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GENERALIZATIONS OF THE SIMPLE TORSION CLASS AND THE SPLITTING PROPERTIES

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In this paper all rings R are associative rings with identity and all modules are members of R-mod, the category of unital left R-modules, unless the contrary is specifically stated.

A subclass $\mathscr{T}$ of R-mod is called a hereditary torsion class if $\mathscr{T}$ is closed under submodules, homomorphic images, direct sums, and extensions [**14**; **15**]. With each hereditary torsion class $\mathscr{T}$, there corresponds a unique class $\mathscr{F}$ such that $(\mathscr{T}, \mathscr{F})$ is a hereditary torsion theory [**2**; **12**; **14**; **15**]. Such a class $\mathscr{F}$ is called a torsion-free class and is closed under submodules, direct products, extensions, and injective hulls. $(\mathscr{T}, \mathscr{F})$ is called stable if $\mathscr{T}$ is closed under injective hulls [**14**; **15**; **17**].

Since simple modules play an important role in ring theory, one hereditary torsion class which is natural to study [**2**; **3**; **5**; **15**; **16**; **17**] is

$$\mathscr{S} = \{M \in R\text{-mod} \mid \text{every non-zero homomorphic image of } M \text{ has non-zero socle}\}.$$

$\mathscr{S}$ is called the simple torsion class. (Elsewhere in the literature (e.g. [**4**; **13**]), modules in $\mathscr{S}$ have also been studied under the name of Loewy modules.)

A hereditary torsion class $\mathscr{T}$ is called a generalization of the simple torsion class $\mathscr{S}$ if $\mathscr{T} \supseteq \mathscr{S}$. (This terminology comes from [**7**].)

The hereditary torsion classes, which arise from Krull dimensions, are important generalizations of $\mathscr{S}$. The Krull dimension of $M \in R$-mod, which will be denoted by $K \dim M$, is defined by transfinite recursion as follows: if $M = 0$, $K \dim M = -1$; if α is an ordinal and $K \dim M \nless \alpha$, then $K \dim M = \alpha$ provided that there is no infinite descending chain $M = M_0 \supset M_1 \supset \ldots$ of submodules M_i of M such that, for $i = 1, 2, \ldots$, $K \dim (M_{i-1}/M_i) \nless \alpha$. (It is of course possible that there is no ordinal α such that $K \dim M = \alpha$.) Given an ordinal α, we can define a hereditary torsion class $\mathscr{T}_\alpha$ by

$$\mathscr{T}_\alpha = \{M \in R\text{-mod} \mid \text{every non-zero homomorphic image of } M \text{ has a non-zero submodule with Krull dimension} < \alpha\}.$$

For any non-zero $M \in R$-mod, $K \dim M = 0$ if and only if M is an Artinian module. Hence it is an easy exercise to see that $\mathscr{T}_1 = \mathscr{S}$. Clearly, if $\alpha < \beta$, then $\mathscr{T}_\alpha \subseteq \mathscr{T}_\beta$; so $\mathscr{T}_\alpha$ is a generalization of $\mathscr{S}$ whenever $\alpha \geqq 1$. For properties of Krull dimension and $\mathscr{T}_\alpha$ the reader should consult [**10**].

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Let $\mathscr{T}$ be a hereditary torsion class with associated torsion theory $(\mathscr{T}, \mathscr{F})$. For $M \in R$-mod, let $\mathscr{T}(M)$ denote the (necessarily) unique largest submodule of M in $\mathscr{T}$. A R-module M is said to split if $\mathscr{T}(M)$ is a direct summand of M. Then $(\mathscr{T}, \mathscr{F})$ is said to have the cyclic splitting property (CSP) if every cyclic module splits. $(\mathscr{T}, \mathscr{F})$ is said to have the finitely generated splitting property ($FGSP$) if every finitely generated module splits. $\mathscr{T}(M)$ is said to have bounded order if $I\mathscr{T}(M) = 0$ for some (left) ideal I such that $R/I \in \mathscr{T}$; hence $(\mathscr{T}, \mathscr{F})$ has the bounded splitting property (BSP) if $\mathscr{T}(M)$ is a direct summand of M whenever $\mathscr{T}(M)$ has bounded order. Finally, $(\mathscr{T}, \mathscr{F})$ is said to have the splitting property (SP) if every module splits. For further discussion of these definitions, the reader is referred to [**15**; **17**].

