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THE CENTRALIZER OF THE GENERAL LINEAR GROUP

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1. Introduction

Let G be a group, written additively with identity 0, but not necessarily abelian and let S be a semigroup of endomorphisms of G. The set $\mathscr{C}(S;G)=\{f:G\to G \mid f\sigma=\sigma f$ for all $\sigma\in S$ and $f(0)=0\}$ is a zero-symmetric near-ring with identity under the operations of function addition and composition, called the centralizer near-ring determined by the pair (S,G). Centralizer near-rings are general, for if N is any zero-symmetric near-ring with identity then there exists a group G and a semigroup $S\subseteq \operatorname{End} G$ such that $N\cong\mathscr{C}(S;G)$. For background material and definitions relative to near-rings in general we refer the reader to the book by Pilz [7]. For material on centralizer near-rings we refer the reader to [4] and [6].

For A, a set of linear transformations on a vector space V with certain conditions, the structure theory of the ring of linear transformations which commute with every element of A has been investigated (e.g., [1], p. 32). In [2], the non-linear analogue for the case in which V is a finite vector space and A is generated by an invertible matrix is studied. This is extended in [4] to include the structure of $\mathscr{C}(A;V)$ where V is a finite vector space and $A\subseteq \operatorname{Aut} V$. For infinite V, the situation is much more difficult. The main structural results for V infinite deal with the question of the simplicity of $\mathscr{C}(A;V)$, $A\subseteq \operatorname{Aut} V$. (See [6] and [8].) It is thus the purpose of this paper to investigate the structure of $\mathscr{C}(\mathscr{U};V)$ where V is an abelian group and $\mathscr{U}$ is the general linear group of size n over a field F with $\mathscr{U}\subseteq \operatorname{Aut} V$. This study then complements and extends the results in [2] and [4] as well as providing structural theory information about the infinite case.

Throughout this paper $\mathscr{U}$ will denote the general linear group $GL_n(F)$ of $n\times n$ matrices over a field F where we always assume $n\geqq 2$, and V will be an abelian group such that $\mathscr{U}\subseteq \operatorname{Aut} V$. Using the fact that the simple ring $R=M_n(F)$, i.e., the ring of $n\times n$ matrices over F, is generated by $\mathscr{U}$, the action of $\mathscr{U}$ on V can be extended so that V becomes a faithful, unital R-module. Since $R=RE_{11}\oplus\cdots\oplus RE_{nn}$ where the E_{ii}, $i=1,2,\ldots,n$, are the orthogonal idempotents E_{ii} with 1 in position (i,i) and 0 elsewhere, it follows that V is the direct sum of irreducible R-modules, $V=\Sigma\oplus RE_\alpha m_\alpha$ where E_α is one of the idempotents E_{ii} and $m_\alpha\in V$. If $E_\alpha=E_{ii}$, then the coefficients of m_α in $RE_\alpha m_\alpha$

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are matrices with at most the ith column different from zero. In representing these elements we will often omit the zero columns and write

$$\begin{bmatrix} a_{1i} \\ \vdots \\ a_{ni} \end{bmatrix} m_\alpha \text{ for } \begin{bmatrix} 0 & a_{1i} & 0 \\ \vdots & \vdots & \vdots \\ 0 & a_{ni} & 0 \end{bmatrix} m_\alpha.$$

We have therefore the situation in which V is a unital R-module where R is a simple ring contained in End V. Since $\mathscr{U} \subseteq R$, $\mathscr{C}(R; V) \subseteq \mathscr{C}(\mathscr{U}; V)$. The centralizer near-ring $\mathscr{C}(R; V)$ where V is a finite, faithful unital module over the finite simple ring R has been the object of study in [3]. It was shown there that $\mathscr{C}(R; V)$ is a simple near-ring, in fact a simple ring unless R is a field and $\dim_R V > 1$. The proof given in [3] also applies to the present situation where $R = M_n(F)$, F not necessarily finite, so here also one has that $\mathscr{C}(R; V)$ is a simple near-ring and is a ring unless R is a field and $\dim_R V > 1$. One is thus lead to consider if these properties are inherited by $\mathscr{C}(\mathscr{U}; V)$. Our work in this paper on the structure theory of $\mathscr{C}(\mathscr{U}; V)$ will show that in general this is not the case.

In the next section we characterize the pairs $(\mathscr{U}; V)$ such that $\mathscr{C}(\mathscr{U}; V)$ is simple. In Section 3 we investigate the left ideal structure of $\mathscr{C}(\mathscr{U}; V)$ which results in characterizations of ν-primitivity for $\mathscr{C}(\mathscr{U}; V)$, $\nu = 0, 1, 2$. In Section 4 we study the radicals, $J_\nu(\mathscr{C}(\mathscr{U}; V))$, $\nu = 0$, 1/2, 1, 2.

2. Structure of $\mathscr{C}(\mathscr{U}; V)$

In this section we obtain several properties of the near-ring $\mathscr{C}(\mathscr{U}; V)$. We first relate the decomposition $V = \sum_\alpha \oplus R E_\alpha m_\alpha$ to the group of units $\mathscr{U}$. Recall from vector space theory that if the ith column of a matrix A is nonzero then there exists a non-singular matrix P such that $AE_{ii} = PE_{ii}$. This establishes the following lemma which suggests that V can be considered as a direct sum of vector spaces of dimension n over F with $\mathscr{U}$ acting on each one naturally.

Lemma 2.1. *Let $R = M_n(F)$ and let V be a faithful R-module. Then $V = \sum_\alpha \oplus \mathscr{U}^0 E_\alpha m_\alpha$ where $\mathscr{U}^0 = GL_n(F) \cup \{0\}$, $E_\alpha \in \{E_{11}, \ldots, E_{nn}\}$ and $m_\alpha \in V$.*

If V is finitely generated over R then the number of nonzero summands in a direct sum decomposition of V into irreducible submodules is unique (see [1], p. 62) so we may call this number $\dim_R V$. Otherwise we say $\dim_R V = \infty$.

Fundamental to our study of $\mathscr{C}(\mathscr{U}; V)$ is the orbit structure of the group V by the group of automorphisms $\mathscr{U}$. We have $V = \{0\} \cup (\bigcup_\lambda \mathscr{U} v_\lambda)$where $\{0\} \cup \{v_\lambda\}$ is a complete set of orbit representatives. The set $\{v_\lambda\}$ is called a *basis* for V over $\mathscr{U}$. For each $v \in V$ we define $\operatorname{stab}(v) = \{A \in \mathscr{U} \mid Av = v\}$. Clearly $\operatorname{stab}(v)$ is a subgroup of $\mathscr{U}$ and for $B \in \mathscr{U}$, $\operatorname{stab} Bv = B \operatorname{stab}(v) B^{-1}$. Let $V^* = V - \{0\}$ and let $\mathscr{S} = \{\operatorname{stab}(v) \mid v \in V^*\}$. Then $\mathscr{S}$ is partially ordered under set inclusion and we say $\operatorname{stab}(v)$ is maximal (minimal) if it is maximal (minimal) in $\mathscr{S}$. The next result due to Betsch (see [6]) points out the importance of the set $\mathscr{S}$ in studying $\mathscr{C}(\mathscr{U}; V)$.

