}^1$-fibration $f\colon X \to \mathbb{A}^1$, where D is a standard linear chain and $X = X_{f,n}$. Furthermore, every* ML_0 *surface arises in this way.*

Proof The steps (A) and (B) above are the only way to create standard chains D on a smooth projective rational surface $\overline{X}$ such that $\overline{X} - D$ is affine. ∎

The following result shows that if an ML_0 surface has positive Picard rank, then it contains an affine open set which is an ML_0 surface with smaller Picard rank.

Lemma 1.8 *Let X be an* ML_0 *surface with $\rho(X) > 0$. Then there exists an irreducible curve A such that*

(i) *A is isomorphic to $\mathbb{A}^1$,*
(ii) *$X' := X - A$ is an* ML_0 *surface with $\rho(X') = \rho(X) - 1$.*

In particular, if $\rho(X) = 1$, then X' is a $\mathbb{Q}$-homology plane.

Proof We choose a standard completion of X as in Lemma 1.6, obtained as in Lemma 1.7. Since $\rho \geq 1$, we have $r \geq 2$. Say A_1 is a (-1)-curve. Let $D' = D + A_1$ and make D' minimal by the contraction of A_1 and successively contractible curves. If a component of A becomes a (-1)-curve in this process, then no feather A_i, $i \geq 2$, is attached to it unless A is a (-2)-curve and $r = 2$. In that case $X - A_1 \cong \mathbb{A}^2$. In the

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general case we end up with a standard minimal boundary $D'' = \ell'' - M'' - A''$ of $X - A_1$ with $A_2, \dots, A_r$ attached to A''. We may assume that D'' is minimal by Lemma 1.6(i). We have $\rho(X - A_1) = r - 1$, and hence $\gamma(X - A_1) = \gamma(X) = 0$ by Lemma 1.3. By Lemma 1.2, $X - A_1$ is an ML_0 surface. ■

We note here that all ML_0 surfaces which are $\mathbb{Q}$-homology planes are described in [5, 17].

2 The Case of $\mathbb{Q}$-Homology Planes

We shall prove the following result which is a generalization of the AMS theorem. The proof below is very similar to a proof of the AMS theorem given in [11]. The arguments in the proof below are used again in the proof of Theorem 3.11.

Theorem 2.1 *Let X be an ML_0 surface with $\rho(X) = 0$. Let C be a curve isomorphic to the affine line on X. Then there exists an $\mathbb{A}^1$-fibration $f\colon X \to B$ such that $B \cong \mathbb{A}^1$ and C is a fiber component of f.*

Proof The proof uses Lemma 1.2 in an essential way. First we will prove the following result.

Claim 2.2 *We have $e(X - C) = 0$ and $\overline{\kappa}(X - C) \leq 1$.*

Since $e(X) = e(C) = 1$, we have $e(X - C) = 0$. Hence $\overline{\kappa}(X - C) \leq 1$ for otherwise $e(X - C) > 0$ by a logarithmic analogue of the Miyaoka–Yau inequality due to R. Kobayashi [16]. For details, see [20, Theorem 6.7.1]. This proves the claim.

We consider each of the cases $\kappa(X - C) = -\infty, 0$ and 1 separately.

Case I: $\overline{\kappa}(X - C) = -\infty$. Since $X - C$ is an affine surface with $\overline{\kappa}(X - C) = -\infty$, there exists an $\mathbb{A}^1$-fibration $f'\colon X - C \to B'$ which extends to an $\mathbb{A}^1$-fibration $f\colon X \to B$ such that C is a fiber component of f.

Case II: $\overline{\kappa}(X - C) = 0$. Since $\mathrm{Pic}(X)$ is a finite cyclic group by Lemma 1.4, there exists an element $u \in \Gamma(X, \mathcal{O}_X)$ such that $\mathrm{Supp}(u) = C$. Consider the morphism $X - C \to \mathbb{C}^*$ which is defined by the function u and hence denoted by u. By replacing u by the Stein factorization $u\colon X - C \to B' \to \mathbb{C}^*$ where B' is necessarily isomorphic to $\mathbb{C}^*$, we may assume that the general fibers of the morphism u are irreducible. By Kawamata's addition formula [20, Lemma 1.14.1], we have

$$0 = \overline{\kappa}(X - C) \geq \overline{\kappa}(\mathbb{C}^*) + \overline{\kappa}(F),$$

where F is a general fiber of u. Then $\overline{\kappa}(F)$ is either $-\infty$ or 0. If $\overline{\kappa}(F) = -\infty$, then u is an $\mathbb{A}^1$-fibration and the argument in Case I applies, and we are done.

So we may assume that $\overline{\kappa}(F) = 0$, *i.e.*, $F \cong \mathbb{C}^*$. By the construction of the morphism u, it extends to a morphism $u\colon X \to B$, where $B \cong \mathbb{A}^1$, such that $u^*(0)_{\mathrm{red}} = C$.

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By Lemma 1.1 applied to $u\colon X - C \to \mathbb{C}^*$, we have

$$0 = e(X - C) = e(\mathbb{C}^*)e(F) + \sum_i (e(F_i) - e(F)),$$

where the F_i exhaust all singular fibers of u. Hence $e(F_i) = e(F)$ for all i. This implies that all the fibers of $u\colon X - C \to \mathbb{C}^*$ are isomorphic to $\mathbb{C}^*$ if taken with reduced structures. Now we consider a smooth normal completion $X \hookrightarrow V$ such that u extends to a $\mathbb{P}^1$-fibration $p\colon V \to \overline{B}$, where $\overline{B} \cong \mathbb{P}^1$. We have two cases to consider.

Case II.A: u is untwisted. This means that there exist two cross-sections H_1, H_2 of p lying in the boundary $D := V - X$. We may assume that the fiber $\ell_\infty = p^{-1}(P_\infty)$ is a smooth fiber and meets H_1 and H_2 in distinct points, where $P_\infty = \overline{B} - B$. The fiber $p^{-1}(P_0)$ contains the closure $\overline{C}$ of C, and $\mathrm{Supp}(p^{-1}(P_0)) - \overline{C}$ is connected and meets both H_1 and H_2 because $C \cong \mathbb{A}^1$. Then the boundary divisor D contains a loop, which is a contradiction because D is a tree if X is a $\mathbb{Q}$-homology plane [23].

Case II.B: u is twisted. There exists a 2-section H of p contained in D. Since $p|_H\colon H \to \overline{B}$ ramifies in two points and $p^{-1}(P)_{\mathrm{red}} \cap X \cong \mathbb{C}^*$ if $P \neq P_0, P_\infty$, it follows that $p|_H$ ramifies over the points P_0 and P_∞ of $\overline{B}$. So we may assume that $p^{-1}(P_\infty) + H$ looks like

$$\begin{array}{ccccc}
 & & (-2)\,\ell'_\infty & & \\
 & & | & & \\
(-2) & — & (-1) & — & H \\
E_1 & & E_2 & &
\end{array}$$

$H + p^*(P_0)_{\mathrm{red}} - \overline{C}$ looks like $H + \Gamma$ where there are three possibilities.

(i) Γ contains at least one branch component for D.

(ii) $\Gamma = F_1 + F_2$ and $H + \Gamma + \overline{C}$ looks like

$$\begin{array}{ccccc}
 & & (-2)\ \overline{C} & & \\
 & & | & & \\
H & — & (-1) & — & (-2) \\
 & & F_2 & & F_1
\end{array}$$

(iii) $\Gamma = \varnothing$ and $\overline{C}$ touches H in a smooth point with order of contact 2.

In case (i), the boundary D has a configuration

$$\begin{array}{ccccccc}
(-2) & — & (-1) & — & H & — & \Gamma \\
 & & | & & & & \\
 & & (-2)\,\ell'_\infty & & & &
\end{array}$$

and D cannot be minimalized to a linear chain because Γ contains a branching component. Hence X is not an ML_0 surface.

In case (ii), contract E_2, E_1, F_2 and F_1 to obtain a relatively minimal $\mathbb{P}^1$-fibration $\overline{p}\colon \overline{V} \to \overline{B}$. Let M_0 and ℓ be the minimal section and a general fiber of $\overline{p}$, respectively.

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Then the image $\overline{H}$ of H is linearly equivalent to $2M_0 + a\ell$ with $a \geq 2n$, where $n = -(M_0{}^2)$. Then $(\overline{H}^2) = 4(a-n)$ and $(H^2) = 4(a-n-1)$. On the other hand, X is ML_0 surface if and only if $H + F_2 + F_1$ is contractible to a point, *i.e.,* $(H^2) = -3$. This implies that D does not minimalize to a linear chain.

In case (iii), D minimalizes to a linear chain if and only if H is a (-1)-curve. With notations as in the case (ii), contract E_1 and E_2 to obtain a relatively minimal $\mathbb{P}^1$-fibration $\overline{p}\colon \overline{V} \to \overline{B}$. As above, $\overline{H} \sim 2M_0 + a\ell$ and hence $(\overline{H}^2) = 4(a-n)$. Meanwhile, $(\overline{H}^2) = 1$ because $(H^2) = -1$, which is a contradiction.

Thus we have shown that the case $\overline{\kappa}(X - C) = 0$ does not occur.

Case III: $\overline{\kappa}(X - C) = 1$.There exists a $\mathbb{C}^*$-fibration $f'\colon X - C \to B'$ by a theorem of Kawamata (see [20]). We claim that $B' \cong \mathbb{A}^1$ or $\mathbb{C}^*$. First of all, since $X - C$ is rational, B' is a smooth rational curve. If $B' \cong \mathbb{C}^{n*}$ $(n \geq 2)$, which is $\mathbb{A}^1$ with n points punctured, then $\gamma(X - C) \geq n$, while $\gamma(X - C) = 1$. Hence $B' \cong \mathbb{C}^*, \mathbb{A}^1$ or $\mathbb{P}^1$. If $B' \cong \mathbb{P}^1$, then $\rho(X - C) > 0$, while $\rho(X - C) = 0$ by Lemma 1.3. So, $B' \cong \mathbb{A}^1$ or $\mathbb{C}^*$. We consider these cases separately. By Lemma 1.1, it follows that all fibers of f' are isomorphic to $\mathbb{C}^*$ if taken with the reduced structures.

Case III.A: $B' \cong \mathbb{A}^1$. We consider the linear pencil Λ spanned by the closures of general fibers of f'. Suppose that the general fibers of f' are closed in X. Then f' extends to a $\mathbb{C}^*$-fibration $f\colon X \to B$. We have the following four cases to consider:

(i) $B \cong \mathbb{P}^1$ and f is untwisted.
(ii) $B \cong \mathbb{P}^1$ and f is twisted.
(iii) $B \cong \mathbb{A}^1$ and f is untwisted.
(iv) $B \cong \mathbb{A}^1$ and f is twisted.

Let $X \hookrightarrow V$ be a smooth normal completion and let $D := V - X$. We may assume that there is a $\mathbb{P}^1$-fibration $p\colon V \to \overline{B}$ such that $p|_X = f$. In the untwisted (resp. twisted) case, there are two cross-sections H_1, H_2 (resp. one 2-section H) in D.