The above splitting properties have been studied for the case $\mathscr{T} = \mathscr{S}$, but not for the case where $\mathscr{T}$ is a generalization of $\mathscr{S}$. In particular, the splitting properties of $(\mathscr{S}, \mathscr{F})$ are discussed for commutative rings in [**3**; **5**; **15**; **17**]; a result [**16**, Theorem 3.5] on SP for $(\mathscr{S}, \mathscr{F})$ has also been obtained for rings which have sufficiently many finitely generated, two-sided ideals. Also [**9**] and [**17**] give some general results on SP which may be applied to $(\mathscr{S}, \mathscr{F})$ under certain restrictive ring conditions.

In section one of this paper, we shall obtain theorems on the various splitting properties of generalizations of $\mathscr{S}$. In section two, these theorems are specialized to the case $\mathscr{T} = \mathscr{S}$; these resulting specializations generalize the main results of [**3**; **5**; **15**; **17**]. An example is given to show that the theorem of section two on SP applies to certain non-local, non-commutative rings that satisfy neither the hypotheses of Gorbachuk's theorems [**9**, Theorems 2 and 3] nor the author's results [**16**, Theorem 3.5].

In order to do this, we will be interested in the following two conditions that R may satisfy for a hereditary torsion class $\mathscr{T}$:

(*) Every two-sided idempotent ideal, which is finitely generated as a left ideal, has the form Re, where $e^2 = e$.

($*\mathscr{T}$) Every non-zero principal left ideal Rx properly contains a two-sided ideal I such that $Rx/I \in \mathscr{T}$.

If R satisfies ($*\mathscr{S}$) and if $\mathscr{T}$ is a generalization of $\mathscr{S}$, then R also satisfies ($*\mathscr{T}$). The following classes of rings satisfy both (*) and ($*\mathscr{S}$):

(1) commutative rings;

(2) von Neumann regular, left duo rings;

(3) von Neumann regular, left semi-artinian rings;

(4) local right perfect rings, where "local" means that the ring has unique maximal left ideal;

(5) left and right noetherian, hereditary integral domains with no two-sided idempotent ideals (e.g. the ring $D[[x]]$ of all power series with coefficients in a division ring D).

Several other interesting examples of rings which satisfy both (*) and ($*\mathscr{S}$) are given in section two.

It is possible that a generalization $\mathscr{T}$ of $\mathscr{S}$ may satisfy $(*\mathscr{T})$, but not $(*\mathscr{S})$. To illustrate this fact, we now show how to construct a ring R which satisfies $(*)$ and $(*\mathscr{T}_{\alpha+2})$ for a given non-limit ordinal α, but does not satisfy $(*\mathscr{T}_\beta)$ for any $\beta \leqq \alpha + 1$.

Example 0.1. Let α be a non-limit ordinal. Let D be a commutative integral domain of Krull dimension α such that D has an automorphism ϕ of infinite period. (The existence of such a domain D is justified in the remark following this example.) Let $T = D[x; \phi]$ be the twisted polynomial ring; i.e. the additive group is the additive group of the polynomial ring $D[x]$, and multiplication in $D[x; \phi]$ is defined by $xd = \phi(d)x$ and its consequences. T is a left Öre domain and hence T is a left order in a division ring F. Let R be the subring of power series ring $F[[y]]$ such that the "constant" term of every member of R is in T; i.e.,

$$R = \left\{ t + \sum_{i=1}^{\infty} a_i y^i \mid t \in T, a_i \in F \right\}.$$

We now outline a proof for showing that R has the desired properties: R satisfies $(*)$ and $(*\mathscr{T}_{\alpha+2})$, but R does not satisfy $(*\mathscr{T}_\beta)$ for any $\beta \leqq \alpha + 1$.

