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EXAMPLES OF FACTORIAL RINGS IN ALGEBRAIC GEOMETRY

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ABSTRACT. We show that the ring of complex-valued regular functions on an affine irreducible nonsingular real algebraic variety X is factorial if $\dim X = 1$ or $\dim X = 2$ and X has no compact connected components or X is compact and the second cohomology group of X with integral coefficients vanishes.

1. **Introduction**. In this note algebraic varieties and regular maps between them are understood in the sense of Serre [10]. Given a real algebraic variety X we denote by $\mathcal{R}(X)$ the ring of real-valued regular functions on X. Factoriality of this ring has been studied in [5] and [11]. Our purpose here is to investigate factoriality of the ring of complex-valued regular functions on X, i.e., the ring

$$\mathcal{R}(X, \boldsymbol{C}) = \{f + ig \mid f, g \in \mathcal{R}(X),\ i^2 = -1\}.$$

Note that if Y is an algebraic subvariety of $\mathbb{R}^n$, $A(Y)$ its coordinate ring (i.e., the ring of polynomial functions on Y) and X a Zariski open subset of Y, then

$$\mathcal{R}(X) = \{f/g \mid f, g \in A(Y),\ g^{-1}(0) \cap X = \emptyset\}.$$

In this setting the ring $\mathcal{R}(X, \boldsymbol{C})$ can be described as follows. Let $Y_{\boldsymbol{C}} \subset \boldsymbol{C}^n$ be the complexification of Y and let $A(Y_{\boldsymbol{C}})$ be the coordinate ring of $Y_{\boldsymbol{C}}$. Then $\mathcal{R}(X, \boldsymbol{C})$ is naturally isomorphic to the localization of $A(Y_{\boldsymbol{C}})$ with respect to the multiplicatively closed set

$$S = \{f \in A(Y_{\boldsymbol{C}}) \mid f^{-1}(0) \cap X = \emptyset\}.$$

Our main result is the following.

THEOREM 1.1. *Let X be an affine nonsingular irreducible real algebraic variety. Then the ring $\mathcal{R}(X, \boldsymbol{C})$ is factorial in each of the following cases:*

(a) $\dim X = 1$;

(b) $\dim X = 2$ and X has no compact connected components;

(c) X is compact with $H^2(X, \boldsymbol{Z}) = 0$.

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Note that, in general, the ring $\mathcal{R}(X, \boldsymbol{C})$ is not factorial. In particular, the assumption "X has no compact connected components" in (b) is essential. Indeed, set

$$S^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} | x_1^2 + \ldots + x_{n+1}^2 = 1\}.$$

The examples below can be easily verified by combining results of section 2 and [7].

EXAMPLE 1.2. The ring $\mathcal{R}(S^2, \boldsymbol{C})$ is not factorial. Its divisor class group is isomorphic to $\boldsymbol{Z}$.

EXAMPLE 1.3. The ring $\mathcal{R}(S^1 \times S^1, \boldsymbol{C})$ is factorial but there exists an affine nonsingular real algebraic variety X diffeomorphic to $S^1 \times S^1$ such that the ring $\mathcal{R}(X, \boldsymbol{C})$ is not factorial.

EXAMPLE 1.4. Let X be an affine nonsingular real algebraic surface. Assume that X is compact connected nonorientable and has odd genus (as a smooth manifold). Then the ring $\mathcal{R}(X, \boldsymbol{C})$ is not factorial. Its divisor class group is isomorphic to $\boldsymbol{Z}/2\boldsymbol{Z}$.

Some information about the factoriality of the affine coordinate ring of a complex variety can be deduced from properties of the real part of this variety. More precisely we have the following.

COROLLARY 1.5. *Let $Z \subset \boldsymbol{C}^n$ be a nonsingular irreducible complex algebraic surface. Assume that $Z \cap \mathbb{R}^n$ is a compact connected nonorientable nonsingular real algebraic surface of odd genus. Then the coordinate ring of Z is not factorial.*

PROOF. The localization homomorphism $A(Z) \to A(Z)_S$, where

$$S = \{f \in A(Z) | f^{-1}(0) \cap \mathbb{R}^n = \emptyset\}$$

composed with the natural isomorphism $A(Z)_S \to \mathcal{R}(X, \boldsymbol{C})$ induce the epimorphism $Cl(A(Z)) \to Cl(\mathcal{R}(X, \boldsymbol{C}))$ of the divisor class groups [8]. By Example 1.4, the group $Cl(A(Z))$ is nontrivial and hence $A(Z)$ cannot be factorial.

Note that here the terms "compact", "connected", etc. refer to the strong (metric) topology on an algebraic variety.

Theorem 1.1 is proved in section 2, where also the Picard group is studied.

2. **The Picard Group of $\mathcal{R}(X, \mathbf{C})$.** We say that a triple $\xi = (E, \pi, X)$, where E, X are real algebraic varieties and $\pi : E \to X$ is a regular map, is a complex algebraic vector bundle of rank k if the following conditions are satisfied:

(i) For each x in X the fiber $\pi^{-1}(x)$ is a k-dimensional complex vector space.

