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Families of finite sets satisfying an intersection condition

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The following theorem is proved.

Let X *be a finite set of cardinality* $n \geq 2$, *and let* $\mathcal{F}$ *be a family of subsets of* X . *Suppose that for* $F_1, F_2, F_3 \in \mathcal{F}$ *we have* $|F_1 \cap F_2 \cap F_3| \geq 2$. *Then* $|\mathcal{F}| \leq 2^{n-2}$ *with equality holding if and only if for two different elements* x, y *of* X , $\mathcal{F} = \{F \subseteq X \mid x \in F, y \in F\}$.

1. Introduction

Let i, j, n, r be positive integers, $n \geq 2$. Let $[i, j]$ denote the set of integers k, $i \leq k \leq j$. Set $X = [1, n]$.

For any pair of non-negative integers t, h , $t \geq 2$, define

$$\mathcal{F}(n, t, h) = \{F \subseteq X \mid |F \cap [1, r+th]| \geq r+(t-1)h\} .$$

Then for $F_1, \ldots, F_t \in \mathcal{F}(n, t, h)$ we have $|F_1 \cap \ldots \cap F_t| \geq r$. Erdös and the author have made the following conjecture.

CONJECTURE. Let $\mathcal{F}$ be a family of subsets of X . If for any $F_1, \ldots, F_t \in \mathcal{F}(n, t, h)$ we have $|F_1 \cap \ldots \cap F_t| \geq r$, then $|\mathcal{F}| \leq \max_h |\mathcal{F}(n, t, h)|$.

The case $r = 1$ is trivial (*cf.* Erdös, Ko, and Rado [1]). For the case $t = 2$ and r arbitrary, the validity of the conjecture follows from Katona [4].

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So the first open case is $t = 3$, $r = 2$. The aim of this paper is to establish the conjecture for this case.

In Frankl [3] the conjecture is proved for $r \le \frac{2^t t}{150}$. We need two preliminary results.

LEMMA 1. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be collections of subsets of* X. *Suppose that for* $A \in \mathcal{A}$, $B \in \mathcal{B}$, $A \cap B \ne \emptyset$. *Then*

(1) $$|\mathcal{A}| + |\mathcal{B}| \le 2^n .$$

Proof. Let us set $\mathcal{A}' = \{X-A \mid A \in \mathcal{A}\}$. Then the condition implies $\mathcal{A}' \cap \mathcal{B} = \emptyset$ whence $|\mathcal{A}| + |\mathcal{B}| = |\mathcal{A}'| + |\mathcal{B}| \le 2^n$. //

LEMMA 2. *Let* $y_1, y_2, \ldots, y_m, \ldots$ *be identically distributed independent random variables defined by* $p(y_i = 1) = 1/2$, $p(y_i = -2) = 1/2$. *Let* s *be a non-negative integer. Then*

$$p\left(\max_m \left(\sum_{i=1}^m y_i\right) \ge s\right) = \left(\frac{\sqrt{5}-1}{2}\right)^s .$$

Proof. The assertion can be easily deduced from the more general theorems in Feller [2], Chapter XII. We use the following corollary of Lemma 2.

COROLLARY 1. *Let* k *be a positive integer and let* $y_1, \ldots, y_k$ *be defined as above. Then*

(2) $$p\left(\max_{m \le k} \left(\sum_{i=1}^m y_i\right) \ge s\right) < \left(\frac{\sqrt{5}-1}{2}\right)^s .$$

2. The main result

THEOREM. *Let* $\mathcal{F}$ *be a collection of subsets of* $X = [1, n]$. *Suppose that for* F_1, F_2, F_3,

(3) $$|F_1 \cap F_2 \cap F_3| \ge 2 .$$

Then $|\mathcal{F}| \le 2^{n-2}$ *and equality holds if and only if for some* $1 \le i < j \le n$, $\mathcal{F} = \{F \subseteq X \mid i \in F, j \in F\}$.