In case (i), $\overline{B} = B$ and H_1 and H_2 do not meet on the fiber $p^{-1}(P)$ $(P \neq P_0)$, where $P_0 = B - B'$, because $f^{-1}(P)_{\mathrm{red}} \cong \mathbb{C}^*$. We may assume that the boundary $D := V - X$ contains no (-1)-curves as fiber components. Let $\overline{C}$ be the closure of the given affine line C. Then $\overline{C}$ is a unique (-1)-curve in the fiber $p^{-1}(P_0)$ and a tip in the fiber. Since D minimalizes to a linear chain, it follows that H_1 and H_2 are not branching components in D. This implies that $f'\colon X - C \to B'$ has at most one multiple fiber. Let $V \to V'$ be a sequence of contractions which brings $p^{-1}(P_0)$ to a smooth fiber ℓ_0 and let D' be the reduced image of $D + \overline{C}$ on V'. Then $\kappa(V, D + \overline{C} + K_V)$ is equal to $\kappa(V', D' + K_{V'})$ and

$$D' + K_{V'} \sim_{\mathbb{Q}} \left\{ -2 + 1 + \left(1 - \frac{1}{m}\right) \right\} \ell + \varepsilon,$$

where ℓ represents a general fiber of p, m is the multiplicity of a possible multiple fiber of f' and ε is an effective $\mathbb{Q}$-divisor with negative-definite intersection matrix which does not affect the calculation of $\kappa(V', D' + K_{V'})$. In fact, $\varepsilon = \frac{1}{m} p^*(P_1) - F_1$, where $m(F_1 \cap X)$ is a multiple fiber of f' and $P_1 = p(F_1)$. See [19, p. 81] for details.

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Then $\overline{\kappa}(X - C) = -\infty$. This is a contradiction.[1]

In case (ii), $p|_H \colon H \to B$ branches over two points P_0, P_1 of B. We may assume that $f^{-1}(P_0)_{\mathrm{red}} = C \cong \mathbb{A}^1$. Since D is connected, it follows that $f'^{-1}(P_1)_{\mathrm{red}} \neq \mathbb{C}^*$, which is a contradiction.

In case (iii), since $B = B' \cong \mathbb{A}^1$, D contains a complete fiber $p^{-1}(P_\infty)$ of p, which we may assume to be a smooth fiber. The fiber $p^{-1}(P_0)$ containing $\overline{C}$ cuts out a fiber of the form either $m_1\mathbb{C}^* + n_1 C$ or $m_1 C' + n_1 C$ on X, where $C' \cong C \cong \mathbb{A}^1$ and C', C meet in one point transversally. We may further assume that there are no (-1) fiber components of p in D.

Consider first the case $p^{-1}(P_0) \cap X = m_1\mathbb{C}^* + n_1 C$. If $m_1 > 1$ there are no other multiple fibers of f', for otherwise both H_1 and H_2 are branching components of D and D cannot be minimalized to a linear chain. If $m_1 = 1$, there is at most one multiple fiber $m_2\mathbb{C}^*$ in f' by the same reason as above. Since $\overline{C}$ is a (-1) curve, contract $\overline{C}$ and consecutively contractible fiber components in $D \cap p^{-1}(P_0)$ and obtain the images V', D' of V, D. Then $\kappa(V, D + \overline{C} + K_V) = \kappa(V', D' + K_{V'})$ and

$$D' + K_{V'} \sim_{\mathbb{Q}} \left\{ -2 + 1 + \left(1 - \frac{1}{m_i}\right) \right\} \ell + \varepsilon = \left(-\frac{1}{m_i}\right) \ell + \varepsilon,$$

where $i = 1$ or 2 according to $m_1 > 1$ or $m_1 = 1$ and ε is, as in case (i) above, an effective $\mathbb{Q}$-divisor which does not affect the calculation of $\overline{\kappa}(V', D' + K_{V'})$. Then $\overline{\kappa}(X - C) = -\infty$. This is a contradiction.

Consider next the case $p^{-1}(P_0) \cap X = m_1 C' + n_1 C$. Suppose $\min(m_1, n_1) > 1$. If there is at least one multiple fiber $m_2\mathbb{C}^*$ in f', then H_1 and H_2 are branching components in D which has only admissible twigs. So this case does not occur. If there is no multiple fiber in f' other than $m_1\mathbb{C}^*$ which is cut out by $m_1 C'$ on $X - C$, a computation similar to the above shows that $\overline{\kappa}(X - C) = -\infty$, which is a contradiction. Hence $\min(m_1, n_1) = 1$. Suppose $m_1 = n_1 = 1$. Let $\overline{C'}$ be the closure of C' on V. Then each of $\overline{C'}$ and $\overline{C}$ meets one of H_1 and H_2. Assume that $\overline{C'}$ meets H_1 and $\overline{C}$ does H_2. In this case, f' has at most one multiple fiber $m_2\mathbb{C}^*$ and $\overline{\kappa}(X - C) = -\infty$, a contradiction. Hence either $m_1 = 1, n_1 > 1$ or $m_1 > 1, n_1 = 1$. Suppose that $m_1 = 1$ and $n_1 > 1$. Then the fiber $p^{-1}(P_0)$ is a linear chain and there is a non-empty linear chain connecting $\overline{C}$ to H_2. Hence there is no other multiple fiber in f', for otherwise H_2 is a branching component. Then $\overline{\kappa}(X - C) = -\infty$ by a computation similar to the above. Hence $m_1 > 1$ and $n_1 = 1$. Then there is a non-empty linear chain in $p^{-1}(P_0)$ connecting $\overline{C'}$ to H_1 and hence there is no other multiple fiber in f'. By the computation of $\overline{\kappa}(X - C)$, this leads to a contradiction as well.

In case (iv), one fiber $f^{-1}(P_0)$ of f must be of the form either $m_1\mathbb{C}^* + n_1 C$ or $m_1 C' + n_1 C$ as in case (iii). Furthermore, we may assume that the fiber at infinity $p^{-1}(P_\infty)$ together with H has the following configuration:

$$\begin{array}{ccccc} (-2) & — & (-1) & — & H \\ & & | & & \\ & & (-2)\ \ell'_\infty & & \end{array}$$

[1]We can use the following simpler argument. Since $\overline{C}$ is a (-1) curve sprouting from D, we have $\kappa(V, D + \overline{C} + K_V) = \kappa(V, D + K_V) = \overline{\kappa}(X) = -\infty$.

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Note that $p|_H : H \to \mathbb{P}^1$ ramifies over P_∞ and a point P_1 of B. Since D is connected, the fiber $p^*(P_1)$ cannot produce a fiber of f which has the form $m_1\mathbb{C}^* + n_1C, m_1C' + n_1C$ (case $P_1 = P_0$) or $m\mathbb{C}^*$. This is a contradiction.

Now suppose that the pencil Λ has a base point P on X. Then P is a one-place point for the general members of Λ, $P \in C$ and C is contained in a member of Λ. The pencil must be parametrized by $\mathbb{P}^1$ because it has a base point in the affine part X. So, mC is a member of Λ for some $m \geq 1$. We consider elimination of base points of Λ as well as a smooth normal completion $X \hookrightarrow V$. If Λ extended to a pencil on V has base points on $V - X$, we replace V by a suitable smooth surface obtained from V by elimination of the base points outside X. Then we obtain a $\mathbb{P}^1$-fibration $p : V \to \overline{B}$ satisfying the following conditions:

(i) $\overline{B} \cong \mathbb{P}^1$.

(ii) The exceptional locus Γ arising from the elimination of the base point P consists of a cross-section of p, which is the last (-1)-curve in the elimination process, and several trees sprouting out of the cross-section.

(iii) The boundary divisor $D := V - X$ has one cross-section H and several trees sprouting out of H.

(iv) The closure $\overline{C}$ of C together with the trees from Γ and D forms a fiber of p.

Furthermore, we may assume that any fiber of p contains no (-1)-curves in $\Gamma + D$. Then we have:

(v) Every degenerate fiber consists of a linear chain of components.

(vi) There is at most one degenerate fiber besides the one containing $\overline{C}$. If it exists, let $m'C'$ be the multiple fiber of f' cut out by this degenerate fiber, where $C' \cong \mathbb{C}^*$.

We shall compute $\overline{\kappa}(X - C)$. We have

$$\overline{C} + D + \Gamma + K_V \sim_Q \varepsilon + \begin{cases} -\ell & \text{if no multiple fibers,} \\ (-2 + 1 + (1 - \frac{1}{m'}))\ell & \text{if one multiple fiber,} \end{cases}$$

where ℓ is a general fiber of p and ε is an effective divisor which does not affect the calculation of $\kappa(V, \overline{C} + D + \Gamma + K_V)$. In both cases, $\overline{\kappa}(X - C) = -\infty$, which is a contradiction to the assumption $\overline{\kappa}(X - C) = 1$.

Suppose that Λ has no base points on C. Then C is a cross-section of Λ and there are no multiple fibers in Λ. Hence Λ defines an $\mathbb{A}^1$-bundle structure on X parametrized by $\mathbb{A}^1$. Hence X is isomorphic to $\mathbb{A}^2$. In this case, $\overline{\kappa}(X - C) = -\infty$, which is a contradiction.

Case III.B: $B' \cong \mathbb{C}^*$. Consider again the pencil Λ generated by the closures on X of general fibers of f'. Then Λ has no base points. In fact, if Λ has a base point, Λ should be parametrized by $\mathbb{P}^1$. Meanwhile, we have only as many members as parametrized by $\mathbb{C}^*$ and one more member corresponding to C. This is a contradiction. If the general members of Λ meet the curve C, then C is a cross-section. But we have one point to which no member of Λ corresponds. So, general members of Λ do not meet the curve C. In this case we argue as in the case $\overline{\kappa}(X - C) = 0$.

Hence we have shown that the case $\overline{\kappa}(X - C) = 1$ does not occur.

We have thus completed the proof of Theorem 2.1 ∎

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Before finishing this section we shall construct a smooth affine surface X satisfying the following conditions:

(i) X is an ML_0 surface with $\rho(X) = r > 0$.
(ii) There exists an affine line C such that $\overline{\kappa}(X - C) \geq 0$. We call such an affine line C an *anomalous* affine line.

In fact, the existence of such an example shows that the initial question is answered in the negative.

We begin by constructing a basic example in case $\rho(X) = 1$. Let Σ_0 be the Hirzebruch surface $\mathbb{P}^1 \times \mathbb{P}^1$. We denote any fiber of the vertical (resp. horizontal) $\mathbb{P}^1$-fibration by ℓ (resp. M) and call it a *fiber* (resp. *section*). Take two horizontal sections M_0, M_1 and three fibers $\ell_0, \ell_1, \ell_\infty$. Let $P_0 := M_0 \cap \ell_0$ and $P_1 := M_1 \cap \ell_1$. Let A be a smooth irreducible curve such that $A \sim M + 2\ell$ and touches M_0 (resp. M_1) at P_0 (resp. P_1) with order of contact 2. Clearly, A is a cross-section of one of the $\mathbb{P}^1$-fibrations on Σ_0 and a 2-section for the other $\mathbb{P}^1$-fibration and $A^2 = 4$. Now A meets ℓ_∞ at a point other than $M_0 \cap \ell_\infty$ and $M_1 \cap \ell_\infty$. Blow up the point P_0 (resp. P_1) and its infinitely near point of the first order lying on M_0 (resp. M_1) to produce irreducible exceptional curves E_1, E_2 (resp. F_1, F_2), where $({E_1}^2) = ({F_1}^2) = -2$ and $({E_2}^2) = ({F_2}^2) = -1$. Then the proper transform A' of A meets E_2 and F_2. We blow up these two intersection points to obtain the exceptional curves E_3 and F_3. We denote the proper transforms of E_i, F_i $(i = 1, 2)$ by the same letters. Now we have $({E_i}^2) = ({F_i}^2) = -2$ for $i = 1, 2$ and $({E_3}^2) = ({F_3}^2) = -1$. Let A' be again the proper transform of A'. Then ${A'}^2 = -2$. Let $\sigma \colon V \to \Sigma_0$ be the composite of these blowing-ups. Let $M_0', M_1', \ell_0', \ell_1', A'$ signify the proper transforms of $M_0, M_1, \ell_0, \ell_1, A$ on V. We set $X := V - D$, where $D := M_0' + M_1' + \sigma^*(\ell_\infty) + E_1 + E_2 + F_1 + F_2$ and $C := A' \cap X$. Then we shall prove the following result.