Lemma 2.2. *Let* $x, y \in V$. *There exists* $f \in \mathscr{C}(\mathscr{U}; V)$ *such that* $f(x) = y$ *if and only if* $\operatorname{stab}(x) \subseteq \operatorname{stab}(y)$.

We consider further the set $\mathscr{S}$. We observe first that for $x \in V$, $x = x_{\alpha_1} + \cdots + x_{\alpha_t}$ where the X_{α_i} come from different summands of the form $RE_\alpha m_\alpha$. If $A \in \operatorname{stab}(x)$ then $x = Ax = Ax_{\alpha_1} + \cdots + Ax_{\alpha_t}$. Hence $A \in \operatorname{stab}(x_{\alpha_i})$ for each i and so

$$\operatorname{stab}(x) = \bigcap_{i=1}^{t} \operatorname{stab}(x_{\alpha_i}).$$

We turn now to a characterization of maximal stabilizers. First consider

$$x = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} m_\alpha.$$

Then

$$\operatorname{stab}(x) = \left\{ \begin{bmatrix} 1 & X_1 \\ 0 & X_2 \end{bmatrix} \Big/ X_1, X_2 \text{ arbitrary, } \det X_2 \neq 0 \right\}.$$

Suppose for $0 \neq y = A_1 E_{\alpha_1} m_{\alpha_1} + \cdots + A_s E_{\alpha_s} m_{\alpha_s}$, $\operatorname{stab}(y) \supseteq \operatorname{stab}(x)$. Let

$$A_j E_{\alpha_j} m_{\alpha_j} = \begin{bmatrix} b_{1j} \\ \vdots \\ b_{nj} \end{bmatrix} m_{\alpha_j}.$$

Since

$$\operatorname{stab}(x) \subseteq \operatorname{stab}(y) \subseteq \operatorname{stab}(A_j E_{\alpha_j} m_{\alpha j})$$

and since X_1 is arbitrary in the elements of $\operatorname{stab}(x)$ one finds that $b_{2j} = \cdots = b_{nj} = 0$. Hence

$$y = \begin{bmatrix} b_{11} \\ 0 \\ \vdots \\ 0 \end{bmatrix} m_{\alpha_1} + \cdots + \begin{bmatrix} b_{1s} \\ 0 \\ \vdots \\ 0 \end{bmatrix} m_{\alpha_s}.$$

But then $\operatorname{stab}(y) \subseteq \operatorname{stab}(x)$. Now let $x \in \mathscr{U} E_\alpha m_\alpha$, say

$$x = A \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} m_\alpha$$

and so

$$\operatorname{stab}(x) = A\left(\operatorname{stab}\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}\mathrm{m}_\alpha\right)\mathrm{A}^{-1}.$$

Hence stab(x) is maximal. Finally let

$$y = A_1E_{\alpha_1}m_{\alpha_1} + \cdots + A_tE_{\alpha_t}m_{\alpha_t}.$$

We note that stab(y) is maximal if and only if stab(y) = stab$(A_iE_{\alpha_i}m_{\alpha_i})$ for $i = 1, 2, \ldots, t$. Moreover, for an appropriate $A \in \mathscr{U}$

$$\operatorname{stab} Ay = \operatorname{stab}\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}m_{\alpha_i} = \left\{\begin{bmatrix}1 & X_1\\0 & X_2\end{bmatrix}\Big/ X_1, X_2 \text{ arbitrary, } \det X_2 \neq 0\right\}.$$

As above this implies

$$AA_jE_{\alpha_j}m_{\alpha_j} = \begin{bmatrix}b_{1j}\\0\\\vdots\\0\end{bmatrix}m_{\alpha_j}, \quad j = 1, 2, \ldots, t$$

$$\text{and so if } A^{-1} = (c_{ij}),\ A_jE_{\alpha_j}m_{\alpha_j} = b_{1j}\begin{bmatrix}c_{11}\\\vdots\\c_{n1}\end{bmatrix}m_{\alpha_j},$$

i.e., all the $A_jE_{\alpha_j}$ are in the same 1-dimensional subspace. Conversely if this is the case then a direct calculation shows that

$$\operatorname{stab}(y) = \operatorname{stab}(A_jE_{\alpha_j}m_{\alpha_j}), \quad j = 1, 2, \ldots, t.$$

Hence stab(y) is maximal.

Theorem 2.3. *Let $y \in V$, $y = A_1E_{\alpha_1}m_{\alpha_1} + \cdots + A_sE_{\alpha_s}m_{\alpha_s}$.* Stab$(y)$ *is maximal if and only if there exists $a_i \neq 0$ in F such that*

$$a_iA_iE_{\alpha_i}m_{\alpha_i} = A_1E_{\alpha_1}m_{\alpha_1}, \qquad i = 1, 2, \ldots, s,$$

i.e., if and only if $\operatorname{rank}[A_1E_{\alpha_1}, \ldots, A_sE_{\alpha_s}] = 1$.

The next lemma will be used later when studying the J_2-radical. Since it involves maximal stabilizers we present it here in a general setting.

Lemma 2.4. *Let* $\mathscr{A} \subseteq \operatorname{Aut} G$ *and let*

$$\Sigma(g) = \{h \in G^* \mid \operatorname{stab}(h) = \operatorname{stab}(g)\} \cup \{0\}$$

where $\operatorname{stab}(g)$ *is maximal. Then* $\Sigma(g)$ *is a subgroup of* G.

Proof. For $h, k \in \Sigma(g)$,

$$\operatorname{stab}(h-k) \supseteq \operatorname{stab}(h) \cap \operatorname{stab}(k) = \operatorname{stab}(g).$$

But $\operatorname{stab}(g)$ is maximal so $\operatorname{stab}(h-k) = \operatorname{stab}(g)$, hence $h-k \in \Sigma(g)$.

Returning to the partially ordered set $\langle \mathscr{S}, \subseteq \rangle$, let $0 \neq w \in V$,

$$w = A_1 E_{\alpha_1} m_{\alpha_1} + \cdots + A_s E_{\alpha_s} m_{\alpha_s}$$

and suppose $\operatorname{rank}[A_1 E_{\alpha_1}, \ldots, A_s E_{\alpha_s}] = j \leqq n$. Without loss of generality we assume the first j columns are independent. Thus there exists an $A \in \mathscr{U}$ such that

$$w = A \left(\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} m_{\alpha_1} + \cdots + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} m_{\alpha_j} \right) + A_{j+1} E_{\alpha_{j+1}} + \cdots + A_s E_{\alpha_s} m_{\alpha_s}.$$

From this,

$$\operatorname{stab} A^{-1} w = \left\{ \begin{bmatrix} I_j & X_{j1} \\ 0 & X_{j2} \end{bmatrix} \Big/ X_{j1}, X_{j2} \text{ arbitrary with } \det X_{j2} \neq 0 \right\}$$

which we henceforth denote by S_j. This shows that for every nonzero w in V, $\operatorname{stab}(w)$ is conjugate to some S_j for a suitable j. Thus the S_j are canonical representatives of the conjugacy classes in $\mathscr{S}$. In particular we see that $\operatorname{stab}(v)$ is maximal if and only if $\operatorname{stab}(v)$ is conjugate to S_1. We also find that $\operatorname{stab}(w)$ is minimal if and only if $\operatorname{stab}(w)$ is conjugate to S_t where $t = \min\{\dim_R V, n\}$ which in turn is equivalent to

$$\operatorname{rank}[A_1 E_{\alpha_1} m_{\alpha_1}, \ldots, A_s E_{\alpha_s} m_{\alpha_s}] = t \quad \text{where} \quad w = \sum_{i=1}^{s} A_i E_{\alpha_i} m_{\alpha_i}.$$

Note that $S_n = \{I\}$, the identity matrix. We complete our discussion of $\mathscr{S}$ by showing that S_j and S_k are not conjugate if $j \neq k$. Thus there will be distinct conjugacy classes if $\dim_R V > 1$.