(1) Each two-sided ideal of T is either generated by an element of the form x^n for some integer n or else contains a nonzero element of D. (Consider an element which has least degree among members of the ideal.)

(2) Let

$$z = t + \sum_{i=1}^{\infty} a_i y^i.$$

If $t \neq 0$, then for each $b \in F$ and each positive integer k, there exists $\sum_{i=1}^{\infty} b_i y^i \in R$ such that

$$\left(\sum_{i=1}^{\infty} b_i y^i\right)\left(t + \sum_{i=1}^{\infty} a_i y^i\right) = by^k.$$

(Solve the coefficient equations inductively.)

(3) By (2), Rz contains the two-sided ideal M generated by the set $\{by \mid b \in F\}$.

(4) If the degree of $t = \sum_{i=0}^{n} d_i x^i \in T$ is positive (i.e., $n \geqq 1$) and $d_0 \neq 0$, then Tt contains no two-sided ideals by (1). Hence, if the degree of t is positive and $d_0 \neq 0$, then every proper two-sided ideal I of R which is contained in Rz is contained in M.

(5) Let $z' = 1 + x \in R$. If I is a two-sided ideal such that $Rz'/I \in \mathscr{T}_\beta$, then by (4), $Rz'/M \in \mathscr{T}_\beta$.

(6) There is a lattice isomorphism between the R-submodules of Rz'/M and the T-submodules of Tz'. Moreover, Tz' is an α-critical T-submodule of T. (See [**10**, Lemma 6.3].) Hence $K \dim_R Rz'/M = K \dim_T Tz' = (K \dim_D D) + 1 = \alpha + 1$, and $K \dim_R N = K \dim_T Tz' = \alpha + 1$ for any submodule N of Rz'/M by [**10**, Proposition 2.3].

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(7) By (6), $Rz'/M \in \mathscr{T}_{\alpha+2}$, but $Rz'/M \notin \mathscr{T}_\beta$ for any $\beta \leqq \alpha + 1$. Hence R does not satisfy $(*\mathscr{T}_\beta)$ for any $\beta \leqq \alpha + 1$ by (5).

(8) By factoring out the highest power of y in z and applying (2), we see that every principal left ideal of R contains M^n for some integer n.

(9) Since $K \dim_R R/M = K \dim_T T = \alpha + 1$, then by (8), $R/L \in \mathscr{T}_{\alpha+2}$ for every non-zero left L of R.

(10) From (8) and (9) it follows that R satisfies $(*\mathscr{T}_{\alpha+2})$.

(11) Let I be an idempotent, two-sided ideal of R which is finitely generated. The coefficients of y^0 of members of I form a finitely generated idempotent ideal I' of T; so by (1), I' must contain an element of D. The coefficients of x^0 of members of I' form a finitely generated idempotent ideal I'' of D. Since D is a commutative domain, $I'' = 0$ or $I'' = D$. If $I'' = D$, then the existence of an element of I' in D implies that $1 \in I'$ and hence $I' = T$; from (2) it now follows that $I \supseteq M$, and hence $I = R$. If $I'' = 0$, then $I' = 0$ and hence $I \subseteq M$; by considering the least positive integer in the set $\{h|y^h$ has nonzero coefficient for some member of $I\}$, it is easy to see that $I^2 = I$ implies $I = 0$. Hence R satisfies (*).

Remark. The example above depends on the existence of certain integral domains D having an automorphism ϕ of infinite period. We now indicate two constructions for such D, one for finite ordinals and one for the general non-limit ordinal case.