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Proof. Let us suppose that $|F| \geq 2^{n-2}$ but F is not of the above form. Let $1 \leq i < j \leq n$ and let H be a collection of subsets of X. The following operation was essentially defined in [1];

$$A_{i,j}(\mathcal{H}) = \{A_{i,j}(H) \mid H \in \mathcal{H}\} ,$$

where

$$A_{i,j}(H) = \begin{cases} (H-\{j\}) \cup \{i\} & \text{if } j \in H,\ i \notin H,\ \left((H-\{j\}) \cup \{i\}\right) \notin \mathcal{H} , \\ H & \text{otherwise.} \end{cases}$$

It can be easily checked that if $\mathcal{H}$ satisfies Condition (3) then $A_{i,j}(\mathcal{H})$ satisfies it as well. Let us apply the operation $A_{i,j}$ iteratedly for all the pairs i, j $(1 \leq i < j \leq n)$ starting with $\mathcal{H} = F$. As X is finite and whenever $A_{i,j}(\mathcal{H}) \neq \mathcal{H}$ then

$$\sum_{H \in A_{i,j}(\mathcal{H})} \sum_{q \in H} q < \sum_{H \in \mathcal{H}} \sum_{q \in H} q ;$$

so after a finite number of steps we obtain a collection G which still satisfies (3), $|G| = |F|$, and for any $1 \leq i < j \leq n$, $A_{i,j}(G) = G$. We divide the proof of the theorem into a series of propositions.

PROPOSITION 1. *If* $G = \{i_1, \ldots, i_s\} \in \mathcal{G}$, $i_1 < i_2 < \ldots < i_s$ *and* $F = \{j_1, \ldots, j_t\}$, $t \geq s$, $j_1 < j_2 < \ldots < j_t$, $i_k \geq j_k$ *for* $k = 1, \ldots, s$, *then* $F \in \mathcal{G}$.

Proof. The assertion follows from $A_{i,j}(\mathcal{G}) = \mathcal{G}$ for any i, j, $1 \leq i < j \leq n$.

PROPOSITION 2. *Let us define*

$$G_1 = \{1, 3, 4, 6, 7, \ldots, 3k, 3k+1, \ldots\} \cap [1, n] .$$

Then $G_1 \notin \mathcal{G}$.

Proof. If G_1 belongs to $\mathcal{G}$ then in view of Proposition 1 so do

$$G_2 = \{1, 2, 4, 5, 7, \ldots, 3k-1, 3k+1, \ldots\} \cap [1, n]$$

and

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$$G_3 = \{1, 2, 3, 5, 6, \ldots, 3k-1, 3k, \ldots\} \cap [1, n] ,$$

but $G_1 \cap G_2 \cap G_3 = \{1\}$, contradicting (3).

PROPOSITION 3. *For any* $G \in \mathcal{G}$ *there exists a non-negative integer* h *such that*

(4) $$|G \cap [1, 2+3h]| \geq 2 + 2h .$$

Proof. The contrary would mean that for some $G \in \mathcal{G}$, $G = \{i_1, \ldots, i_q\}$, $i_1 < i_2 < \ldots < i_q$, and any $h \geq 0$, $i_{2+2h} \geq 3 + 3h$. Hence for any $h \geq 0$, $i_{2(h+1)+1} \geq 3(h+1) + 1$, and these inequalities along with $i_1 \geq 1$ and Proposition 1, imply $G_1 \in \mathcal{G}$, contradicting Proposition 2.

Let us define the random variables $x_1, x_2, \ldots, x_n$ on a subset F of X by $x_i = 1$ if $i \in F$ and $x_i = -2$ if $i \notin F$. Then the x_i's are independent and $p(x_i = 1) = p(x_i = -2) = 1/2$.

Proposition 3 yields immediately

PROPOSITION 4. *For every* $G \in \mathcal{G}$, $\max\limits_{1 \leq j \leq n} \left(\sum\limits_{i=1}^{j} x_i\right) \geq 2$.

Let us set $\mathcal{G}_{1,2} = \{G \in \mathcal{G} \mid 1 \in G, 2 \in G\}$.

PROPOSITION 5.

(5) $$|\mathcal{G}_{1,2}| \leq 2^{n-3} .$$

Proof. If we knew that for $G, H \in \mathcal{G}_{1,2}$, $(G-[1, 2]) \cap (H-[1, 2]) \neq \emptyset$, then the assertion would follow from Lemma 1, $\left(\mathcal{A} = \mathcal{B} = \{G-[1, 2] \mid G \in \mathcal{G}_{1,2}\}\right)$. But if for some $G, H \in \mathcal{G}_{1,2}$, $(G-[1, 2]) \cap (H-[1, 2]) = \emptyset$, then $G \cap H = [1, 2]$ implies, in view of (3), that for any $G' \in \mathcal{G}$, $[1, 2] \subseteq G'$. As $|\mathcal{G}| = |\mathcal{F}| \geq 2^{n-2}$ so necessarily $\{1, 2\} \in \mathcal{G}$. Now by the definition of the operation $A_{i,j}$ it follows that there is a 2-element set $\{i, j\}$ which belongs to $\mathcal{F}$ which in turn implies $\mathcal{F} = \{F \subseteq X \mid i \in F, j \in F\}$, a contradiction.