To this end suppose for some $j \neq k$, $j<k$, S_j is conjugate to S_k. Observe that all matrices in S_k have 1 as an eigenvalue of multiplicity at least k and in S_j there are matrices which have 1 as an eigenvalue of multiplicity exactly j. Since eigenvalues are preserved under conjugation, S_j cannot be conjugate to S_k.

Summarizing the above, we note that the partially ordered set $\{\mathscr{S}, \subseteq\}$ of stabilizer subgroups has a rather nice structure. Indeed $\langle \mathscr{S}, \subseteq \rangle$ can be thought of as being stratified into t conjugacy layers, $t = \min\{\dim_R V, n\}$, each layer being uniquely determined by a suitable S_j.

In investigating centralizer near-rings over infinite groups Zeller [8] found the following finiteness condition very useful.

Definition 2.5. ([8]) Let G be a group and A a group of automorphisms of G. The pair (A, G) is said to satisfy the finiteness condition (F.C.) if $\operatorname{stab}(x) \subseteq \operatorname{stab}(\alpha x)$ implies $\operatorname{stab}(x) = \operatorname{stab}(\alpha x)$ for $x \in G$, $\alpha \in A$.

Theorem 2.6. *$\mathscr{C}(\mathscr{U}; V)$ satisfies (F.C.).*

Proof. Let $v \in V$ and suppose $\operatorname{stab}(v) \subseteq \operatorname{stab}(Av)$ for some $A \in \mathscr{U}$. From our discussion about $\mathscr{S}$, we know there exists a $B \in \mathscr{U}$ such that $\operatorname{stab} Bv = S_k$ for some k and there are components in Bv having column coefficients of the form

$$\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

where the last column vector has a 1 in the kth row. Then

$$\operatorname{stab}(Bv) \subseteq \operatorname{stab} BAv = \operatorname{stab} BAB^{-1}Bv.$$

Let $Bv = v_0$ and $BAB^{-1} = C$. If

$$Cv_0 = A_1 E_{\alpha_1} m_{\alpha_1} + \cdots + A_t E_{\alpha_t} m_{\alpha_t}$$

then, since $S_k \subseteq \operatorname{stab}(Cv_0)$, we have

$$A_i E_{\alpha_i} = \begin{bmatrix} a_{1i} \\ \vdots \\ a_{ki} \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

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Let

$$C=\begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix}$$

where C_1 is a $k\times k$ matrix. Then because of the form of the above column coefficients in v_0 and because of the form of the column coefficients in Cv_0 we conclude that $C_3=0$. Therefore C^{-1} has the same form and consequently $\operatorname{stab} Cv_0=CS_kC^{-1}\subseteq S_k$. Hence $\operatorname{stab} Cv_0=\operatorname{stab} v_0$ which in turn gives $\operatorname{stab}(v)=\operatorname{stab}(Av)$ as desired.

Zeller [8] also showed that if (A,G) satisfies (F.C.) and there are at least two conjugacy classes of stabilizers then the centralizer near-ring determined by (A,G) is not simple. From the above theorem and the fact that if $\dim_R V>1$ there are distinct conjugacy classes we have the following.

Corollary 2.7. *If* $\dim_R V>1$ *then* $\mathscr{C}(\mathscr{U};V)$ *is not simple.*

The converse of this corollary is also true.

Theorem 2.8. *If* $\dim_R V=1$ *then* $\mathscr{C}(\mathscr{U};V)$ *is simple and in this case* $\mathscr{C}(\mathscr{U};V)=\mathscr{C}(R;V)=\operatorname{End}_R V\cong F$.

Proof. Since $\dim_R V=1$, $V=\mathscr{U}^0E_{\alpha_1}m_{\alpha_1}=\mathscr{U}E_{\alpha_1}m_{\alpha_1}\cup\{0\}$. Thus there is one nonzero orbit. From this and the fact that $\mathscr{C}(\mathscr{U};V)$ satisfies F.C. we find that every nonzero f in $\mathscr{C}(\mathscr{U};V)$ is a bijection, hence $\mathscr{C}(\mathscr{U};V)$ is a near-field. Suppose

$$f(E_{\alpha_1}m_{\alpha_1})=\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} m_{\alpha_i} \qquad \text{and} \qquad E_{\alpha_i}=E_{ii}.$$

For $j\neq i$, E_{1j} is nilpotent, so $I+E_{ij}\in\mathscr{U}$. Further $(I+E_{ij})E_{ii}m_{\alpha_1}=E_{ii}m_{\alpha_1}$ while

$$(I+E_{ij})f(E_{ii}m_{\alpha_i})=\begin{bmatrix} a_1 \\ \vdots \\ a_{i-1} \\ a_i+a_j \\ a_{i+1} \\ \vdots \\ a_n \end{bmatrix}.$$

From this we conclude that

$$f(E_{ii}m_{\alpha_1})=\begin{bmatrix} 0 \\ \vdots \\ 0 \\ a_i \\ 0 \\ \vdots \\ 0 \end{bmatrix} m_{\alpha_1}$$

or $f(E_{ii}m_{\alpha_i})=a_iE_{ii}m_{\alpha_i}$, $a_i\in F^*$. Thus for $A\in\mathscr{U}$, $f(AE_{ii}m_{\alpha_i})=a_iAE_{ii}m_{\alpha_1}$ which implies that $f=\lambda_{a_i}$, i.e. f is just left multiplication by a_i. Hence under the mapping $f\to\lambda_{a_i}$ we have $\mathscr{C}(\mathscr{U};V)\cong F$. Thus $\mathscr{C}(\mathscr{U};V)$ is simple. Since $\lambda_{a_i}\in\operatorname{End}_R V$ we have $\mathscr{C}(\mathscr{U};V)\subseteq\operatorname{End}_R V$. On the other hand since $\mathscr{U}\subseteq R$, $\mathscr{C}(R;V)\subseteq\mathscr{C}(\mathscr{U};V)$ and clearly $\operatorname{End}_R V\subseteq\mathscr{C}(R;V)$.

Recall that the Kern of a near-ring N is the set

$$\operatorname{Kern} N=\{a\in N\mid a(b+c)=ab+ac \text{ for all } b,c\in N\}.$$

In the case that $\langle N,+\rangle$ is abelian, Kern N is a subring of N. We conclude this section by characterizing $\operatorname{Kern}(\mathscr{C}(\mathscr{U};V))$.

Theorem 2.9. $\operatorname{Kern}(\mathscr{C}(\mathscr{U};V))=\operatorname{End}_R V=\mathscr{C}(R;V)$.