Theorem 2.3 *The following assertions hold.*

(i) *X is an* ML_0 *surface with* $\rho(X) = 1$.
(ii) $\overline{\kappa}(X - C) = 0$ *and hence C is an anomalous affine line.*
(iii) *$X - C$ is a $\mathbb{Q}$-homology plane.*

Proof (i) The boundary $D := V - X$ consists of a linear chain

$$\begin{array}{ccccccccccccc} E_1 & & E_2 & & M_0' & & \sigma^*(\ell_\infty) & & M_1' & & F_2 & & F_1 \\ (-2) & — & (-2) & — & (-2) & —— & (0) & —— & (-2) & — & (-2) & — & (-2) \end{array}$$

Since D is a linear chain we see that X is an ML_0 surface with $\rho(X) = 1$, and C is an affine line on X. $\mathrm{Pic}(X)$ is generated by $E_3^* = E_3 - D$ and $F_3^* = F_3 - D$ with $2E_3^* = 2F_3^* = C$. Hence $\rho(X) = 1$ and $\rho(X - C) = 0$. This gives (i) and (iii). By the adjunction formula we check easily that

$$4K_V \sim -3M_0' - 3M_1' - E_1 - 2E_2 - F_1 - 2F_2 - 4l_\infty' - 2A'.$$

Hence

$$4(K_V + D + A') \sim M_0' + M_1' + 3E_1 + 2E_2 + 3F_1 + 2F_2 + 2A'.$$

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Since the reduced divisors $M_0' + E_1 + E_2$, $M_1' + F_1 + F_2$ and A' are disjoint from each other and since each of them has negative definite intersection matrix, it follows that $\dim |4(A' + D + K_V)| = 0$. Hence $\overline{\kappa}(X - C) = 0$ and C is an anomalous affine line. ■

We recall the definition of a half-point attachment in order to produce further examples of ML_0 surfaces admitting anomalous affine lines. Let $X \hookrightarrow V$ be a normal completion of a smooth affine surface and let $D := V - X$. Let P be a point on D which is not an intersection point of two irreducible components of D. Let $\sigma \colon V' \to V$ be the blowing-up of P, let $E = \sigma^{-1}(P)$, let $D' := \sigma'(D)$ and let $X' := V' - D'$. We say that the affine line $E \cap X'$ is a *half-point* and that X' is obtained by a *half-point attachment* with center P. In fact, X is a Zariski open set of X'. We are interested only in the case where X' is again affine. An operation of attaching a feather, which we defined in section one, is a kind of half-point attachment.

Theorem 2.4 *Let $X \hookrightarrow V$ be the same as in Theorem 2.3. Choose r points $P_1, \dots, P_r$ on the boundary component E_1 and let X' be a smooth surface obtained by half-point attachments with centers $P_1, \dots, P_r$. Then the following assertions hold:*

(i) *X' is an ML_0 surface with $\rho(X) = r + 1$ for any $r \geq 0$.*
(ii) *The curve C in X is also an anomalous affine curve in X'.*

Proof (i) Let $\tau \colon V' \to V$ be the blowing-ups of $P_1, \dots, P_r$. We denote the proper transforms on V' of the components of D by the same letters. Then $({E_1}^2) = -(r+2)$. Let

$$L = 4E_1 + (4r + 9)E_2 + (8r + 15)M_0' + (12r + 22)\sigma^*(\ell_\infty) + (9r + 15)M_1' + (6r + 9)F_2 + (3r + 4)F_1.$$

By the Nakai–Moishezon test, L is an effective ample divisor on V' with $\operatorname{Supp} L = \tau'(D)$, the proper transform of D. Hence X' is affine. Since $\tau'(D)$ is a linear chain, X' is an ML_0 surface with $\gamma(X') = 0$. Hence by Lemma 1.3, we have $\rho(X') = \rho(X) = r + 1$.

(ii) It is known that a half-point attachment does not change the logarithmic Kodaira dimension, *cf.* [15]. So, $\overline{\kappa}(X' - C) = \overline{\kappa}(X - C) = 0$ and C is anomalous in X' too. ■

The above theorems give counterexamples to the question posed at the beginning of the article. The divisor class of C has infinite order in $\operatorname{Pic}(X)$ by Theorem 2.3(iii). Similarly, C has infinite order in $\operatorname{Pic}(X')$.

So we shall reformulate the question as follows.

Question Let X be an ML_0 surface and let C be a curve on X isomorphic to the affine line $\mathbb{A}^1$. Suppose that C has torsion divisor class in $\operatorname{Pic}(X)$. Does there exist an $\mathbb{A}^1$-fibration $f \colon X \to B$ such that C is a fiber component of f, where $B \cong \mathbb{A}^1$?

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3 ML_1 Surfaces

Let X be a smooth affine surface. We say that X is *unruled* (resp. *simply ruled*, *multi-ruled*) if X has no $\mathbb{A}^1$-fibrations (resp. only one $\mathbb{A}^1$-fibration, two independent $\mathbb{A}^1$-fibrations). If there is no fear of confusion, we say that X is 0-*ruled*, 1-*ruled*, 2-*ruled* if X is unruled, simply-ruled, multi-ruled, respectively. Note that an $\mathbb{A}^1$-fibration on X does not necessarily have as base an affine curve. If X is an ML_0 surface then it is 2-ruled. We shall see later that the converse is not necessarily the case.

We shall begin with giving a criterion for an affine surface to be an ML_1 surface. Let V be a smooth projective surface and let D be an effective reduced divisor with simple normal crossings. We assume that D consists of smooth rational curves. Let $\Gamma(D)$ be the associated weighted graph. In the following, *we only consider D which are trees.* Blowing up an intersection point of two components will add one more component of weight -1 and decrease by -1 the weights of the components concerned; we call this blowing-up *subdivisional*. Blowing up a point on a single component also adds one component of weight (-1) and decrease by (-1) the weight of the concerned component; we call this blowing-up *sprouting*. See [7, 19] for further details of the terminology. Modelled on $\Gamma(D)$, we consider a connected weighted graph Γ in general. We say that Γ is *minimal* if any (-1) component meets at least three other components, *i.e.,* it is a branching component. Let Γ, Γ' be two weighted graphs. If there exist a weighted graph Γ'' and morphisms $\varphi : \Gamma'' \to \Gamma, \varphi' : \Gamma'' \to \Gamma'$, which are, by definition, composites of subdivisional or sprouting blowing-ups, we say that Γ and Γ' are pre-equivalent. Let $\mathfrak{G}$ be the set of all connected weighted graphs, which are trees, together with the equivalence relation generated by the pre-equivalence relations. Given two graphs Γ, Γ' which are equivalent, we say that Γ' is a *modification* of Γ and vice versa.

Lemma 3.1 *Let $\Gamma, \Gamma', \Gamma''$ be the connected weighted graphs and let $\varphi \colon \Gamma'' \to \Gamma$, $\varphi' \colon \Gamma'' \to \Gamma'$ be morphisms. Assume that no (-1) component of Γ'' which is φ-exceptional is φ'-exceptional, and vice versa. Then the following assertions hold.*

(i) *A branch component of Γ does not become φ'-exceptional on Γ''.*

(ii) *If φ contains a sprouting blowing-up on a non-tip component D_i, then D_i becomes a branching component and hence does not become φ'-exceptional.*

(iii) *One can create φ'-exceptional (-1) components on Γ'' by applying successively one of the following operations:*

  (a) *subdivisional blowing-up on a non-branching component with non-negative weight,*

  (b) *sprouting blowing-up on a tip component with non-negative weight.*

Proof We leave the proof to the reader. ∎

Let Γ be a weighted graph and let L be a connected part of Γ which is a linear chain. We say that L is *admissible* if all components of L have weight ≤ -2.

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Corollary 3.2

(i) *The modifications between Γ and Γ' occur on*

(a) *non-admissible linear chains,*
(b) *non-admissible maximal twigs T_i or*
(c) *non-admissible linear chains L_j connecting two branching components.*

Under the modifications, admissible maximal twigs and admissible maximal linear chains between branching components are unaffected.

(ii) *If all the maximal twigs of Γ are admissible, then the same is true of Γ'.*
(iii) *If Γ is a minimal non-admissible linear chain and if Γ' is a minimal graph equivalent to Γ, then Γ' is also a non-admissible linear chain.*

Proof The assertions (i) and (ii) are more or less obvious. As for the assertion (iii), if one tries to modify the graph, a non-admissible component is a tip component or has two components adjacent to it. If it is a tip component, it is clear that Γ' is a linear chain. If it has two adjacent components and the blowing-up is sprouting, then it becomes a branch component and hence not contractible. This means that we have to contract back exactly the same exceptional components we obtained by the blowing-ups. If the blowing-up is subdivisional, we obtain a linear chain. ■

We need one more lemma.

Lemma 3.3 *Suppose that the weighted graph Γ is associated to an effective reduced divisor with simple normal crossings consisting of rational curves on a smooth projective rational surface V. If D_i, D_j are two components of D such that $(D_i{}^2) \geq 0, (D_j{}^2) \geq 0$, then either D_i and D_j are adjacent or $(D_i{}^2) = (D_j{}^2) = 0$ and $D_i \sim D_j$.*

Proof Suppose that D_i and D_j are not adjacent. By the Hodge index theorem, $(D_i{}^2) = (D_j{}^2) = 0$. Since $|D_i|$ is a linear pencil and $D_i \cap D_j = \varnothing$, D_j is a member of $|D_i|$. Hence $D_i \sim D_j$. ■

We can now state a result which leads to a criterion for ML_1 surfaces.

Theorem 3.4 *Let X be a smooth affine rational surface and let $X \hookrightarrow V$ be a minimal normal completion. Let $D := V - X$ and $\Gamma = \Gamma(D)$. Then the following conditions are equivalent.*

(i) *X has an $\mathbb{A}^1$-fibration $f \colon X \to B$, where B is an open set of $\mathbb{A}^1$.*
(ii) *Γ as well as any other minimal modification of Γ has a non-admissible twig.*
(iii) *There exists a modification Γ' of Γ which has a tip with weight 0.*

Proof (i) $\Rightarrow$ (ii): We may assume that the $\mathbb{A}^1$-fibration extends to a $\mathbb{P}^1$-fibration $p \colon V \to \overline{B}$, where $\overline{B} \cong \mathbb{P}^1$. We may also assume that the fibers of p over the points $\overline{B} - B$ are smooth fibers. Then, in $\Gamma(D)$, these fibers represent non-admissible twigs.

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(ii) $\Rightarrow$ (iii): If a non-admissible twig has a non-negative component which is not a tip component of the twig, then we can modify the graph by subdivisional blowing-ups and blowing-downs so that the tip component has weight 0. Let D_0 be a non-admissible tip component and let D_1 be the component adjacent to D_0. Then, applying subdivisional blowing-ups with centers at $D_0 \cap D_1$ and its infinitely near points lying on D_0, we can ensure that the proper transform D_0' has weight 0. Then the obtained weighted graph Γ' has D_0' as a tip.

(iii) $\Rightarrow$ (i): We may assume that D has a tip component D_0 with weight 0. Then $|D_0|$ is a linear pencil of rational curves, and the adjacent component, say D_1, is a cross-section. Hence the $\mathbb{P}^1$-fibration $p = \Psi_{|D_0|} : V \to \overline{B}$ restricts to an $\mathbb{A}^1$-fibration $f : X \to B$ with $B \subset \mathbb{A}^1$. ■

Corollary 3.5 *Let the notations and assumptions be the same as in Theorem* 3.4. *Then the following conditions are equivalent.*

(i) *X is an* ML_1 *surface which is different from* $\mathbb{A}^1 \times \mathbb{C}^*$.
(ii) *Let $X \hookrightarrow V$ be a smooth minimal normal completion and let $D := X - V$. Then $\Gamma(D)$ has a non-admissible twig and $\Gamma(D)$ is not a linear chain.*

Proof (i) $\Rightarrow$ (ii): Since X is an ML_1 surface, there is an $\mathbb{A}^1$-fibration $f : X \to B$ with an open set $B \subset \mathbb{A}^1$. Then, by Theorem 3.4, $\Gamma(D)$ has a non-admissible twig. Suppose that $\Gamma(D)$ is a linear chain. Then we take a different completion of the same kind if necessary and assume that a tip component, say D_1, has weight 0. Consider the $\mathbb{A}^1$-fibration f defined by $|D_1|$. Name the components of $\Gamma(D)$ as $D_1 - D_2 - \cdots - D_n$. We may assume that $({D_2}^2) = 0$ and $D_3 - \cdots - D_n$ contains no (-1) components if $n \geq 3$. If $n = 2$, $X \cong \mathbb{A}^2$ and X is an ML_0 surface. Suppose $n \geq 3$. If $D_3 - \cdots - D_n$ is not negative-definite, then $n = 3$ and $({D_3}^2) = 0$ because $D_3 - \cdots - D_n$ is contained in one and the same fiber of f. Hence $X \cong \mathbb{A}^1 \times \mathbb{C}^*$. This case is excluded. Hence $D_3 - \cdots - D_n$ is negative-definite. Then the base of f is isomorphic to $\mathbb{A}^1$. Hence $\gamma(X) = 0$. Then X is an ML_0 surface by Lemma 1.2. So $\Gamma(D)$ is not a linear chain.