(ii) For each point x in X there exist a Zariski neighborhood U of x and a regular isomorphism $\varphi : \pi^{-1}(U) \to U \times \boldsymbol{C}^k$ such that for each point y in U, φ induces a $\boldsymbol{C}$-linear isomorphism between $\pi^{-1}(y)$ and $\{y\} \times \boldsymbol{C}^k$.

Here $\boldsymbol{C}$ is considered as a real algebraic variety. The notions of a morphism or an isomorphism of complex algebraic vector bundles can be defined in the usual way. Vector bundles of rank 1 will be called line bundles.

A complex algebraic vector bundle $\xi = (E, \pi, X)$ is said to be strongly algebraic if there exists an algebraic complex vector bundle η over X such that the Whitney sum $\xi \oplus \eta$ is algebraically isomorphic to a product vector bundle $X \times \boldsymbol{C}^m$ for some m and if X is an affine variety. There are several equivalent definitions (cf. [2], [3], [4], [6], where mainly real vector bundles are considered but most proofs can be trivially modified to cover the complex case too). In this note we shall make use of the following.

PROPOSITION 2.1 *(cf. [2]). Every smooth section of a strongly algebraic complex vector bundle ξ over a compact affine nonsingular real algebraic variety X can be approximated in the C^∞ topology by regular sections. More precisely, if s is a smooth section of ξ, F a finite subset of X, and $\{k_x\}_{x \in F}$ a collection of nonnegative integers, then every neighborhood of s contains a regular section u whose k_x-jet at x is the same as the k_x-jet at x of s for all x in F.*

PROOF. By definition, one can find an algebraic complex vector bundle η over X such that $\xi \oplus \eta$ is algebraically isomorphic to a product vector bundle $X \times \boldsymbol{C}^m$. Denote by z the zero section of η. The Weierstrass theorem implies that there exists a regular section of $\xi \oplus \eta$ arbitrarily close to $s \oplus z$ and whose k_x-jet at x is the same as the k_x-jet of $s \oplus z$ at x for all x in F. The conclusion follows since the projection $\xi \oplus \eta \to \xi$ is an algebraic morphism. □

REMARK 2.2. If one drops the assumption that X is nonsingular, then every continuous section of ξ can be approximated by regular sections in the C^0 topology.

Given an affine real algebraic variety X we denote by $V(X, \boldsymbol{C})$ the semiring of algebraic isomorphism classes of complex strongly algebraic vector bundles over X with addition and multiplication induced by the Whitney sum and tensor product of vector bundles, respectively. It will be convenient to compare $V(X, \boldsymbol{C})$ with the semiring $Proj(\mathcal{R}(X, \boldsymbol{C}))$ of isomorphism classes of finitely generated projective $\mathcal{R}(X, \boldsymbol{C})$-modules. Note that the map φ_X which assigns to the isomorphism class $[\xi]$ of a vector bundle ξ the isomorphism class $[\Gamma(\xi)]$ of the $\mathcal{R}(X, \boldsymbol{C})$-module of regular global sections of ξ is a homomorphism of semirings from $V(X, \boldsymbol{C})$ into $Proj(\mathcal{R}(X, \boldsymbol{C}))$.

PROPOSITION 2.3. *The map φ_X is an isomorphism.*

PROOF. With any finitely generated projective $\mathcal{R}(X, \boldsymbol{C})$-module P we can associate, in the usual way, an algebraic complex vector bundle $\xi(P) = (E(P), \pi(P), X)$, whose fiber over x in X is equal to P/M_xP, where M_x is the maximal ideal of $\mathcal{R}(X, \boldsymbol{C})$ of all functions vanishing at x. Since the bundles $\xi(P_1 \oplus P_2)$ and $\xi(P_1) \oplus \xi(P_2)$ are naturally isomorphic, $\xi(P)$ is a strongly algebraic vector bundle. We claim that the natural homomorphism $h(P): P \to \Gamma(\xi(P))$, defined by $(h(P)(e))(x) = e + M_xP$ for e in P and x in X, is bijective. Indeed, injectivity of $h(P)$ is obvious. To show surjectivity, pick up a finitely generated $\mathcal{R}(X, \boldsymbol{C})$-module Q such that $P \oplus Q$ is a free module. Clearly, $h(P \oplus Q)$ is an isomorphism. Hence also $h(P) \oplus h(Q): P \oplus Q \to \Gamma(\xi(P))$

$\oplus\, \Gamma(\xi(Q))$ is an isomorphism and the claim follows. By construction, $[P] \rightarrow [\xi(P)]$ is the inverse map of φ_X. □

REMARK 2.4. Bijectivity of $h(P)$ is equivalent to bijectivity of the natural homomorphism from P into the $\mathcal{R}(X, \boldsymbol{C})$-module of sections of the sheaf over X whose stalk over x in X is equal to $P \otimes \mathcal{R}(X, \boldsymbol{C})_{M_x}$. Note that, contrary to the familiar situation from algebraic geometry over an algebraically closed field, the last statement is not true for an arbitrary finitely generated $\mathcal{R}(X, \boldsymbol{C})$-module [4].