Proof. From the generalization of Theorem 1 of [3] as mentioned in the introduction we know $\operatorname{End}_R V=\mathscr{C}(R;V)$ so it remains to verify the first equality. If $\dim_R V=1$ then the result follows from the previous theorem. Thus we suppose $\dim_R V>1$. Let $v_i\in\mathscr{U}^0E_{\alpha_i}m_{\alpha_i}$, $v_j\in\mathscr{U}^0E_{\alpha_j}m_{\alpha_j}$, $i\neq j$ and let $v=v_i+v_j$. Then $\operatorname{stab}(v)\subseteq\operatorname{stab}(v_i)$, $\operatorname{stab}(v)\subseteq\operatorname{stab}(v_j)$ so there exists functions h_i, $h_j\in\mathscr{C}(\mathscr{U};V)$, $h_i(v)=v_i$, $h_j(v)=v_j$. For $d\in\operatorname{Kern}(\mathscr{C}(\mathscr{U};V))$,

$$d(v_i+v_j)=d(h_i(v)+h_j(v))=d(h_i+h_j)(v)=(dh_i+dh_j)(v)=d(v_i)+d(v_j).$$

Now let v_i, $v_j\in\mathscr{U}^0E_{\alpha_i}m_{\alpha_i}$. Then there exists $w_j\in\mathscr{U}^0E_{\alpha_j}m_{\alpha_j}$ $j\neq i$ such that $\operatorname{stab}(v_j)=\operatorname{stab}(w_j)$. Let $w=v_i+w_j$. As above there exist g_i,g_j in $\mathscr{C}(\mathscr{U};V)$ such that $g_i(w)=v_i$, $g_j(w)=w_j$. Since $\operatorname{stab}(w_j)=\operatorname{stab}(v_j)$, there exists $g\in\mathscr{C}(\mathscr{U};V)$ such that $g(w_j)=v_j$. Hence $\bar{g}_j=gg_j$ takes w to v_j. Again if $d\in\operatorname{Kern}(\mathscr{C}(\mathscr{U};V))$ then $d(v_i+v_j)=d(v_i)+d(v_j)$. This suffices to show $d\in\operatorname{End} V$. Since $d\in\mathscr{C}(\mathscr{U};V)$ we now have $d\in\operatorname{End}_{\mathscr{U}} V$. The converse is clear so $\operatorname{Kern}(\mathscr{C}(\mathscr{U};V))=\operatorname{End}_{\mathscr{U}} V$. Since R is generated by $\mathscr{U}$, $\operatorname{End}_R V=\operatorname{End}_{\mathscr{U}} V$.

3. Left Ideals in $\mathscr{C}(\mathscr{U};V)$

In this section we examine various left ideals in $\mathscr{C}(\mathscr{U};V)$. We determine all minimal left ideals and then use our characterization to show that there are no nonzero nilpotent left ideals in $\mathscr{C}(\mathscr{U};V)$. We further use our characterization of minimal left ideals to establish when $\mathscr{C}(\mathscr{U};V)$ is ν-primitive, $\nu=0,1,2$.

Notation. For the remainder of this paper we use N to denote the near-ring $\mathscr{C}(\mathscr{U};V)$.

For an arbitrary centralizer near-ring $\mathscr{C}(A;G)\equiv M$, let e_x denote the idempotent mapping in M which fixes the orbit Ax and maps all other orbits to 0. In [5] it is shown that if L is a minimal left ideal of M then $L\subseteq Me_x$ for some $x\in G$, and under certain conditions related to x, the left ideal Me_x is minimal. Here we find that all minimal left ideals of $N\equiv\mathscr{C}(\mathscr{U};V)$ are of the form Ne_x.

We first give an easy but useful technical result.

Lemma 3.1. *Let L be a left ideal of $N\equiv\mathscr{C}(\mathscr{U};V)$ contained in Ne_x for some $x\in V$. Let*

$$T(x)=\{v\in V\mid \operatorname{stab}(v)\supseteq\operatorname{stab}(x)\}$$

and let

$$L(x) = \{w \in V \mid w = l(x) \quad \textit{for some } l \in L\}.$$

Then for each

$$y \in T(x) - L(x),\ y + L(x) \subseteq \mathscr{U} y.$$

Proof. We first note that $T(x)$ is a subgroup of V. Now let $y \in T(x) - L(x)$ and assume for some v in $L(x)$ that $y + v \notin \mathscr{U} y$. Thus $y, y+v$ are in different orbits so there exists an f in N such that $f(y) = y$ and $f(y+v) = 0$. Further there exist $l \in L$, $g \in N$ such that $l(x) = v$ and $g(x) = y$. Since L is a left ideal of N, $h = f(g + l) - fg \in L$ and so $h(x) \in L(x)$. But

$$h(x) = f(y+v) - f(y) = -y.$$

This is a contradiction since $L(x)$ is group and $y \notin L(x)$.

For x in V. $Ne_x = \mathrm{Ann}(1 - e_x) = \mathrm{Ann}(V - \mathscr{U} x)$ so clearly Ne_x is a left ideal of N. Further $\mathrm{Ann}\, e_x = \mathrm{Ann}(x)$ is a left ideal of N with $Ne_x \oplus \mathrm{Ann}\, e_x = N$, hence Ne_x is N-isomorphic to $N/\mathrm{Ann}\, e_x$. Consequently, Ne_x is a minimal (strictly minimal) left ideal if and only if $\mathrm{Ann}(e_x)$ is a maximal (strictly maximal) left ideal. Further, Ne_x is strictly minimal if and only if stab(x) is maximal. For if stab(x) is maximal this is indeed the case. If stab(x) is not maximal then stab$(x) \subsetneq$ stab(y) for some $y \in V^*$. Hence there exists a mapping $f \in N$ defined by $f(x) = y$ and $f(w) = 0$ if $w \notin \mathscr{U} x$. But then $f \in Ne_x$ and Nf is an N-subgroup of Ne_x, $(0) \subsetneq Nf \subsetneq Ne_x$.

Theorem 3.2. *For each* $x \in V^*$, Ne_x *is a minimal left ideal.*

Proof. Let L be a nonzero left ideal in Ne_x. Hence $L(x) \neq 0$, say $0 \neq y \in L(x)$ where $y = y_{\alpha_1} + \cdots + y_{\alpha_t}$ with $y_{\alpha_i} \neq 0$ for at least one i, say y_{α_1}. Since stab$(y_{\alpha_1}) \supseteq$ stab(y), $y_{\alpha_1} \in L(x)$. If stab(x) is maximal then we know Ne_x is minimal and $L = Ne_x$. If stab(x) is not maximal, $x = x_{\beta_1} + \cdots x_{\beta_s}$ then from Theorem 2.3, there must be at least two non-zero components. For x_{β_j}, $\beta_j \neq \alpha_1$, if $x_{\beta_j} \notin L(x)$ then since $x_{\beta_j} \in T(x)$ we have from Lemma 3.1, $x_{\beta_j} + L(x) \subseteq \mathscr{U} x_{\beta_j}$. Hence $x_{\beta_j} + y_{\alpha_1} = Ax_{\beta_j}$ for some $A \in \mathscr{U}$. But then $(A - I)x_{\beta_j} = y_{\alpha_1}$ which contradicts the fact that $RE_{\alpha_1} m_{\alpha_1} \cap RE_{\beta_j} m_{\beta_j} = (0)$ for $\alpha_1 \neq \beta_j$. Thus we have $x_{\beta_j} \in L(x)$ for $\beta_j \neq \alpha_1$. For x_{β_j} where $\beta_j = \alpha_1$ we have $\beta_i \neq \beta_j$ such that $x_{\beta_i} \neq 0$ and $x_{\beta_i} \in L(x)$. Therefore as above if $x_{\beta_j} (\equiv x_{\alpha_1}) \notin L(x)$, $x_{\beta_j} + x_{\beta_i} = Bx_{\beta_j}$, $B \in \mathscr{U}$, again leading to a contradiction. From this we find that $x_{\beta_j} \in L(x)$ for each β_j and so

$$x = \sum_{\alpha=1}^{s} x_{\beta_j} \in L(x).$$

Thus there exists h in L such that $h(x) = x$, i.e., $e_x \in L$ and so $L = Ne_x$.