(ii) $\Rightarrow$ (i): By changing the completion V, we may assume that a tip component, say D_1, has weight 0. Let $f : X \to B$ be the $\mathbb{A}^1$-fibration defined by $|D_1|$. Then B is an open set of $\mathbb{A}^1$. If $B \cong \mathbb{A}^1$, then $\gamma(X) = 0$ and X is not an ML_0 surface because $\Gamma(D)$ is not a linear chain. If $B \subsetneqq \mathbb{A}^1$, then $\Gamma(D) - D_1 - D_2$ contains a non-admissible connected component, say Γ_2, where D_2 is the component adjacent to D_1. Since Γ_2 is contained in a member of $|D_1|$, the minimality of $\Gamma(D)$ shows that Γ_2 consists of a single component of weight 0. Hence $\gamma(X) \neq 0$ and X is not an ML_0 surface by Lemma 1.5. Since $\Gamma(D)$ is not a linear chain, $X \not\cong \mathbb{A}^1 \times \mathbb{C}^*$ as well. ■

Our interest lies in clarifying the interdependence between the ML_i property and j-ruledness on a smooth rational affine surface X. We shall first consider a smooth affine rational surface X which is an ML_2 surface, *i.e.,* X has no G_a actions. Note that almost all smooth affine rational surfaces are ML_2 and 0-ruled. For example, if $\overline{\kappa}(X) \geq 0$, X satisfies these properties.

As for an example of X which is ML_2 and 1-ruled, we have the following result [13, Theorem 4.1].

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Theorem 3.6 *Let $f\colon X \to B$ be an $\mathbb{A}^1$-fibration on a smooth affine surface X with base B a smooth curve such that every fiber of f is irreducible. Assume further that B is isomorphic to $\mathbb{A}^1$ or $\mathbb{P}^1$ and that f has at least two (resp. three) multiple fibers if $B \cong \mathbb{A}^1$ (resp. $B \cong \mathbb{P}^1$). Then X has no other $\mathbb{A}^1$-fibrations whose general fibers are transverse to f.*

If, in this theorem, we take B to be $\mathbb{P}^1$, X is ML_2 and 1-ruled. We define a *Platonic $\mathbb{A}^1$-fiber space* as a smooth affine surface X with an $\mathbb{A}^1$-fibration $f\colon X \to B$ such that $B \cong \mathbb{P}^1$, all fibers of f are irreducible and there are three multiple fibers m_1F_1, m_2F_2, m_3F_3, where $\{m_1, m_2, m_3\}$ is a Platonic triplet $\{2, 2, n\}$, $\{2, 3, 3\}$, $\{2, 3, 4\}$, $\{2, 3, 5\}$, up to permutations.

Claim 3.7 *By modifying the surface X in Theorem 2.3, we can construct a smooth affine surface X_1 which is ML_2 and at least 1-ruled. With the notations in Theorem 2.3, we blow up the points $E_3 \cap A'$ and $F_3 \cap A'$. Let V_1 be the obtained projective surface. To avoid complicated notations, we denote the proper transform of $\sigma^*(\ell_\infty)$ by ℓ_∞ and those of M_0', M_1', E_i, F_i, A' by the same letters, where $1 \le i \le 3$. Now $({E_3}^2) = ({F_3}^2) = -2$. Let E_4 and F_4 be the new (-1) curves. Let $D_1 = D + E_3 + F_3$ and $X_1 := V_1 - D_1$. Now E_2 and F_2 are branching in D_1 and there are no non-admissible maximal twigs in D_1. Hence X is ML_2*

We shall consider the question in Section 1 for ML_1 surfaces. The answer is negative as shown by the following claim.

Claim 3.8 *Let V_0 be a Hirzebruch surface of degree $n = 0$ or 1 with the $\mathbb{P}^1$-fibration $p_0\colon V_0 \to \mathbb{P}^1$. Let M_0 and ℓ be respectively the minimal section and a general fiber. Let H_0 be a smooth curve such that $H_0 \sim 2M_0 + \ell$ (resp. $H_0 \sim 2(M_0 + \ell)$) if $n = 0$ (resp. $n = 1$). Let P_0, P_∞ be the points of the base curve of p_0 over which $p_0|_{H_0}\colon H_0 \to \mathbb{P}^1$ ramifies and let $\ell_0 = p_0^{-1}(P_0)$ and $\ell_\infty = p_0^{-1}(P_\infty)$. Let $\sigma\colon V \to V_0$ be the blowing-ups of the point $\ell_\infty \cap H_0$ and its infinitely near point on H_0 which produce a (-2) curve E_1 and a (-1) curve E_2. Let $H = \sigma'(H_0), L = \sigma'(\ell_\infty)$ and $\overline{C} = \sigma'(\ell_0)$. Let $X = V - (H + E_1 + E_2 + L)$ and let $C = \overline{C} \cap X$. Then the following assertions hold.*

(i) *$(H^2) = 2$. Let $\tau\colon V' \to V$ be the blowing-ups of the point $H \cap E_2$ and its infinitely near point on H which produce a (-2) curve E_3 and a (-1) curve E_4. Denote $\tau'(E_1), \tau'(E_2)$ again by E_1, E_2 and let $H' = \tau'(H)$. Then $({H'}^2) = 0$ and $|H'|$ defines a $\mathbb{P}^1$-fibration $\overline{f}\colon V' \to \mathbb{P}^1$ such that $f = \overline{f}|_X\colon X \to \mathbb{A}^1$ is an $\mathbb{A}^1$-fibration. In the fibration $\overline{f}$, E_4 is a cross-section and $E_3 + L + 2(E_2 + E_1 + A)$ (resp. $E_3 + E_1 + 2(E_2 + L + A)$) is a fiber of $\overline{f}$ if $n = 0$ (resp. $n = 1$), where A is a (-1) curve meeting X.*

(ii) *X is an ML_1 surface with $\rho(X) = 0$ and one multiple fiber of multiplcity 2.*

(iii) *C is an affine line lying transversally to f and $\overline{\kappa}(X - C) = 0$.*

Proof (i) If $n = 0$, let M be a section of p_0 such that $M \sim M_0$ and M passes through the point $H_0 \cap \ell_\infty$. Then M meets ℓ_0 in a point other than $H_0 \cap \ell_0$. If $n = 1$, let $M = M_0$. It meets ℓ_0, ℓ_∞ in the points other than $H_0 \cap \ell_0, H_0 \cap \ell_\infty$. Then the proper transform of M on V is the (-1) curve A in the statement.

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(ii) The boundary divisor $D = V - X$ is $H + E_1 + E_2 + L$, where E_2 is a branching component and H is a non-admissible twig. Hence X is an ML_1 surface by Corollary 3.5. A unique $\mathbb{A}^1$-fibration on X is given by f.

(iii) In fact, C is a 2-section of f. It is easy to see that $\overline{\kappa}(X - C) = 0$ since $X - C$ has a $\mathbb{C}^*$-fibration over $\mathbb{C}^*$ with all fibers reduced and irreducible. ■

Example 3.9 Let V_0 be a Hirzebruch surface of degree $n \geq 0$ with the $\mathbb{P}^1$-fibration $p_0 : V_0 \to \mathbb{P}^1$ with general fiber ℓ. Let M_0 and M_1 be disjoint sections (so $M_0^2 = -M_1^2$ and $|M_i^2| = n$). Choose three fibers $\ell_0, \ell_1, \ell_\infty$. Let $\sigma\colon V \to V_0$ be a sequence of blowing-ups which produce the following degenerate fibers Γ_i from ℓ_i for $i = 0, 1$:

$$\Gamma_0 : \quad M_0' — \underset{\overline{C}}{(-m_1)} — \underset{E_0}{(-1)} — \underset{E_1}{(-2)} — \cdots — \underset{E_{m_1-1}}{(-2)} — M_1'$$

$$\Gamma_1 : \quad M_0' — (-a_1) — \cdots — (-a_s) — \underset{F_0}{(-1)} — (-b_t) — \cdots — (-b_1) — M_1'$$

where $a_i \geq 2$ $(1 \leq i \leq s), b_j \geq 2$ $(1 \leq j \leq t), \overline{C} = \sigma'(\ell_0)$ and $M_k' = \sigma'(M_k)$ for $k = 0, 1$. Let m_2 be the multiplicity of the component F_0 in the fiber $\sigma^*(\ell_1)$, let $D = M_0' + M_1' + \ell_\infty + (\sigma^*(\ell_0)_{\mathrm{red}} - (\overline{C} + E_0)) + (\sigma^*(\ell_1)_{\mathrm{red}} - F_0)$, and let $X = V - D$. Finally, let $C = \overline{C} \cap X$. Then we have:

Proposition 3.10 *If $m_1 \geq 2$ and $m_2 \geq 2$, then the following assertions hold.*

(i) *X is an* ML_1 *surface.*
(ii) *C is an affine line, and it lies transversally to a unique $\mathbb{A}^1$-fibration $f\colon X \to \mathbb{A}^1$.*
(iii) *$\overline{\kappa}(X - C) = 0$ if and only if $m_1 = m_2 = 2$ and $\overline{\kappa}(X - C) = 1$ otherwise.*
(iv) *If $m_1 = m_2 = 2$, X is isomorphic to the surface constructed in Claim* 3.7.

Proof (i) In the divisor D, the component M_1' is a branching component and has a non-admissible twig

$$(-a_s) - \cdots - (-a_1) - M_0' - \ell_\infty - M_1'$$

where $({\ell_\infty}^2) = 0$. Hence X is an ML_1 surface by Corollary 3.5.

(ii) When we make the end component of the above twig a (0) curve A by blowing-ups and blowing-downs (see [19, Corollary 2.4.3]), the image $\widetilde{C}$ of $\overline{C}$ meets the end component A. Hence the pencil $|A|$ defines a unique $\mathbb{A}^1$-fibration $f\colon X \to \mathbb{A}^1$ and $\widetilde{C} \cap X$ is an affine line which lies transversally to the $\mathbb{A}^1$-fibration f.

(iii) Consider the $\mathbb{P}^1$-fibration on V defined by the pencil $|\ell_\infty|$. It induces a $\mathbb{C}^*$-fibration $f'\colon X - C \to \mathbb{A}^1$ which has two multiple fibers $m_1\mathbb{C}^*, m_2\mathbb{C}^*$. By the formula used in the proof of Theorem 2.1, $\overline{\kappa}(X - C) = 1$ if and only if

$$-2 + 1 + \left(1 - \frac{1}{m_1}\right) + \left(1 - \frac{1}{m_2}\right) > 0.$$

Namely, $\overline{\kappa}(X - C) = 1$ if and only if $(m_1 - 1)(m_2 - 1) > 1$. Similarly, $\overline{\kappa}(X - C) = 0$ if and only if $m_1 = m_2 = 2$.