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Let us set

$F_i = \{G-[1, 5] \mid G \in G,\ G \cap [1, 5] = \{i, 3, 4, 5\}\}$ $(i = 1, 2)$.

PROPOSITION 6. *For* $F_1 \in F_1$, $F_2 \in F_2$, $F_1 \cap F_2 \neq \emptyset$.

Proof. If for some $F_1 \in F_1$, $F_2 \in F_2$, $F_1 \cap F_2 = \emptyset$, then, according to the definition of the F_i's , there exist $H_1, H_2 \in G$ such that $H_1 \cap H_2 = [3, 5]$. Using (3) it follows that for any $G \in G$,

(6) $$|G \cap [3, 5]| \geq 2 .$$

If for some $G \in G$, $|G \cap [1, 5]| < 4$, then, by Proposition 1, $G' = \big((G-[1, 5]) \cup [1, 3]\big) \in G$; but $G' \cap [3, 5] = \{3\}$ contradicting (6). Hence for any $G \in G$, $|G \cap [1, 5]| \geq 4$; that is, $G \subseteq F(n, 3, 1)$. But $|F(n, 3, 1)| = 6.2^{n-5} < 2^{n-2}$, a contradiction.

PROPOSITION 7.

(7) $$|F_1| + |F_2| \leq 2^{n-5} .$$

Proof. (7) follows immediately from Proposition 6 and Lemma 1.

Now let us set $G_i = \{G \in G \mid [1, 2] \nsubseteq G,\ |G \cap [1, 5]| = i\}$, $i = 0, 1, 2, 3$. Then

(8) $$|G| = |G_{1,2}| + |F_1| + |F_2| + |G_0| + |G_1| + |G_2| + |G_3| .$$

PROPOSITION 8. *For* $G \in G_i$ $(i = 0, 1, 2, 3)$,

(9) $$\max_{6 \leq s \leq n} \left(\sum_{i=6}^{s} x_i\right) \geq 3(4-i) .$$

Proof. It is an immediate consequence of Proposition 3 and 4.

PROPOSITION 9.

(10) $$|G_3| \leq 7.2^{n-5}\left(\frac{\sqrt{5}-1}{2}\right)^3 .$$

Proof. Let us set $F_3 = \{G-[1, 5] \mid G \in G_3\}$. As there are 7 subsets A of $[1, 5]$ satisfying $|A \cap [1, 5]| = 3$ and $[1, 2] \nsubseteq A$ so $|G_3| \leq 7|F_3|$.

In view of (9) and Corollary 1, $(y_i = x_{6-i})$, we have

$$\frac{|F_3|}{2^{n-5}} \le \left(\frac{\sqrt{5}-1}{2}\right)^3 ,$$

so (10) follows.

PROPOSITION 10.

$$|G_2| \le 9.2^{n-5}\left(\frac{\sqrt{5}-1}{2}\right)^6 , \tag{11}$$

$$|G_1| \le 5.2^{n-5}\left(\frac{\sqrt{5}-1}{2}\right)^9 , \tag{12}$$

$$|G_0| \le 2^{n-5}\left(\frac{\sqrt{5}-1}{2}\right)^{12} . \tag{13}$$

Proof. These inequalities can be proven in exactly the same way as inequality (10).

Now summing up the inequalities (5), (7), (10), (11), (12), (13) in view of (8) we obtain

$$|G| \le 2^{n-2}\left(\frac{1}{2} + \frac{1}{8} + \frac{7}{8}\left(\frac{\sqrt{5}-1}{2}\right)^3 + \frac{9}{8}\left(\frac{\sqrt{5}-1}{2}\right)^6 + \frac{5}{8}\left(\frac{\sqrt{5}-1}{2}\right)^9 + \frac{1}{8}\left(\frac{\sqrt{5}-1}{2}\right)^{12}\right) < 0,91.2^{n-2} .$$

This final contradiction finishes the proof of the theorem.

References

[1] P. Erdös, Chao Ko, and R. Rado, "Intersection theorems for systems of finite sets", *Quart. J. Math. Oxford Ser.* 12 (1961), 313-320.

[2] William Feller, *An introduction to probability theory and its applications*, Volume II (John Wiley & Sons, New York, London, Sydney, Toronto, 1966; second edition, 1971).

[3] Peter Frankl, "Families of finite sets satisfying a union-condition", submitted.

[4] Gy. Katona, "Intersection theorems for systems of finite sets", *Acta Math. Acad. Sci. Hungar.* 15 (1964), 329-337.

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