We now turn to the problem of showing that $\mathscr{C}(\mathscr{U}; V)$ has no nonzero nilpotent left ideals.

Theorem 3.3. *Let L be a left ideal of N containing no nonzero idempotent elements. Then for each f in L, for each* $x \in V$ *if* $f(x) \neq 0$, *then* stab$(x) \subsetneq$ stab $f(x)$.

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Proof. We know $\text{stab}(x) \subseteq \text{stab}\, f(x)$ for each $f \in L$. If the theorem is false, there exists an $f \in L$ and $x \in V^*$ such that $\text{stab}(x) = \text{stab}(f(x))$. Let $y = f(x)$. Thus there exists a map e_{xy} in N such that $e_{xy}(y) = x$ and $e_{xy}(w) = 0$ for $w \notin \mathcal{U}y$. Since $e_{xy} \in N$, $e_{xy} f \in L$ so we may assume $f(x) = x$. Similarly $e_x f \in L$ so we may also assume that $f(V) \subseteq \mathcal{U}^0 x$. Let $K = \{v \in V \mid f(v) \in \mathcal{U}x\}$. Then $K \neq \varnothing$ since $\mathcal{U}x \subseteq K$. If $K = \mathcal{U}x$ then $f(v) = v$ for $v \in \mathcal{U}x$ and $f(v) = 0$ for $v \notin \mathcal{U}x$; i.e., $f = e_x$ which is a contradiction. Thus there exist $v \in K$, $v \notin \mathcal{U}x$. Thus for some $A \in \mathcal{U}$, $f(v) = Ax$. Let $f_1 = e_x(-e_v + f) + e_x e_v$ which is in L since L is a left ideal and $f \in L$. Now $f_1(x) = x$, $f_1(v) = e_x(-v + f(v)) = e_x(-v + Ax)$ and $f_1(y) = f(y)$ if $y \notin (\mathcal{U}x \cup \mathcal{U}v)$. Assume $-v + Ax \in \mathcal{U}x$. Then $f_1(v) = -v + Ax$ so $(f - f_1)(v) = Ax - (-v + Ax) = v$ while $(f - f_1)w = 0$ if $w \notin \mathcal{U}v$. Again this is impossible so we have for all $v \in K - x$, $-v + Ax \notin \mathcal{U}x$ where $f(v) = Ax$. We now define a new function h as follows. Let v_0 be arbitrary but fixed in $K - x$. Define h by $h(y) = y$ if $y \in \mathcal{U}x \cup \mathcal{U}v_0$, $h(y) = -y + f(y)$ if $y \in K - (\mathcal{U}x \cup \mathcal{U}v_0)$ and $h(y) = 0$ if $y \notin K$. Let $g_1 = e_x h - e_x(h - f_1)$ and $g_2 = e_x(f_1 - h) + e_x h$. Then $g_1, g_2 \in L$. Now

$$g_1(y) = \begin{cases} 0 \text{ if } y \notin K \\ y \text{ if } y \in \mathcal{U}x \\ 0 \text{ if } y \in \mathcal{U}v_0 \\ y \text{ if } y \in K - (\mathcal{U}x \cup \mathcal{U}v_0) \text{ and } -y \in \mathcal{U}x \\ 0 \text{ if } y \in K - (\mathcal{U}x \cup \mathcal{U}v_0) \text{ and } -y \notin \mathcal{U}x \end{cases}$$

and $g_2 g_1 = e_x$ which is again impossible. Thus the result is established.

Theorem 3.4. *Let L be a nonzero left ideal of N containing no nonzero idempotent elements. Then there exists some $x \in V^*$ such that $L \cap Ne_x \neq (0)$.*

Proof. Let f be nonzero in L with say $f(x) = y \neq 0$. From the previous theorem $\text{stab}(x) \subsetneq \text{stab}(y)$. Since $e_y f \in L$ we suppose without loss of generality that $f(V) \subseteq \mathcal{U}^0 y$. Let $K = \{v \in V \mid f(v) \in \mathcal{U}y\}$. Then $y \notin K$; for if $f(y) = Ay$ for some $A \in \mathcal{U}$, we would have $\text{stab}(y) \subsetneq \text{stab}\, f(y) = \text{stab}(Ay)$ which contradicts the finiteness condition of Theorem 2.6. A similar argument shows that $y \notin \mathcal{U}x$. Let $f_1 = e_y(-e_x + f) + e_y e_x$. Then $f_1 \in L$ with $f_1(x) = e_y(-x + y)$ and $f_1(w) = f(w)$ for $w \notin \mathcal{U}x$. If $-x + y \in \mathcal{U}y$ then $f_1(x) = -x + y$ and consequently $e_x = f - f_1 \in L$ which is a contradiction. Therefore $-x + y \notin \mathcal{U}y$ so $f_1(x) = 0$. But then $(f - f_1)w = f(w)$ if $w \in \mathcal{U}x$ while $(f - f_1)w = 0$ if $w \notin \mathcal{U}x$. Hence $0 \neq f - f_1 = fe_x$ is in $Ne_x \cap L$.

Corollary 3.5. *If L is a nonzero left ideal of N then L contains an idempotent. Further there are no nonzero nilpotent left ideals in N.*

Proof. Suppose L is a nonzero left ideal that does not contain an idempotent. From the above theorem, $L \cap Ne_x \neq (0)$ for some $x \in V^*$. But for each $x \in V^*$, Ne_x is a minimal left ideal so that $Ne_x = L \cap Ne_x \subseteq L$. This contradiction establishes the desired result.

Corollary 3.6. *Let L be a left ideal of N. L is a minimal left ideal if and only if $L = Ne_x$ for some $x \in V^*$.*

Proof. If $L=Ne_x$ then from Theorem 3.2 L is minimal. Conversely, from the above corollary $e_x \in L$ for some $x \in V^*$ and so $Ne_x \subseteq L$. Since L is minimal $L=Ne_x$.

We remark that Lemma 3.1 as well as Theorem 3.3 and Theorem 3.4 do not use the structure of $\mathscr{C}(\mathscr{U}; V)$ in their proofs and therefore are valid in a more general setting. Indeed these results will hold in any centralizer near-ring $\mathscr{C}(A; G)$, $A \subseteq \operatorname{Aut} G$, in which the Ne_x are minimal, for $x \in G^*$ and such that the finiteness condition (F.C.) is satisfied.

We further apply Theorem 3.2 to obtain information about the ν-primitivity of $\mathscr{C}(\mathscr{U}; V)$, $\nu = 0, 1, 2$. For the necessary definitions and background material on this topic we again refer the reader to Pilz [7].