(iv) Straightforward. ■

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Notwithstanding the above examples, we can prove the following result.

Theorem 3.11 *Let X be a $\mathbb{Q}$-homology plane. Suppose that X is an* ML_1 *surface and not isomorphic to one of the surfaces constructed in Claim* 3.8 *and Claim* 3.9. *Then any affine line on X is a fiber of the unique $\mathbb{A}^1$-fibration $f\colon X \to \mathbb{A}^1$. In other words, there are no affine lines which lie transversally to the unique $\mathbb{A}^1$-fibration $f\colon X \to \mathbb{A}^1$.*

Proof Our proof follows essentially the same arguments as in the proof of Theorem 2.1. The reason the arguments can be applied to the present case is that if the boundary divisor D becomes a linear chain when minimalized, then we obtain contradictions by the computations of $\overline{\kappa}(X - C)$, and if D is not a linear chain when minimalized, then all maximal twigs of D are admissible, whence X is not even an ML_1-surface by Corollary 3.5. We shall use the same notations as in the proof of Theorem 2.1 and indicate the points which need special attention.

Step I. We note that $\mathrm{Pic}(X)$ is a finite group by [23, Lemma 1.1]. Let C be an affine line on X. Then there exists an element $u \in \Gamma(X, \mathcal{O}_X)$ such that $mC = (u)$ for some $m > 0$, which we take to be minimal. Then u defines a morphism $u\colon X \to \mathbb{A}^1$ such that $u^*(0) = mC$ and the general fibers are irreducible. Since $e(X - C) = 0$, we have $\overline{\kappa}(X - C) \leq 1$. If $\overline{\kappa}(X - C) = -\infty$, then $X - C$ has an $\mathbb{A}^1$-fibration which extends to such an $\mathbb{A}^1$-fibration on X with C as a fiber component. The base of the fibration is $\mathbb{A}^1$ since X is a $\mathbb{Q}$-homology plane. By the uniqueness of $\mathbb{A}^1$-fibration on X, it coincides with the given f. Hence C is a fiber of f. We consider below the cases $\overline{\kappa}(X - C) = 0, 1$ separately.

Step II. Suppose that $\overline{\kappa}(X - C) = 0$. Consider $u\colon X - C \to B'$, where $B' \cong \mathbb{C}^*$. With the notations in Case II in the proof of Theorem 2.1, if $\overline{\kappa}(F) = -\infty$, u is an $\mathbb{A}^1$-fibration over B' which, extended to X, coincides with f and hence C is a fiber of f. Suppose that $\overline{\kappa}(F) = 0$. Then u is a $\mathbb{C}^*$-fibration. The case where u is untwisted is treated in the same way as in Theorem 2.1. In the case where u is twisted and Γ contains at least one branching component (case (i)), we see readily that D has only admissible twigs. So, X is not ML_1 by Corollary 3.5. In the case where u is twisted and case (ii) or (iii) occurs, contract the curves E_2, E_1 (and F_2, F_1 in case (ii)) to obtain a relatively minimal $\mathbb{P}^1$-fibration $\overline{p}\colon \overline{V} \to \overline{B}$. Then $\kappa(V, D + K_V) = -\infty$ and $\kappa(V, D + \overline{C} + K_V) = 0$ if and only if $a = n + 1$, where $a \geq 2n$ because $(\overline{H} \cdot M_0) \geq 0$. Hence $n = 0, 1$. Then it is easy to show that X is isomorphic to the surface constructed in Claim 3.8 and C is as given in the same example.

Step III. Suppose that $\overline{\kappa}(X - C) = 1$. The arguments for the proof of Theorem 2.1 can be applied without change when D is minimalized to a linear chain, and the criterion in Corollary 3.5 can be applied to the non-ML_1 property of X when D does not minimalize to a linear chain. The only exception occurs in case (iii) where $B \cong \mathbb{A}^1$, f is untwisted and $p^{-1}(P_0) \cap X = m_1 C' + n_1 C$. Suppose that one of the cases (1) $\min(m_1, n_1) > 1$ and (2) $m_1 = 1, n_1 \geq 1$ occurs. In order that $\overline{\kappa}(X - C) = 1$, we need multiple fibers $m_2\mathbb{C}^*, \ldots, m_r\mathbb{C}^*$ such that the following inequality holds:

$$-2 + 1 + \Big(1 - \frac{1}{m_1}\Big) + \sum_{i=2}^{r}\Big(1 - \frac{1}{m_i}\Big) > 0.$$

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Hence $r \geq 2$ in case (1) and $r \geq 3$ in case (2). Then H_1 and H_2 are branching components (with the fiber at infinity ℓ_∞ taken into account) and all the twigs of D are admissible. So X is not an ML_1 surface by Corollary 3.5. Suppose that $m_1 > 1$ and $n_1 = 1$. Since $p^{-1}(P_0)$ is a linear chain, its graph looks like Γ_0 in Claim 3.9 with E_0, M_0', M_1' corresponding respectively to $\overline{C'}, H_2, H_1$.

Since $\overline{\kappa}(X - C) = 1$, we have $r \geq 2$ by the above inequality. If $r \geq 3$, then H_1 and H_2 are branching components of D and there are only admissible twigs in D. Hence X is not an ML_1 surface. So $r = 2$. Let $m_2\mathbb{C}^*$ be the second multiple fiber of f' over a point $P_1 \in B'$. Then $\overline{\kappa}(X-C) = 1$ if and only if $(m_1-1)(m_2-1) > 1$. Furthermore, the fiber $p^{-1}(P_1)$ looks like the fiber Γ_1 in Claim 3.9 with M_0', M_1' corresponding to H_2, H_1 respectively, and F_0 is the component with multiplicity m_2 and meets X. Now we are in the same situation as in Claim 3.9. Hence X is isomorphic to the surface constructed in that example. ■

4 Ascent and Descent of the ML_i Property

We begin with recalling the following result [17].

Lemma 4.1 *Let $\varphi\colon X \to Y$ be an étale finite morphism of smooth affine surfaces. Then any G_a-action on Y lifts to a G_a-action on X. In particular, if Y is ML_0, then so is X.*

We ask if the converse holds, and obtain the following descent result for the ML_0 property. Note that there is no restriction on ρ (see Theorem 4.3).

Theorem 4.2 *Let $\varphi\colon X \to Y$ be a finite morphism of smooth affine surfaces with X an ML_0 surface. Assume that either φ is étale or φ is a Galois (possibly ramified) covering. Then Y is an ML_0 surface.*

Proof If φ is étale, then $\pi_1(X)$ is a subgroup of finite index in $\pi_1(Y)$. Hence by taking a normal subgroup of finite index in $\pi_1(Y)$ contained in $\pi_1(X)$ we can find a smooth affine surface Z and an étale finite morphism $\psi\colon Z \to X$ such that $\varphi\circ\psi\colon Z \to Y$ is an étale Galois covering. Since X is ML_0, it follows by Lemma 4.1 that Z is also an ML_0 surface. After replacing X by Z if necessary, we can assume that φ is a (possibly ramified) Galois covering with Galois group G.

By the equivariant completion theorem of Sumihiro [26, 27] and G-equivariant resolution of singularities, we can find a smooth normal G-completion $X \hookrightarrow V$, where G acts on the boundary divisor $D := V - X$. If the completion is minimal, then D is a linear chain because X is ML_0 (Lemma 1.2). We shall show that V can be chosen so that D is linear.

Assume that D is not minimal. Then D has an irreducible component D_1 such that D_1 is a (-1) curve and meets at most two other components. Then all the conjugates of D_1 in D have the same property. Let $D_1, D_2, \ldots, D_r$ be all the conjugates of D_1. If $D_i \cap D_j = \varnothing$ for every pair (i, j) with $1 \leq i < j \leq r$, then we can contract all of them simultaneously and obtain a new normal G-completion. Assume that $D_i \cap D_j \neq \varnothing$ for some pair (i, j), say $(i, j) = (1, 2)$. Let Γ_1 be the connected component of $D - D_2$

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containing D_1 and let Γ_2 be the connected component of $D - D_1$ containing D_2. Then Γ_1 (resp. Γ_2) does not contain D_2 (resp. D_1), and Γ_1 and Γ_2 are also conjugate. By the assumption, D_1 meets only one other irreducible component of Γ_1 and similarly for D_2. If Γ_1 contains a branch component of D, then so does Γ_2. If we contract D_1 and any subsequent (-1) curves which meet at most two other irreducible components, then we reach a minimal divisor with simple normal crossings which is a tree but still has two branching components. This contradicts the assumption that X is ML_0. Hence we can assume that D is a G-stable linear chain.

Consider the quotient surface $V/\!/G$ which contains Y as an open set. Then $V/\!/G$ is normal and $D/\!/G$ is a simply connected divisor. Furthermore, $V/\!/G$ has at most quotient singular points on $D/\!/G$ which are the images of intersection points of irreducible components of D.

We shall show that Y has a smooth normal completion W such that $W - Y$ is a linear chain of smooth rational curves. First consider the case when D is irreducible. Then the isotropy subgroup of any point in D is finite cyclic. This implies that $V/\!/G$ has at most cyclic quotient singular points.

Now consider the general case. Let H be a subgroup of G which keeps all the irreducible components of D stable. Then H has index at most 2 in G. Any intersection point of two irreducible components of D is fixed by H. From this we deduce that $V/\!/H$ has cyclic quotient singularities. By taking a minimal G/H-equivariant resolution of singularities, we obtain a normal completion $X/\!/H \hookrightarrow U$ such that U is smooth along $U - X/\!/H$ and $U - X/\!/H$ is a linear chain of smooth rational curves. We note here that $X/\!/H$ may have singular points if φ is not étale.

Now we can assume that $G \cong \mathbb{Z}/2\mathbb{Z}$ and the generator of G permutes the end components of D which is a linear chain. If D is irreducible, then at a fixed point of G we can find local coordinates x, y such that the action is given by $(x, y) \mapsto (\pm x, \pm y)$. Then taking a minimal resolution of singularities of $U/\!/(\mathbb{Z}/(2))$ we get a smooth completion W of Y such that $W - Y$ is a linear chain of smooth rational curves. Hence Y is an ML_0 surface by Lemma 1.2.

Suppose that D is not irreducible. Then a local analysis at a possible fixed point on D shows that G-action is given by $(x, y) \mapsto (y, x)$ with respect to a suitable system of local coordinates at the fixed point and hence that $U/\!/(\mathbb{Z}/2\mathbb{Z})$ is, in fact, smooth. Let $W := U/\!/(\mathbb{Z}/2\mathbb{Z})$. Then $W - Y$ is a linear chain of smooth rational curves. Hence Y is an ML_0 surface by Lemma 1.2 because $\gamma(X) = 0$ implies $\gamma(Y) = 0$. ■

In the case of ML_0 surfaces with $\rho = 0$ we have the following general result.

Theorem 4.3 *Let $f\colon X \to Y$ be a finite morphism of smooth affine surfaces. Suppose that X is an ML_0 surface with $\rho(X) = 0$. Then Y is also an ML_0 surface with $\rho(Y) = 0$.*

Proof The proof of this result is inspired by results about pseudoconcave spaces in the theory of analytic spaces, but we will not use any definitions and results from this theory.

For any normal affine surface S, let $\pi_1'(S)$ denote the fundamental group at infinity for S as defined in [25].

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Clearly X is a $\mathbb{Q}$-homology plane. Since f is proper, we see easily that $b_1(Y) = 0 = \rho(Y)$ and hence $e(Y) = 1$. Since X is an ML_0 surface, by Lemma 1.2 and the well-known result of Mumford on the presentation of $\pi_1'(X)$ in terms of the intersection form on the divisor at infinity for a normal completion of X, $\pi_1'(X)$ is finite cyclic. The next result clearly implies that $\pi_1'(Y)$ is finite cyclic.