Let $V^1(X, \boldsymbol{C})$ be the subset of $V(X, \boldsymbol{C})$ of isomorphism classes of line bundles. Clearly, $V^1(X, \boldsymbol{C})$ is a group with multiplication.

COROLLARY 2.5. *For every affine real algebraic variety X, φ_X induces an isomorphism from $V^1(X, \boldsymbol{C})$ onto the Picard group $Pic(\mathcal{R}(X, \boldsymbol{C}))$ of $\mathcal{R}(X, \boldsymbol{C})$. In particular, the ring $\mathcal{R}(X, \boldsymbol{C})$ is factorial if and only if $V^1(X, \boldsymbol{C})$ is a trivial group, provided that X is irreducible and nonsingular.*

PROOF. Obvious. □

COROLLARY 2.6. *Let U be a Zariski open subset of an affine irreducible nonsingular real algebraic variety X. Then the homomorphism*

$$\rho: V^1(X, \boldsymbol{C}) \rightarrow V^1(U, \boldsymbol{C})$$

induced by the restriction of vector bundles is surjective.

PROOF. Since the diagram of natural homomorphisms

$$\begin{array}{ccc} V^1(X, \boldsymbol{C}) & \xrightarrow{\rho} & V^1(U, \boldsymbol{C}) \\ \downarrow & & \downarrow \\ Pic(\mathcal{R}(X, \boldsymbol{C})) & \rightarrow & Pic(\mathcal{R}(U, \boldsymbol{C}) \end{array}$$

is commutative and vertical arrows are isomorphisms, it suffices to observe that $\mathcal{R}(U, \boldsymbol{C})$ is naturally isomorphic to the localization of $\mathcal{R}(X, \boldsymbol{C})$ with respect to the multiplicatively closed subset

$$S = \{f \in \mathcal{R}(X, \boldsymbol{C}) | f^{-1}(0) \subset X - U\}$$

and apply [1], p. 144. □

Now we are ready to prove Theorem 1.1.

PROOF OF THEOREM 1.1. Proofs of (a) and (b) are similar so we shall only show (b) which is more interesting. We may assume that X is an algebraic subset of $\mathbb{R}^n$. Let S^n be the unit n-sphere and let $p: S^n - \{a\} \rightarrow \mathbb{R}^n$ be the stereographic projection from the "north pole" $a = (0, \ldots, 0, 1)$. Denote by X^* the Zariski closure of $p^{-1}(X)$ in $\mathbb{R}^{n+1}$. Topologically X^* is the one point compactification of X and $X^* - \{a\}$ is algebraically isomorphic to X. Let $\pi: \tilde{X} \rightarrow X^*$ be the Hironaka desingularisation of X^* [9]. Then $\tilde{X}$ is a compact affine nonsingular real algebraic variety, $\pi^{-1}(a)$ is a union of finitely many nonsingular real algebraic curves in $\tilde{X}$, and the induced map $\pi | \tilde{X} - \pi^{-1}(a): \tilde{X}$

$-\pi^{-1}(A) \to X^* - \{a\}$ is a regular isomorphism. Clearly, the intersection of any connected component of $\tilde{X}$ and $\pi^{-1}(a)$ is an infinite set. By Corollaries 2.5 and 2.6, it suffices to prove that given a complex strongly algebraic line bundle ξ over $\tilde{X}$ its restriction to $\tilde{X} - \pi^{-1}(a)$ is algebraically trivial. Take a real analytic section s of ξ transversal to the zero section. Since $\tilde{X}$ is compact and $\dim \tilde{X} = \operatorname{rank}_R \xi$, the set $Z = \{x \in X | s(x) = 0\}$ is finite. By [6] there exists a real analytic diffeotopy $\sigma_t : \tilde{X} \to \tilde{X}$, $t \in [0, 1]$, of $\tilde{X}$ such that $\sigma_1^{-1}(Z) \subset \pi^{-1}(a)$. Since the pullback vector bundle $\sigma_1^* \xi$ is analytically isomorphic to ξ and the set of zeros of the pullback section $\sigma_1^* s$ is equal to $\sigma_1^{-1}(Z)$, we may assume that Z is contained in $\pi^{-1}(a)$. Fix a positive integer k. One can find a regular section u of ξ which is arbitrarily close to s in the C^∞ topology and has the same k-jet as s at each point in Z. If k is sufficiently large, then the set of zeros of u is equal to Z and hence the restriction of ξ to $\tilde{X} - \pi^{-1}(a)$ is algebraically trivial.

(c). We have to show that $V^1(X, \boldsymbol{C})$ is a trivial group. Note that the natural homomorphism ψ_X from $V^1(X, \boldsymbol{C})$ into $H^2(X, \boldsymbol{Z})$ induced by the map which assigns to each complex vector bundle over X its first Chern class is injective. Indeed, if the isomorphism class of a complex strongly algebraic line bundle ξ is sent, via ψ_X, onto 0, then ξ is topologically trivial. Hence, by Remark 2.2, ξ is also algebraically trivial and ψ_X is a monomorphism. Since $H^2(X, \boldsymbol{Z}) = 0$, the conclusions follow. □

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