Theorem 3.7. *For $N=\mathscr{C}(\mathscr{U}; V)$ the following are equivalent:*

(i) *N is simple,*

(ii) *N is 2-primitive,*

(iii) *N is 1-primitive.*

Proof. The equivalence of (ii) and (iii) follows from the general results in [7] (p. 104) since N has an identity.

(i)→(ii). Since N is simple, from Theorem 2.8, N is a field and so is 2-primitive on $\langle N, + \rangle$.

(ii)→(i). It is known that when a near-ring M is 2-primitive with a minimal left ideal then all minimal left ideals are M-isomorphic [Pilz, p. 130]. In our situation if N is not simple this is impossible. For if N is not simple, $\dim_R V \geqq 2$. Thus if $v = E_{\alpha_1} m_{\alpha_1}$ then stab(v) is maximal and hence Ne_v is a strictly minimal left ideal. On the other hand for

$$w = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} m_{\alpha_1} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} m_{\alpha_2}$$

stab(w) is not maximal and so as we have seen the minimal left ideal Ne_w is not strictly minimal. Hence $Ne_v \ncong Ne_w$ as N-groups.

To complete the characterizations of ν-primitivity it remains to consider the case for $\nu = 0$. Here the situation is quite different. In fact $\mathscr{C}(\mathscr{U}; V)$ is always 0-primitive.

Theorem 3.8. *$\mathscr{C}(\mathscr{U}; V)$ is 0-primitive.*

Proof. We separate the proof into two cases depending on $\dim_R V$.

Case 1: $\dim_R V \geqq n$. As we have seen

$$v = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} m_{\alpha_1} + \cdots + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} m_{\alpha_n}$$

is such that $\text{stab}(v)=\{I\}$. But then $Ne_v=V$ is a minimal left ideal, monogenic and clearly the left annihilator of V in N is $\{0\}$. Hence the N-module V is of type 0, i.e., $\mathscr{C}(\mathscr{U};V)$ is 0-primitive in this case.

Case 2: $\dim_R V=t<n$. If

$$x=\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}m_1+\cdots+\begin{bmatrix}0\\\vdots\\0\\1\\0\\\vdots\\0\end{bmatrix}m_t$$

then we know $\text{stab}(x)=S_t$ and S_t is minimal in $\mathscr{S}$. Moreover for any $y\in V$ there exists a $B\in\mathscr{U}$ such that $\text{stab}(By)=S_k$ for some k and so $S_t\subseteq S_k$. Now Ne_x is a minimal left ideal, and hence an N-group of type 0, clearly monogenic since for each $f\in Ne_x$, $f=fe_x$. Let $h\in\text{Ann}(Ne_x)$ and let y be arbitrary in V^*. As we showed above, $\text{stab}(x)\subseteq\text{stab}(By)$ for some $B E\mathscr{U}$ so there exists a $g\in N$ with $g(x)=By$. But then $0=h(ge_x)$ implies $0=h(By)=Bh(y)$ so $h(y)=0$. Since y was arbitrary $h\equiv 0$. Hence N is 0-primitive on Ne_x.

4. Radicals in $\mathscr{C}(\mathscr{U};V)$

In this section we investigate the structure of the various radicals $J_v(N)$, $v=0,1/2,1,2$ for the near-ring $N=\mathscr{C}(\mathscr{U};V)$. For the necessary definitions we again refer to [7]. As in the case of primitivity, since N contains an identity $J_1(N)=J_2(N)$.

From Theorem 3.2, Ne_x is a minimal left ideal for each $x\in V^*$. Thus $\text{Ann}\,e_x$ is a maximal left ideal for $x\in V^*$. Therefore

$$J_{1/2}(N)=\cap\{K\,|\,K \text{ is a maximal left ideal of } N\}\subseteq\bigcap_{x\in V^*}\text{Ann}\,e_x=\{0\}.$$

Thus $J_{1/2}=(0)$ and since $J_0(N)\subseteq J_{1/2}(N)$, $J_0(N)=(0)$. Of course this latter result was known already since N is 0-primitive.

It remains to consider $J_2(N)$. We first establish some bounds. Let $\mathscr{B}=\{x_\lambda\}\cup\{0\}$ be a basis for V over $\mathscr{U}$. Let $M=\{x_\lambda\in\mathscr{B}^*\,|\,\text{stab}(x_\lambda) \text{ is maximal in } \mathscr{S}\}$ and let $\bar{M}=\mathscr{B}-M$. For $x_\lambda\in M$, $\text{Ann}(x_\lambda)$ is a strictly maximal left ideal so $J_2(N)\subseteq\bigcap_{x_\lambda\in M}\text{Ann}(x_\lambda)$. We note that $\bigcap_{x_\lambda\in M}\text{Ann}\,x_\lambda=Ne_{\bar{M}}$ where $e_{\bar{M}}(x)=x$ if $x\in\bar{M}$ and $e_{\bar{M}}(x)=0$ if $x\in M$.

If L is a strictly maximal ideal not of the form $\text{Ann}\,x_\lambda$ for $x_\lambda\in M$ then for each x, $L+Ne_x=N$. If for some x, $L\cap Ne_x=(0)$ then one finds that $L=\text{Ann}(x)$. Since Ne_x is minimal, $\text{Ann}(x)$ is maximal so $\text{Ann}(x)=L$, a contradiction. Thus for each x, $L\cap Ne_x\neq(0)$ so $Ne_x\subseteq L$. This also follows from results in [5]. Consequently for every strictly maximal ideal L not of the form $\text{Ann}\,x_\lambda$, for $x_\lambda\in M$, we have $L\supseteq\sum_{x\in V}Ne_x$. Further, $Ne_{\bar{M}}\supseteq\sum_{\lambda\in\bar{M}}\oplus Ne_{x_\lambda}$. Since $J_2(N)$ is the intersection of all strictly maximal left ideals of N we have $J_2(N)\supseteq\sum_{x_\lambda\in\bar{M}}\oplus Ne_{x_\lambda}$.

Theorem 4.1. $\sum_{x_\lambda \in \bar{M}} \oplus Ne_{x_\lambda} \subseteq J_2(N) \subseteq Ne_{\bar{M}}$.

Corollary 4.2. *If $\bar{M}$ is finite, $J_2(N) = Ne_{\bar{M}} = \sum_{x_\lambda \in \bar{M}} \oplus Ne_{x_\lambda}$.*

The left ideal $\sum_{x_\lambda \in \bar{M}} \oplus Ne_{x_\lambda}$ is precisely the collection of functions f in $Ne_{\bar{M}}$ with finite support, i.e., $\text{supp}(f) < \infty$ where

$$\text{sup}(f) = \{x \in V \mid f(x) \neq 0\} \cap \mathscr{B} = \{x \in \mathscr{B} \mid f(x) \neq 0\}.$$

We now characterize when this set is $J_2(N)$.