Claim 4.4 *The homomorphism $\pi_1'(X) \to \pi_1'(Y)$ is a surjection.*

Since f is proper there is a natural induced homomorphism $\pi_1'(X) \to \pi_1'(Y)$. Finiteness of f also implies that the image H of this homomorphism has finite index in $\pi_1'(Y)$ [24, Lemma 1.5]. Suppose that H is a proper subgroup of $\pi_1'(Y)$. Let $K \subset Y$ be a suitable big compact set such that the closure of $U := Y \setminus K$ in Y is an orientable real 4-manifold with boundary a compact 3-manifold whose fundamental group is $\pi_1'(Y)$. Let N be the inverse image of U in X. We can assume that $\pi_1(N) = \pi_1'(X)$. There is a covering $\widetilde{U} \to U$ corresponding to the subgroup H of $\pi_1(U)$. By covering space theory we have a complex analytic map $N \to \widetilde{U}$ which factors the map $N \to U$.

Let $Y \subset W$ be a smooth projective embedding such that $D := W \setminus Y$ is a simple normal crossing divisor. The complex space $W \setminus K$ is 0-concave in the sense of [1]. We can find an open embedding $\widetilde{U} \subset T$ such that T is a normal complex analytic space with a proper analytic map with finite fibers $\varphi : T \to W \setminus K$ extending the map $\widetilde{U} \to U$. Since Y is affine, D supports an effective divisor Δ which is very ample for W. Let $B := \varphi^{-1}(D)$ and $\widetilde{B} = \varphi^*\Delta$. Our aim is to show that T is an open subset of a normal projective surface Z such that $Z \setminus \widetilde{B}$ is a smooth affine surface.

Let V be the normalization of W in the function field of X and let $f'\colon V \to W$ be the natural proper morphism with finite fibers. Let $\overline{N} = N \cup A$, where A is the inverse image of D in V. Now $\overline{N}$ is also 0-concave. There exists a natural complex analytic map $\psi : \overline{N} \to T$ which factors $\overline{N} \to W \setminus K$. The divisor $f'^{-1}(D) = \psi^{-1}(B)$ supports the ample divisor $\psi^*(\widetilde{B})$ on V.

Let p, q be distinct points in T and let p' be a point in $\overline{N}$ lying over p. There exists $m \gg 0$ and a section $L \in H^0(V, \psi^* m\widetilde{B})$ such that $L(p') = 0$ and L is not zero at any point lying over q. If $d = \operatorname{degree} \psi$, then $\psi_*(L|_{\overline{N}}) \sim dm\widetilde{B}$ and this divisor vanishes at p, but not at q. This shows that sections of multiples of $\widetilde{B}$ separate points in T.

Since T is essentially compact, we can find a finite number of open sets T_i in T such that for each i there exists m_i and a section $\sigma_i \in H^0(T, m_i\widetilde{B})$ which does not vanish at any point in T_i. Taking M to be the product of all the m_i, we can find sections $h_0, \dots, h_R$ of $|M\widetilde{B}|$ such that the map $h\colon T \to \mathbb{P}^R$ given by sending any point $p \in T$ to $[h_0(p), \dots, h_R(p)]$ is holomorphic. Since $\varphi\colon T \to W \setminus K$ is finite and W is projective, we know that the field of meromorphic functions on T has transcendence degree 2 over $\mathbb{C}$. The meromorphic functions h_i/h_0 generate a field of transcendence degree 2 over $\mathbb{C}$. From this and the fact that $\widetilde{B} = \varphi^*\Delta$ we can further assume that M is so large that the image of h is contained in a projective algebraic surface $S \subset \mathbb{P}^R$. Let q be a general point in $h(T)$. By the argument above, we can find a multiple M' of M and sections $h_0', \dots, h_{R'}$ of $|M'\widetilde{B}|$ such that the analogous map $h'\colon T \to \mathbb{P}^{R'}$ is injective on the inverse image of q in T. It is clear that the map h' is bimeromorphic. Since φ is a finite map, by the projection formula no closed curve in T can contract to a point under h'. From these observations we conclude that $h'\colon T \to h'(T)$ is

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both finite and bimeromorphic. Further, $h'(T)$ is contained in a projective surface $S_n \subset \mathbb{P}^n$. Then the normalization Z of S_n is biholomorphic to T. Hence T can be embedded as an open subset in a normal projective surface.

By Hartog's theorem, the analytic map ψ extends to a proper analytic map with finite fibers from Z to W which is then also algebraic. Let S be the inverse image of Y in Z. Since $\widetilde{U} \to U$ is an étale map the morphism $S \to Y$ is ramified only at finitely many points in S. By the purity of branch locus it follows that the morphism $S \to Y$ is actually étale. Hence S is a smooth affine surface. We have a finite morphism $X \to S$ which induces a surjection $\pi_1'(X) \to \pi_1'(S)$. It follows that $\pi_1'(S)$ is finite cyclic. Since X is a $\mathbb{Q}$-homology plane so is S. Hence $e(S) = 1$. But the degree of the étale (finite) map $S \to Y$ is > 1. Hence $e(S) > 1$. This contradiction shows that $H = \pi_1'(Y)$.

Hence $\pi_1'(Y)$ is finite cyclic. Since Y is a $\mathbb{Q}$-homology plane, we conclude by [13, Theorem 2.10] that Y is an ML_0 surface. This completes the proof of the theorem. ∎

Remark The argument above proves the following general result. Let $f\colon X_1 \to X_2$ be a proper morphism between normal affine surfaces. Then there is a factorization of f in the form $X_1 \to S \to X_2$, where S is a normal affine surface and $X_1 \to S$ is a proper morphism such that $\pi_1'(S)$ is isomorphic to the image of the homomorphism $\pi_1'(X_1) \to \pi_1'(X_2)$.

We deduce the following result using the proof of the above theorem. This result was proved in [21] using the proof of the cancellation theorem for $\mathbb{C}^2$ and in [14] using the Mumford–Ramanujam method and Milnor's classification of finite groups acting freely on a homotopy 3-sphere.

Corollary 4.5 *Let Y be a smooth affine surface with a finite morphsim $f\colon \mathbb{C}^2 \to Y$. Then Y is isomorphic to $\mathbb{C}^2$.*

Proof Clearly $\mathbb{C}^2$ is an ML_0 surface with $\rho(\mathbb{C}^2) = 0$. The above proof shows that the fundamental group at infinity of Y is trivial. Hence by the well-known topological characterization of $\mathbb{C}^2$ [25], we know that Y is isomorphic to $\mathbb{C}^2$. ∎

Remark More generally, if $\mathbb{C}^2 \to Y$ is a proper morphism onto a normal affine surface Y, then we can show by similar arguments that Y is isomorphic to a quotient $\mathbb{C}^2/G$ for a finite group of automorphisms of $\mathbb{C}^2$ [14, 22].

Next we shall consider the ascent and descent of the ML_i-property for $i = 1, 2$. We have the following result.

Theorem 4.6 *Let $\varphi\colon X \to Y$ be an étale finite morphism. Then Y is ML_i $(i = 1, 2)$ if and only if X is also.*

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Proof Consider first the ML_1-property. Suppose that Y is ML_1. Then X is ML_1 or ML_0 by Lemma 4.1. If X is ML_0, then Y is ML_0 by Theorem 4.2. This is a contradiction. Hence X is ML_1. Conversely, suppose that X is ML_1. As in the proof of Theorem 4.2, there exists a Galois étale finite covering $\psi\colon Z \to X$ such that $\varphi \circ \psi\colon Z \to Y$ is a Galois étale covering. Since Z is ML_1 by what we have just proved, we may assume that $\varphi\colon X \to Y$ is a Galois étale finite covering with group G. Let $f\colon X \to B$ be an $\mathbb{A}^1$-fibration with $B \cong \mathbb{A}^1$. Since this $\mathbb{A}^1$-fibration is unique on X, the G-action preserves f. Namely, the G-translates of a fiber of f are again fibers of f. Hence f induces an $\mathbb{A}^1$-fibration $g\colon Y \to \mathbb{A}^1$. So, Y is ML_1 or ML_0. If Y is ML_0, then X is ML_0 by Theorem 4.2, a contradiction. Hence Y is ML_1.

The case for the ML_2-property follows readily if one uses the ascent and descent of ML_i-property for $i = 0, 1$. ■

Remark The ML_1-property does not descend under a ramified Galois covering morphism. In fact, let X be the hypersurface in $\mathbb{A}^3$ defined by $x^r y = z^n - 1$, where $r, n \geq 2$. Then X is ML_1. Meanwhile, the projection $(x, y, z) \mapsto (x, y)$ defines a ramified Galois covering morphism $X \to \mathbb{A}^2 = \operatorname{Spec} \mathbb{C}[x, y]$ with group $\mathbb{Z}/n\mathbb{Z}$. Hence the ML_1-property is not preserved.

5 Derksen Invariants and ML_0 Surfaces

We shall prove the following somewhat surprising result.

Theorem 5.1 *Let X be an ML_0 surface with $\rho(X) > 0$. Then there exists a surjective $\mathbb{A}^1$-fibration $X \to \mathbb{P}^1$.*

In order to prove this result, we introduce the *Derksen invariant* $\mathrm{Dk}(X)$ of an affine variety $X = \operatorname{Spec} R$, which is defined as the subalgebra of R generated over $\mathbb{C}$ by all $\operatorname{Ker} \delta$, where δ runs over the locally nilpotent derivations of R [6]. If X is a rational affine surface with $\gamma(X) = 0$, which we always assume tacitly, then $\operatorname{Ker} \delta$ is a polynomial ring $\mathbb{C}[u]$, where $\operatorname{Spec} \mathbb{C}[u]$ is the parameter space of the $\mathbb{A}^1$-fibration on X associated to δ. Note that an $\mathbb{A}^1$-fibration on X parametrized by $\mathbb{A}^1$ is always associated with a G_a-action on X, hence a locally nilpotent derivation on R. Hence $\mathrm{Dk}(X)$ is generated by all the elements u such that $\operatorname{Spec} \mathbb{C}[u]$ parametrizes an $\mathbb{A}^1$-fibration on X. We sometimes write $\mathrm{Dk}(R)$ instead of $\mathrm{Dk}(X)$ when R is the coordinate ring of X. We do not know if $\mathrm{Dk}(X)$ is finitely generated over $\mathbb{C}$. On the other hand, we also do not know of an ML_0 surface X with $Dk(X) \neq R$.

We shall first prove a preparatory result.

Lemma 5.2 *Let X be an ML_0 surface with $\rho(X) > 0$. Let $C_1, \ldots, C_r$ be affine lines such that $U := X - (C_1 \cup \cdots \cup C_r)$ is also an ML_0 surface. Assume that $\Gamma(U, \mathcal{O}_U)$ is integral over $\mathrm{Dk}(U)$. Then there exists an $\mathbb{A}^1$-fibration $\widetilde{f}\colon X \to \mathbb{P}^1$.*

Proof We shall prove the case $r = 1$ and write $C_1 = C$. The general case is treated in a similar way. Let $f\colon U \to B$ be an $\mathbb{A}^1$-fibration. Then it extends to an $\mathbb{A}^1$-fibration

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$\widetilde{f}: X \to \widetilde{B}$. If either $B \cong \mathbb{P}^1$ or $B \cong \mathbb{A}^1$ and $\widetilde{B} \cong \mathbb{P}^1$, then we are done. Suppose that all $\mathbb{A}^1$-fibrations on U are parametrized by $\mathbb{A}^1$ and such $\mathbb{A}^1$-fibrations extend to $\mathbb{A}^1$-fibrations on X parametrized by $\mathbb{A}^1$. Let $f: U \to B$ be an $\mathbb{A}^1$-fibration with $B \cong \mathbb{A}^1$ and let $\widetilde{F}: X \to B$ be its extension on X. Then C is contained in some fiber of $\widetilde{f}$. This implies that if $B = \operatorname{Spec} \mathbb{C}[u]$, the function u is constant along C. In particular, any function of $\mathrm{Dk}(U)$ is constant on C. On the other hand, $\Gamma(X, \mathcal{O}_X) \subset \Gamma(U, \mathcal{O}_U)$, and $\Gamma(U, \mathcal{O}_U)$ is integral over $\mathrm{Dk}(U)$ by the assumption. Hence $\Gamma(X, \mathcal{O}_X)$ is integral over $\mathrm{Dk}(U)$. Let ξ be any element of $\Gamma(X, \mathcal{O}_X)$. Then we have a monic relation

$$\xi^n + a_1\xi^{n-1} + \cdots + a_{n-1}\xi + a_n = 0,$$

where $a_1, \ldots, a_n \in \mathrm{Dk}(U)$, which are constants on C. Hence ξ is also a constant on C. This is a contradiction because any two points of C are separated by a function of $\Gamma(X, \mathcal{O}_X)$. ∎

We know by Lemma 1.8 that if X is an ML_0 surface with $\rho(X) > 0$, then there exists an affine line C such that $X - C$ is still an ML_0 surface. So, in order to prove Theorem 5.1, we have only to show the following result.

