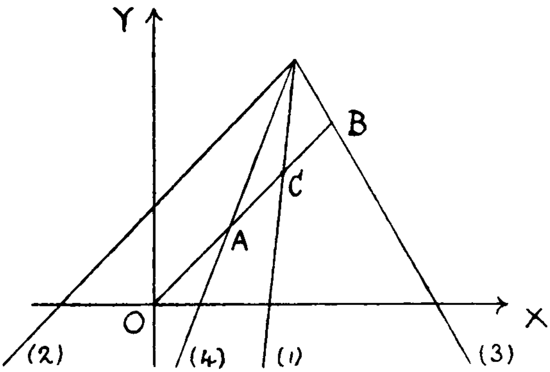
and since $B(x_2, y_2)$ lies on (3)

$$a_1 x_2 + b_1 y_2 + c_1 = +k c_2 .$$

Adding, and dividing by 2,

$$a_1 \frac{x_1 + x_2}{2} + b_1 \frac{y_1 + y_2}{2} + c_1 = 0.$$

Therefore the mid-point of AB lies on (1).



N. M'ARTHUR.

Trigonometrical Ratios of the half-angles of a Triangle (Geometrical Proofs).

1. ABC is a triangle; bisect angle A by AE; produce AB; draw BDF and CEG perpendicular to AE; join FG.

$GCFB$ is a cyclic trapezium

$$\therefore \quad GC \,.\, FB + CF \,.\, BG = BC \,.\, FG,$$

$$\therefore \quad 2\,EC \,.\, 2\,DB + (b-c)^2 = a^2,$$

$$\therefore \quad 4\,EC \,.\, DB = a^2 - (b-c)^2$$

$$= (a - b + c)(a + b - c)$$

$$= 4(s-b)(s-c),$$

$$\therefore \quad EC \,.\, DB = (s-b)(s-c).$$

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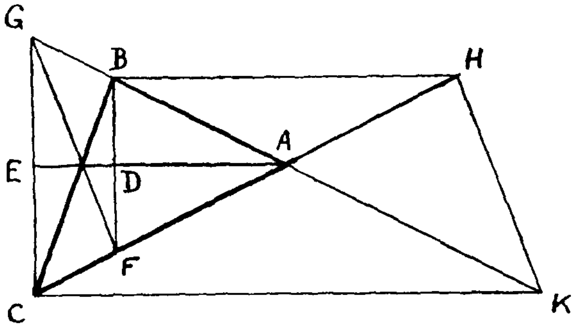
Now $\sin \frac{A}{2} = \frac{EC}{AC} = \frac{DB}{AB}$,

$$\therefore \quad \sin^2 \frac{A}{2} = \frac{EC \cdot DB}{AC \cdot AB} = \frac{(s-b)(s-c)}{bc},$$

$$\therefore \quad \sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}.$$



2. Produce CA to H making $AH = AB$; produce BA to K making $AK = AC$; join HK, BH, CK.

$CKHB$ is a cyclic trapezium

$$\therefore \quad CK \cdot BH + BC \cdot HK = BK \cdot HC,$$

$$\therefore \quad 2\,AE \cdot 2\,AD + a^2 = (b+c)^2,$$

$$\therefore \quad 4\,AE \cdot AD = (b+c)^2 - a^2$$

$$= (b+c+a)(b+c-a)$$

$$= 4\,s(s-a),$$

$$\therefore \quad AE \cdot AD = s(s-a).$$

Now $\cos \frac{A}{2} = \frac{AE}{AC} = \frac{AD}{AB}$,

$$\therefore \quad \cos^2 \frac{A}{2} = \frac{AE \cdot AD}{AC \cdot AB} = \frac{s(s-a)}{bc},$$

$$\therefore \quad \cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}.$$

3. Again $\tan\dfrac{A}{2}=\dfrac{EC}{AE}=\dfrac{DB}{AD}$,

$$\therefore\quad \tan^2\frac{A}{2}=\frac{EC\,.\,DB}{AE\,.\,AD}=\frac{(s-b)(s-c)}{s(s-a)},$$

$$\therefore\quad \tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

4. The formula of the area of a triangle in terms of the sides may be obtained by the direct use of these ratios, thus

$$\begin{aligned}\triangle ABC &= \triangle BCK - \triangle ACK\\ &= \tfrac{1}{2}CK(CE+DB)-\tfrac{1}{2}CK\cdot CE\\ &= \tfrac{1}{2}CK\,.\,DB\\ &= AE\,.\,DB\\ &= b\cos\frac{A}{2}\,.\,c\sin\frac{A}{2}\\ &= bc\sqrt{\frac{s(s-a)}{bc}}\sqrt{\frac{(s-b)(s-c)}{bc}}\\ &= \sqrt{s(s-a)(s-b)(s-c)}.\end{aligned}$$

ALEX. D. RUSSELL.

Note re *Prior Publication.*

We are in receipt of the following communication from Mr A. D. Russell, dated 20/12/20:—

"I have a p.c. from Mr R. C. Archibald, Editor of the *American Math. Monthly*, in which he points out that my "Proof of the Law of Tangents," *Proc. Edin. Math. Soc.*, Vol. XXXVIII., p. 58 (Nov. 1920), is identical with that by Mr Cheney published in the *Amer. Math. Monthly*, Feb. 1920. I need scarcely say that the publication of Mr Cheney's proof was quite unknown to me, but at the same time I should like to state that I have used the proof in my classes for a few years now. I find, for example, from some former pupils that their notebooks fully establish the fact that I gave the proof to a class in Falkirk Science and Art School on 6th Feb. 1918."