Theorem 4.3. (i) *Let F be an infinite field. Then $J_2(N) = \sum_{x_\lambda \in \bar{M}} \oplus Ne_{x_\lambda}$ if and only if $\dim_R V \leqq 2$.*

(ii) *Let F be a finite field. Then $J_2(N) = \sum_{x_\lambda \in \bar{M}} \oplus Ne_{x_\lambda}$ if and only if $\dim_R V$ is finite.*

Proof. (i) If $\dim_R V = 0$ then $V = (0)$ while if $\dim_R V = 1$, $\mathscr{C}(\mathscr{U}; V)$ is simple so in both of these cases $J_2(N) = (0) = \sum_{x_\lambda \in \bar{M}} \oplus Ne_{x_\lambda}$ since $\bar{M} = \{0\}$. Thus suppose $V = RE_{\alpha_1} m_1 \oplus RE_{\alpha_2} m_2$. From our investigations of the set $\mathscr{S}$ we know that in this case $v \in \bar{M}$ if and only if stab(v) is conjugate to S_2. But this means there exists $A \in \mathscr{U}$ such that stab(Av) $= S_2$ and

$$Av = v_0 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} m_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} m_2.$$

This in turn implies that if stab(w) is not maximal then $w \in \mathscr{U} v_0$. Thus $\bar{M}$ has one nonzero element so from Corollary 4.2, $J_2(N) = \sum_{z_\lambda \in \bar{M}} \oplus Ne_{x_\lambda}$.

Conversely suppose $\dim_R V \geqq 3$. For $a \in F^*$, let

$$x_a = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} m_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} m_2 + \begin{bmatrix} a \\ 0 \\ \vdots \\ 0 \end{bmatrix} m_3.$$

We claim $\mathscr{U} x_a \neq \mathscr{U} x_b$ if $a \neq b$. Otherwise there would exist $A = [a_{ij}]$, $B = [b_{ij}]$ in $\mathscr{U}$ such that $Ax_a = Bx_b$. Thus

$$Ax_a = \begin{bmatrix} a_{11} \\ \vdots \\ a_{n1} \end{bmatrix} m_1 + \begin{bmatrix} a_{12} \\ \vdots \\ a_{n2} \end{bmatrix} m_2 + \begin{bmatrix} aa_{11} \\ \vdots \\ aa_{n1} \end{bmatrix} m_3 = \begin{bmatrix} b_{11} \\ \vdots \\ b_{n1} \end{bmatrix} m_1 + \begin{bmatrix} b_{12} \\ \vdots \\ b_{n2} \end{bmatrix} m_2 + \begin{bmatrix} bb_{11} \\ \vdots \\ bb_{n1} \end{bmatrix} m_3.$$

From the uniqueness of representation of elements in V we find that $a=b$, a contradiction. From Theorem 2.3, stab(x_a) is not maximal. We use the x_a, $a\in F^*$ as part of a basis $\mathscr{B}$ for V over $\mathscr{U}$. Since F^* is infinite, so is $\bar{M}$. We define a function f in N as follows. For each $x_a\in\bar{M}$ let

$$f(x_a)=\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}m_1+\begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix}m_2\equiv w_0$$

and let $f(x)=x$ for $x\in\bar{M}-\{x_a\}_{a\in F^*}$. Finally define $f(y)=0$ for $y\in M$. Then $f\in N$ and $f(V)\subseteq\mathscr{U}\bar{M}$. Further, since $w_0\in\bar{M}$, $e_{w_0}\in J_2(N)$ and so $e_{w_0}f\in J_2(N)$. But supp $e_{w_0}f=\{x_a\}_{x\in F^*}$ is infinite so $e_{w_0}f\in J_2(N)-\sum_{x_\lambda\in\bar{M}}\oplus Ne_{x_\lambda}$.

(ii) Let F be a finite field. Then R is finite. If $\dim_R V$ is finite then V is finite and the result follows from Corollary 4.2. If $\dim_R V$ is not finite then for $j\geqq 3$, the elements $x_j=E_{\alpha_1}m_1+E_{\alpha_2}m_2+E_{\alpha_j}m_j$ are in distinct orbits and so can be used as part of a basis. Also $x_j\in\bar{M}$ and $\{x_j\}_{j\geqq 3}$ is infinite. As in the first part of the proof one can find a function $0\neq g\in J_2(N)-\sum_{x_\lambda\in\bar{M}}\oplus Ne_{x_\lambda}$.

If $J_2(N)\neq\sum_{x_\lambda\in\bar{M}}\oplus Ne_{x_\lambda}$, what can be said about the functions in $J_2(N)$? We give a partial answer to this question. Thus for the remainder of this section we take $\bar{M}$ to be infinite and $\dim_R V\geqq 3$.

Lemma 4.4. *Let L be a strictly maximal left ideal of N. Either e_M or $e_{\bar{M}}$ is in L.*

Proof. Suppose $e_M\notin L$. Since L is strictly maximal, $L+Ne_M=N$ so there exist $s\in L$, $n\in N$ such that $s+ne_M=1$. Let $s_1=e_{\bar{M}}s$. Then $s_1(V)\subseteq\mathscr{U}\bar{M}$ and since $s(x)=x$ for $x\in\bar{M}$, s_1 is a nonzero element in L. Let $h=e_{\bar{M}}(s_1+e_M)-e_{\bar{M}}e_M$. Then $h=e_{\bar{M}}(s_1+e_M)$ is in L with $h(x)=x$ for $x\in\bar{M}$ while $h(y)=e_{\bar{M}}(s_1(y)+y)$ for $y\in M$. Since $y\in M$, stab(y) is maximal so stab $f(y)=$ stab(y). From Lemma 2.4, $s_1(y)+y\in\mathscr{U}M$ so $h(y)=0$. Thus $e_{\bar{M}}=g\in L$.

Theorem 4.5. *$J_2(N)=\cap\{L\,|\,L$ is a strictly maximal left ideal containing $e_M\}\cap Ne_{\bar{M}}$.*

Proof. Let $\Sigma=\{L_\alpha\,|\,L_\alpha$ is a strictly maximal left ideal$\}$. By Lemma 4.4, $\Sigma=\Sigma_1\cup\Sigma_2$ where $\Sigma_1\cap\Sigma_2=\varnothing$ and $\Sigma_1=\{L_\sigma\in\Sigma\,|\,e_M\in L_\alpha\}$, $\Sigma_2=\{L_\sigma\in\Sigma\,|\,e_{\bar{M}}\in L_\sigma\}$. By definition $J_2(N)=\bigcap_{\sigma\in\Sigma}L_\sigma$ and since $Ne_{\bar{M}}\supseteq J_2(N)$ we have $J_2(N)=\bigcap_{\sigma\in\Sigma}L_\sigma\cap Ne_{\bar{M}}$. For $L_\sigma\in\Sigma_2$, $Ne_{\bar{M}}\subseteq L_\sigma$ so $Ne_{\bar{M}}=L_\sigma\cap Ne_{\bar{M}}$. Thus

$$J_2(N)=\bigcap_{\sigma\in\Sigma_1}L_\sigma\cap\left(\bigcap_{\sigma\in\Sigma_2}L_\sigma\right)\cap Ne_{\bar{M}}=\bigcap_{\alpha\in\Sigma_1}L_\alpha\cap Ne_{\bar{M}}$$

as desired.

Let $f\in N$. We define the *rank* of f to be the cardinality of the set $f(V)\cap\mathscr{B}^*$ where $\mathscr{B}$ is a basis for V.