Lemma 5.3 *Let X be an ML_0 surface of $\rho(X) = 0$. Then $\Gamma(X, \mathcal{O}_X)$ is integral over* $\mathrm{Dk}(X)$.

Proof Let $\widetilde{X}$ be the universal covering of X. It is known [5, 17] that $\widetilde{X}$ is a hypersurface in $\mathbb{A}^3$ defined by an equation $xy = z^n - 1$ and that X is the quotient of $\widetilde{X}$ by the Galois group $G \cong \mathbb{Z}/n\mathbb{Z}$ which acts as $(x, y, z) \mapsto (\omega x, \omega^{-1} y, \omega^i z)$ for some $0 < i < n, \gcd(n, i) = 1$, where ω is a primitive n-th root of unity and G is identified with the multiplicative group $\{\omega^j | 1 \le j \le n\}$. Furthermore, $\widetilde{X}$ has two independent $\mathbb{A}^1$-fibrations $f_x: \widetilde{X} \to \mathbb{A}^1$ and $f_y: \widetilde{X} \to \mathbb{A}^1$ which are defined by locally nilpotent derivations δ_x and δ_y, respectively, where

$$\begin{array}{lll} \delta_x(x) = 0, & \delta_x(y) = nz^{n-1}, & \delta_x(z) = x, \\ \delta_y(x) = nz^{n-1}, & \delta_y(y) = 0, & \delta_y(z) = y. \end{array}$$

So $\mathbb{C}[x, y] \subseteq \mathrm{Dk}(\widetilde{X})$. Hence $\Gamma(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$ is integral over $\mathrm{Dk}(\widetilde{X})$.

On the other hand, the G-action maps a fiber $f_x^{-1}(a)$ to $f_x^{-1}(\omega a)$ if $a \neq 0$ and permutes the n lines $A_1, \ldots, A_n$ of $f_x^{-1}(0)$, where A_j is defined by $x = z - \omega^j = 0$, where $1 \le j \le n$. Hence the $\mathbb{A}^1$-fibration f_x descends down to an $\mathbb{A}^1$-fibration on X. So $x^n \in \mathrm{Dk}(X)$. Similarly, $y^n \in \mathrm{Dk}(X)$. Thus, $\mathbb{C}[x^n, y^n] \subseteq \mathrm{Dk}(X)$. Note that we have the natural inclusion $\mathrm{Dk}(X) \subseteq \mathrm{Dk}(\widetilde{X})$ because any G_a-action lifts to a G_a-action on $\widetilde{X}$.

Now we have the following inclusion relations.

$$\mathbb{C}[x^n, y^n] \subseteq \mathbb{C}[x, y] \subseteq \mathrm{Dk}(\widetilde{X}) \subseteq \Gamma(\widetilde{X}, \mathcal{O}_{\widetilde{X}}),$$

$$\mathbb{C}[x^n, y^n] \subseteq \mathrm{Dk}(X) \subseteq \Gamma(X, \mathcal{O}_X) \subseteq \Gamma(\widetilde{X}, \mathcal{O}_{\widetilde{X}}).$$

Since $\Gamma(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$ is integral over $\mathbb{C}[x, y]$, so also over $\mathbb{C}[x^n, y^n]$. Hence $\Gamma(X, \mathcal{O}_X)$ is integral over $\mathrm{Dk}(X)$. ∎

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6 ML_0 Surfaces Not Containing $\mathbb{C}^2$

Let X be an ML_0 surface. In Theorem 2.1 and Theorem 2.4, we saw that there is a difference between the cases $\rho(X) = 0$ and $\rho(X) > 0$. In this section we shall point out some more differences in these cases.

We recall that a smooth algebraic surface X is isomorphic to the affine plane $\mathbb{A}^2$ if and only if $\mathrm{Pic}(X) = (0), \gamma(X) = 0$ and $\overline{\kappa}(X) = -\infty$ [20]. This characterization of the affine plane is equivalent to saying that a smooth affine surface is isomorphic to $\mathbb{A}^2$ if and only if X is $\mathrm{ML}_0, \mathrm{Pic}(X)_{\mathrm{tor}} = (0)$ and $\rho(X) = 0$. Hence one can ask if a smooth affine surface X contains $\mathbb{A}^2$ as an open set provided X is ML_0 and $\mathrm{Pic}(X)_{\mathrm{tor}} = (0)$. It will be shown later that the answer is negative. In order to construct a counterexample, we shall describe birational transformations of smooth completions of X which do not affect X. For this purpose, we employ the terminology and notation in §1 after Lemma 1.6 and the beginning of §3.

Let D be a weighted linear chain. We denote by $w(D_i)$ (or simply w_i) the weight of a component D_i of D. Let $Q(D)$ be the intersection form of the components of D. Namely, if D is a linear chain $D_1 - D_2 - \cdots - D_n$, then $Q(D)$ is an $(n \times n)$-matrix such that its (i, j)-entry is given by $w(D_i)$ (resp. 1 or 0) if $i = j$ (resp. $j = i \pm 1$ or otherwise). Let $d(D) = \det(-Q(D))$. Given a tip D_1 of D, we put $(d, d') = (d(D), d(D'))$, where $D' = D - D_1$ and call it the *pair* of D seen from D_1. We have the following result, which can be verified as an easy exercise.

Lemma 6.1 *Let D be a weighted linear chain. The following assertions then hold.*

(i) *Suppose that D is admissible,* i.e., $w(D_i) \le -2$ *for all i. With the above notations, we have $d' < d$ and* $\gcd(d, d') = 1$. *Let D_n be the tip of D on the opposite side of D_1 and let $d'' = d(D'')$ with $D'' = D - D_n$. Then $d'd'' \equiv 1 \pmod{d}$* [7, Lemma 3.6, (2)].

(ii) *Suppose that D has the form $U - E - V$, where U, V are linear chains meeting the component E at their end components. Let (u, u') (resp. (v, v')) be the pair of U seen from E, where u' (resp. v') is $d(U')$ (resp. $d(V')$) with U' (resp. V') being U (resp. V) minus the end component adjacent to E. Let $w = w(E)$. Then we have*

$$d(D) = -wuv - uv' - u'v.$$

(cf. [31, Lemma 3.1]*).*

(iii) *Let c, p be relatively prime integers with $1 \le p < c$. Let Q be a point on a* (0)*-curve ℓ on a smooth projective surface. Apply the Euclidean transformation σ with center Q with respect to the data* (c, p) [18] *and let D be the linear chain consisting of irreducible components of $\sigma^*(\ell)$,*

$$D = B - E - A$$

where E is the last (-1) *curve and B contains the proper transform $\sigma'(\ell)$ as the end component. More precisely, let $c/p = [b_1, b_2, \ldots, b_n]$ be the expansion of c/p as a*

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continued fraction

$$\frac{c}{p} - b_1 - \cfrac{1}{b_2 - \cfrac{1}{b_3 - \cfrac{1}{\ddots - \cfrac{1}{b_n}}}}$$

and let $c/(c-p) = [a_1, a_2, \dots, a_m]$. *Then* D *is of the form*

$$\begin{array}{c}
\underset{\sigma'(\ell)}{(-b_1)} — (-b_2) — \cdots — (-b_n) \\
\hspace{8em}| \\
\hspace{8em}(-1) \\
\hspace{8em}| \\
(-a_1) — (-a_2) — \cdots — (-a_m)
\end{array}$$

where B, A *have respectively the pairs* (c, p) *and* $(c, c-p)$, *seen from the tips of* B *and* A *with weights* $-b_1$ *and* $-a_1$.

With the notations of Lemma 1.7, we have the following result.

Lemma 6.2 *Let* $D = \ell - M - A$ *be a standard chain with* $(M^2) = n_0$. *Assume that* A *has the pair* (a, a') *seen from* M. *Suppose we blow up a point* $Q \in \ell$ *not on* M *according to the pair* (c, p), i.e., *the Euclidean transformation with center* Q *and data* (c, p). *Let* E *be the last* (-1) *curve and let* D' *be the resulting weighted chain (containing* M *and* A*). Let* D^* *be the chain obtained from* D' *by changing the weight of* E *from* -1 *to* 0. *Then we have* $d(D^*) = -a + c(n_0ca + ca' + pa)$.

Proof The chain D' is of the form $U - E - V$ with $w(E) = -1$. Let U, V have pairs $(u, u'), (v, v')$ seen from E. Then we may assume that V has the form $W - M - A$, where W has the pair (c, p) seen from M and $(M^2) = n_0$. Hence

$$v = d(V) = -n_0ca - ca' - pa$$

by Lemma 6.1. We have also $u = d(U) = c$. Now $d(D') = uv - uv' - u'v$ and $d(D^*) = -uv' - u'v$, whence $d(D^*) = d(D') - uv$. On the other hand, $d(D) = -a$ and $d(D') = d(D)$ since D' is obtained by a sequence of blowing-ups from D. The result then follows readily. ■

We shall now fix the setting. *Assume that an* ML_0 *surface* X *has* $\rho(X) = 1$ *and contains an open set* U *which is isomorphic to* $\mathbb{A}^2$. Then we have the following.

(i) $X - U$ is an irreducible curve C which is isomorphic to $\mathbb{A}^1$ by Lemma 1.6.

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(ii) Let $f\colon X \to \mathbb{A}^1$ be an $\mathbb{A}^1$-fibration and $\overline{X}$ be a standard completion of X with boundary $D = \ell - M - A$ and $\mathbb{P}^1$-fibration $\overline{f}\colon \overline{X} \to \mathbb{P}^1$ induced by ℓ which extends f. Then A has two feathers A_1, A_2 of respective multiplicities m_1, m_2. Since X contains $\mathbb{A}^2$, it is simply connected and hence $\operatorname{Pic}(X)_{\mathrm{tor}} = (0)$. So, $\gcd(m_1, m_2) = 1$ by Lemma 1.6. Let (a, a') be the pair of A seen from M.

(iii) We assume that for any standard completion $\overline{X}$ of X as above we have $\min(m_1, m_2) > 1$.

(iv) $\operatorname{Pic}(X - A_1) \cong \mathbb{Z}/m_2\mathbb{Z}$ and $\operatorname{Pic}(X - A_2) \cong \mathbb{Z}/m_1\mathbb{Z}$. This implies that $A_i \neq C$ for $i = 1, 2$.

Our aim is to show that the above assumption is realizable for a special choice of the linear chain A and leads to a contradiction. We proceed in several steps.

Claim 6.3 *Let $\overline{C}$ be the closure of C in $\overline{X}$.*

(i) *Then $\overline{C}$ meets ℓ in a single point Q which is a one-place point of $\overline{C}$.*
(ii) *After suitable elementary transformations on ℓ, we may assume that $Q \neq \ell \cap M$.*
(iii) *Let $\widetilde{c} = (\overline{C} \cdot \ell)$ and let $\widetilde{p}$ be the multiplicity of $\overline{C}$ at Q. It cannot occur that $\widetilde{c} = \widetilde{p} = 1$. We may further assume that $\widetilde{c} > \widetilde{p}$.*

Proof (i) $\overline{C} - C$ is a one-place point of $\overline{C}$ and hence $\overline{C}$ meets D in one point. If $(\overline{C} \cdot \ell) = 0$, then C is a fiber component of f. Since $C \neq A_i$ for $i = 1, 2$, C is a smooth fiber of f. Hence there is a surjective morphism from $\mathbb{A}^2 \to \mathbb{C}^*$. This is a contradiction. So, $\overline{C}$ meets ℓ.

(ii) If $Q = \ell \cap M$, then perform elementary transformations with center Q and its infinitely near points lying on M until the proper transform of $\overline{C}$ is separated from the proper transform of M.

(iii) Suppose $\widetilde{c} = \widetilde{p} = 1$. Then C cannot meet A_1 and A_2 because $m_1 > 1$ and $m_2 > 1$. Since $\overline{C}$ meets ℓ, $\overline{C}$ does not meet A either, hence the fiber of $\overline{f}$ supported by $A + A_1 + A_2$. This is a contradiction. If $\widetilde{c} = \widetilde{p}$ and $\widetilde{p} > 1$, perform blowing-ups with centers at Q and its infinitely near points lying on $\overline{C}$ until the proper transform of $\overline{C}$ meets the last (-1) curve, say E, with the intersection number greater than the multiplicity of singularity. Then contract the proper transform of ℓ and all the exceptional curves but E. Then we have $\widetilde{c} > \widetilde{p}$. ■

We call such a standard completion $\overline{X}$ *normalized* for C and let $n_0 = (M^2)$. Let $\Gamma = D + \overline{C}$, which is the boundary divisor of $U \cong \mathbb{A}^2$ in $\overline{X}$. Let $c = \widetilde{c}/\gcd(\widetilde{c}, \widetilde{p})$ and $p = \widetilde{p}/\gcd(\widetilde{c}, \widetilde{p})$. Let $\widetilde{\Gamma} = \widetilde{D} + \widetilde{C}$ be the boundary divisor of U which results from the Euclidean transformation with center Q with respect to the pair (c, p) and let E be the last (-1) curve. Then $\widetilde{C}$ meets E in a point not on the proper transform of ℓ or any other exceptional curves. Further, let $\Gamma^* = D^* + C^*$ be the boundary divisor obtained by the minimal blowing-ups such that the proper transform of Γ together with the exceptional curves has only simple normal crossings. This process involves the above Euclidean transformation. We then have the following two possibilities:

(i) $\widetilde{C}$ is a smooth curve meeting normally E and $(\widetilde{C}^2) = -1$.

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(ii) Either $\widetilde{C}$ is singular or $\widetilde{C}$ is a smooth curve meeting E normally with $(\widetilde{C}^2) \neq -1$.

Suppose the second case above occurs. Then we have the following.

Claim 6.4

(i) *Write $\widetilde{D}$ as $W + E + B_3$, where W and B_3 are subchains and W contains $M + A$. Then $n_0 = (M^2) = -1$ and W is contractible to a smooth point.*

(ii) *Suppose that $\widetilde{C}$ is singular. By blowing up, if necessary, the point $E \cap \widetilde{C}$ (resp. $E \cap B_3$) and its infinitely near points lying on $\widetilde{C}$ (resp. B_3) and contracting the proper transform of E and a part of exceptional curves, we have a new standard normalized completion $\overline{X}'$ with a standard boundary $D' = \ell' - M' - A'$ with the image of $\widetilde{C}$ (still possibly singular but the singularity better than C) meeting ℓ' at a point Q' not on M'.*

(iii) *In the case* (ii) *above, the linear chain A' is the transpose ${}^tA$ of A, which is, by definition, the same A read from the other tip not meeting M.*

(iv) *The case where $\widetilde{C}$ is smooth with $(\widetilde{C}^2) \neq -1$ cannot occur.*

Proof (i) In the divisor Γ^*, the proper transform of E is a branching component with three branches B_1, B_2, B_3, where B_1 contains the proper transform of ℓ, B_2 contains C^* and B_3 is an admissible chain. In fact, it is always the case that the last (-1)-curve of the Euclidean transformation with data (c, p) is a branching component with the image of C as one branch if $c > p$. Note that since $\Gamma^* + C^*$ is the boundary divisor with simple normal crossings it is minimalized to a linear chain by a theorem of Ramanujam [25]. Hence one of B_1 and B_2 is contractible to a point. Suppose B_2 is contractible. Then C^* is a (-1) curve meeting a component of B_2 normally which becomes a 0-curve after the contraction of C^*. This is a contradiction to the assumption that B_2 is contractible. Hence B_1 is contractible in Γ^* and already contractible in $\widetilde{\Gamma}$. In particular, $n_0 = -1$. In fact, we blow up at least twice on ℓ, so only M can be a (-1) curve in B_1.

(ii) After the contraction of B_1, the image of $\widetilde{\Gamma}$ is of the form $\Gamma' = D'' + \widetilde{C}$, where D'' is a chain with the image E' of E as a tip and $({E'}^2) \geq 0$. If $({E'}^2) > 0$, we blow up the point $E' \cap B_3$ and its infinitely near points lying on E' until the proper transform ℓ' of E' becomes a (0)-curve. Let B_3' be the inverse image of B_3 at this stage and M' the component of B_3' meeting ℓ' and $A' = B_3' - M'$, the rest of B_3'. Then $D' = \ell' - M' - A'$ is the boundary divisor of a standard completion of X. After performing an elementary transformation involving ℓ' if necessary, we may assume that it is normalized for C, *i.e.,* we have $(\overline{C}' \cdot \ell') >$ (multiplicity of $\overline{C}'$ at $\overline{C}' \cap \ell'$), where $\overline{C}'$ is the proper transform of $\widetilde{C}$.

(iii) We refer to [4, Theorem 3.12].

(iv) When $\widetilde{C}$ is smooth with $(\widetilde{C}^2) \neq -1$, the component E is also a branching component of $\widetilde{\Gamma}$ and the branch B_1 contracts to a point. As in the case (ii), we have a standard completion $\overline{X}'$ of X with the boundary divisor $D' = \ell' - M' - A'$, where $\widetilde{C}$ meets normally ℓ'. This contradicts (iii) in Claim 6.3. ■

We call the operation of changing the standard normalized completions $\overline{X} \to \overline{X}'$ a *flip*. After repeating flips several times, we reach to the case (i) described after

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Claim 6.3. Namely, we may assume that $\widetilde{C}$ is a (-1) curve meeting E normally. Contract $\widetilde{C}$ to a point and obtain the image of E having intersection number 0. Let $\overline{\Gamma}$ be the image of $\widetilde{\Gamma}$. Then $\overline{\Gamma}$ is a linear chain which is the boundary divisor of $\mathbb{A}^2$ with simple normal crossings. Hence we have $d(\overline{\Gamma}) = \pm 1$. By Lemma 6.2, we have one of the following two equalities, where $n_0 = (M^2)$:

$$a \pm 1 = c((n_0 c + p)a + ca'), \quad a \pm 1 = c((n_0 c + p)a + ca'').$$

Construction of a counterexample. We construct an ML_0 surface X as in Lemma 1.7 by constructing a chain

$$\begin{array}{ccccccc} (-3) & — & (-1) & — & (-2) & — & (-2) \\ \ell_0 & & E_3 & & E_2 & & E_1 \end{array}$$

and attaching feathers A_1, A_2 to E_3 and E_2 of multiplicities 3 and 2, respectively. The corresponding chain A is now

$$\begin{array}{ccccccc} (-3) & — & (-2) & — & (-3) & — & (-2) \\ \ell_0' & & E_3' & & E_2' & & E_1' \end{array}$$

with $a = 19$, $a' = 8$, and $a'' = 12$. It is readily verified that the only way to attach feathers to A or its transpose $A' = {}^tA$ so as to produce a complete fiber in a $\mathbb{P}^1$-fibration is to attach them to E_3 and E_2. Hence the requirements (ii) and (iii) before Claim 6.3 are satisfied. It is also readily verified that there are no solutions of c, n_0 and p satisfying one of the above equalities. So, X does not contain $\mathbb{A}^2$. On the other hand, $\mathrm{Pic}(X) \cong \mathbb{Z}$ is torsion-free. Thus we have constructed an example of an ML_0 surface X with $\mathrm{Pic}(X)_{\mathrm{tor}} = (0)$ and $\rho(X) = 1$ which does not contain an open set isomorphic to $\mathbb{A}^2$.

Recall that an $\mathbb{A}^1$-fibration on an ML_0 surface with Picard number zero has at most one multiple fiber (which is irreducible) and the multiplicity is the order of the Picard group. Hence this multiplicity is invariant for any $\mathbb{A}^1$-fibration on the surface. This is not the case for the multiplicities of the components of singular fibers of an ML_0 surface with positive Picard number. We exhibit this by giving an example.

Example 6.5 Let C be a smooth conic on $\mathbb{P}^2$ and let $Q \in C$ be a point. Let $\sigma\colon V_0 \to \mathbb{P}^2$ be the blowing-up of Q with $M_0 := \sigma^{-1}(Q)$. The surface V_0 is the Hirzebruch surface of degree 1, M_0 is the minimal section and the standard $\mathbb{P}^1$-fibration $p_0\colon V_0 \to \mathbb{P}^1$ is given by the pencil of lines through Q. Let $C' = \sigma'(C)$ and let $X = V_0 - C'$. Then the following assertions hold.

(i) X is an ML_0 surface with $\rho(X) = 1$ and $\mathrm{Pic}(X)_{\mathrm{tor}} = (0)$. Furthermore, X is a half-point attachment of $\mathbb{P}^2 - C$. In particular, $\mathbb{P}^2 - C$ is an open set of X.

(ii) $f_0 := p_0|_X : X \to \mathbb{P}^1$ is an $\mathbb{A}^1$-fibration such that every fiber is irreducible and reduced.

(iii) Let $P \in C$ be a point other than Q and let T be the tangent line to C at P. The pencil generated by $2T$ and C defines an $\mathbb{A}^1$-fibration on $\mathbb{P}^2 - C$ which extends to an $\mathbb{A}^1$-fibration $f_1\colon X \to \mathbb{P}^1$. Then f_1 has one singular fiber which is irreducible of multiplicity 2 and all other fibers are irreducible and reduced.

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(iv) There exists an irreducible quintic Y on $\mathbb{P}^2$ which has six consecutive cusps of multiplicity 2 infinitely near to the point P with five of them lying on C [29]. The pencil generated by $2Y$ and $5C$ induces an $\mathbb{A}^1$-fibration on $\mathbb{P}^2 - C$ which extends to an $\mathbb{A}^1$-fibration $f_2 \colon X \to \mathbb{P}^1$ such that $Y' := \sigma(Y) \cap X$ and $M_0' := M_0 \cap X$ support multiple fibers of f_2 with respective multiplicities 2 and 5 and all other fibers are irreducible and reduced.

The proofs are straightforward.

Remark The construction of X in Example 6.5 can be generalized as follows. By a result in [3], a smooth affine surface $\Sigma_s - S$, where Σ_s is the Hirzebruch surface of degree s and S is an ample section with $(S^2) = k+1$, contains an open set U which is an ML_0 surface with $\mathrm{Pic}(U) = \mathbb{Z}/k\mathbb{Z}$. Concretely, U can be realized as the complement of the zero locus C of a weighted homogeneous poynomial $f := X_0X_2 - X_1^{k+1}$ in the weighted projective plane $\mathbb{P}$ with weight $(1, 1, k)$ and $\Sigma_s - S$ as the complement of the proper transform C' of C in the blowing-up of $\mathbb{P}$ at the (singular) point $Q = (0, 0, 1)$. Again, there exist $\mathbb{A}^1$-fibrations of $\mathbb{P}$ with C and another curve Y as singular fibers.

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