Theorem 4.6. *$J_2(N)\supseteq\{f\in N\,|\,\mathrm{supp}(f)\subseteq\bar{M}$ and rank f is finite$\}$.*

Proof. Let $f\in Ne_{\bar{M}}$ such that f has finite rank. Let $x_{\lambda_1},\ldots,x_{\lambda_k}$, $x_{\lambda_{k+1}},\ldots,x_{\lambda_t}$ be the basis elements in $f(V)$ where $x_{\lambda_i}\in M$, $i=1,2,\ldots,k$ and x_{λ_j}, $j=k+1,\ldots,t$ are in $\bar{M}$. Thus f can be represented as $f=f_1+f_2$ where $f_1=\sum_{i=1}^{k}e_{x_{\lambda_i}}f$ and $f_2=\sum_{j=k+1}^{t}e_{x_{\lambda_j}}f$. Since $e_{x_{\lambda_j}}\in J_2(N)$, $j=k+1,\ldots,t$ so does f_2. Hence if $f\notin J_2(N)$ then $f_1\notin J_2(N)$. This in turn implies that one of the summands, say without loss of generality $e_{x_{\lambda_1}}f$, is not in $J_2(N)$. Let $g=e_{x_{\lambda_1}}f$ and let $\mathrm{supp}\,g=\bar{M}_1$. If $\bar{M}_1$ is finite then $g\in J_2(N)$ so we assume $\bar{M}_1$ is infinite. Since $g\in Ne_{\bar{M}}$ but is not in $J_2(N)$ there exists a strictly maximal left ideal L in Σ_1 such that $g\notin L$. Hence $L+Ng=N$ and so there exist $s\in L$, $n\in N$ such that $s+ng=1$. Therefore $s(x)=x$ for $x\notin\bar{M}_1$ and for $x\in\bar{M}_1$ $s(x)=x-n(Ax_{\lambda_1})$ for some $A\in\mathcal{U}$. Clearly $n(Ax_{\lambda_1})\neq 0$. Thus $n(Ax_{\lambda_1})\in\mathcal{U}M$ since $x_{\lambda_1}\in M$. Let $h=e_M(e_{\bar{M}_1}-s)-e_Me_{\bar{M}_1}=e_M(e_{\bar{M}_1}-s)$ since $\bar{M}_1\subseteq\bar{M}$. Then $h\in L$ and for $x\in\bar{M}_1$, $h(x)=e_M(x-s(x))=x-s(x)$ since $n(Ax_1)\in\mathcal{U}M$ while for $x\notin\bar{M}_1$

$$h(x)=e_M(-x)=\begin{cases}-x \text{ if } x\in M\\ 0 \text{ if } x\in\bar{M}-\bar{M}_1\end{cases}.$$

Since $h\in L$, h_1+s is also in L and

$$h_1(x)=\begin{cases}x \text{ if } x\in\bar{M}_1\\ 0 \text{ if } x\in M\\ x \text{ if } x\in\bar{M}-\bar{M}_1.\end{cases}$$

Therefore $e_{\bar{M}}=h_1\in L$ which is a contradiction. Consequently $f\in J_2(N)$.

In a similar manner we now show that $J_2(N)$ contains all functions with support in $\bar{M}$ and range in $\mathcal{U}M\cup\{0\}$.

Theorem 4.7. *$J_2(N)\supseteq\{f\in N\,|\,f\in Ne_{\bar{M}}$ and $f(\bar{M})\subseteq\mathcal{U}M\cup\{0\}\}$.*

Proof. Let $f\in Ne_{\bar{M}}$ with $f(\bar{M})\subseteq\mathcal{U}M\cup\{0\}$. Further let $\bar{M}_1=\mathrm{supp}\,f$. If $f\notin J_2(N)$ then as in Theorem 4.6 there exists a strictly maximal left ideal L with $e_M\in L$ and $s\in L$, $n\in N$ with $s+nf=1$. Since $f(x)\in\mathcal{U}M$ so does $nf(x)$ for all $x\in\bar{M}_1$. Now $h=e_M(e_{\bar{M}_1}-s)-e_Me_{\bar{M}_1}=e_M(e_{\bar{M}_1}-s)$ is in L. As above $h_1=h+s$ is in L and $h_1=e_{\bar{M}}$ a contradiction. Thus $f\in J_2(N)$.

The problem of characterizing the elements in $J_2(N)$ remains open. That the above two results do not give this characterization is pointed out in the following example in which we give a function f in $J_2(N)$ with $f(\bar{M})\subseteq\mathcal{U}\bar{M}$ and f is *not* of finite rank.

Example 4.8. Let $\dim_R V$ be at least 3 and F an infinite field. Let

$$x_a=\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}m_1+\begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix}m_2+\begin{bmatrix}a\\0\\\vdots\\0\end{bmatrix}m_3,$$

$a \in F^*$ as in Theorem 4.3. Define f_1 by

$$f_1(x_a) = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} m_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} m_2$$

and f_1 to be zero on the other basis elements. Since f_1 is of finite rank, $f_1 \in J_2(N)$. Define f_2 by

$$f_2(x_a) = \begin{bmatrix} a \\ 0 \\ \vdots \\ 0 \end{bmatrix} m_3$$

and f_2 to be zero on the other basis elements. Since $f_2(\bar{M}) \subseteq \mathcal{U}M \cup \{0\}$, $f_2 \in J_2(N)$. Hence $f = f_1 + f_2 \in J_2(N)$ where f is the identity on $\{x_a\}_{a \in F^*}$ and f is zero on the other basis elements.

We conclude with a definite result for the situation in which $\dim_R V$ is finite.

Theorem 4.9. *Let* $\dim_R V$ *be finite and let* $f \in Ne_{\bar{M}}$. *Then* $f \in J_2(N)$ *if and only if* f *is the sum of rank* 1 *functions.*

Proof. Suppose $f = \sum_{j=1}^{n} f_j$ where f_j is a rank 1 function. Since each f_j is in $J_2(N)$, so is f. Conversely let $f \in J_2(N)$ and let π_i be the ith projection map $i = 1, 2, \ldots, t$ where $t = \dim_R V$. Since $\pi_i \in \operatorname{End}_R V$, $\pi_i \in N$ so $\pi_i f \in J_2(N)$ and $\pi_i f$ is of rank 1. But $f = \sum_{i=1}^{t} \pi_i f$ so the proof is complete.

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REFERENCES

1. N. Jacobson, *Structure of Rings* (Amer. Math. Soc. Coll. Publ. Vol. 37, Providence, RI, 1964).

2. C. J. Maxson and K. C. Smith, The centralizer of a group automorphism, *J. Algebra* **54** (1978), 27–41.

3. C. J. Maxson and K. C. Smith, Simple near-ring centralizers of finite rings, *Proc. Amer. Math. Soc.* **75** (1979), 8–12.

4. C. J. Maxson and K. C. Smith, The centralizer of a set of group automorphisms, *Comm. in Alg.* **8** (1980), 211–230.

5. C. J. Maxson and K. C. Smith, Centralizer near-rings: Left ideals and 0-primitivity, *Proc. Royal Irish Acad.* **81** (1981), 187–199.

6. J. D. P. Meldrum and A. Oswald, Near-rings of mappings, *Proc. Royal Soc. Edinburgh* **83A** (1978), 213–223.

7. G. Pilz, *Near-rings* (North-Holland, New York, 1977).

8. M. Zeller, *Centralizer near-rings on infinite groups* (Ph.D. Dissertation, Texas A&M University, College Station, 1980).

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