(1) Let C be the algebraic closure of Z_2, the field of two elements. Let ρ be the automorphism of C defined by $\rho(a) = a^2$ for each $a \in C$. If $\alpha = n$ is a finite ordinal, extend ρ to an automorphism ϕ of the polynomial ring $D = C[x_1, x_2, \ldots, x_n]$ by $\phi(x_i) = x_i$ for $i = 1, 2, \ldots, n$. Then $K \dim D = n = \alpha$, and ϕ has infinite period. (Note: if $\alpha = 0$, let $D = C$ and $\phi = \rho$.)

(2) Let α be a non-limit ordinal. By [**10**, Theorem 9.6], there exists a commutative integral domain C with Krull dimension $\alpha - 1$. By examining the proof of [**10**, Theorem 9.6], we also see that if the base field in the construction for C has characteristic zero, then so does C. (The construction of C is done by forming a big polynomial ring over the base field, localizing at a prime ideal generated by a "gang" of indeterminates, and then passing to a homomorphic image.) Now let $D = C[u]$, the polynomial ring in the indeterminate u. Then $K \dim D = \alpha$, and

$$\phi: D \to D: \sum_{i=1}^{n} c_i u^i \to \sum_{i=1}^{n} c_i (u+1)^i$$

is the desired automorphism of infinite period.

Now let $(\mathscr{T}, \mathscr{F})$ be a hereditary torsion theory of R-modules, and let I be a two-sided ideal of R. Then $(\mathscr{T}, \mathscr{F})$ induces a torsion theory $(\mathscr{T}', \mathscr{F}')$ of R/I-modules in a natural way: $\mathscr{T}' = \{M \in R/I\text{-mod}|M \in \mathscr{T}$, where M is viewed as an R-module via $xm = (x + I)m$ for all $x \in R$ and $m \in M\}$. Since an R/I module is a simple R/I-module if and only if it is simple as an R-module in the natural way, then the torsion class $\mathscr{S}'$ induced by $\mathscr{S}$, is just the simple torsion class for R/I-mod.

LEMMA 0.2. *Let* $(\mathscr{T}, \mathscr{F})$ *be a hereditary torsion theory for* R-mod, *let* I *be a two-sided ideal of* R, *and let* $(\mathscr{T}', \mathscr{F}')$ *be the torsion theory of* R/I-mod *induced by* $(\mathscr{T}, \mathscr{F})$.

(1) *If* $(\mathscr{T}, \mathscr{F})$ *has CSP (FGSP, BSP, SP), then* $(\mathscr{T}', \mathscr{F}')$ *has CSP (FGSP, BSP, SP) for* R/I*-modules.*

(2) *If* I *is a finitely generated idempotent left ideal and if* R *satisfies* (*) *and* $(*\mathscr{T})$, *then* R/I *satisfies* (*) *and* $(*\mathscr{T}')$.

Proof. (1) is known (e.g., see [**17**, p. 72] or [**15**, p. 452]). Both (1) and (2) are straight forward to prove from the appropriate definitions.

1. Splitting properties for generalizations of $\mathscr{S}$. Before we can give characterizations of the splitting properties for generalizations of $\mathscr{S}$, we need several elementary lemmas.

LEMMA 1.1. *Let* $\mathscr{T}$ *be a generalization of* $\mathscr{S}$, *and let* $(\mathscr{T}, \mathscr{F})$ *be a hereditary torsion theory with CSP. If* $R/I \in \mathscr{F}$, *then* $I^2 = I$.

Proof. Replace $\mathscr{S}$ by $\mathscr{T}$ in the proof of [**15**, Proposition 2.1].

LEMMA 1.2. *Let* $\mathscr{T}$ *be a generalization of* $\mathscr{S}$, *and let* $(\mathscr{T}, \mathscr{F})$ *be a hereditary torsion theory for* R-mod. *Let* $R \in \mathscr{F}$, *and let* R *satisfy* $(*\mathscr{T})$. *If* $R/I \in \mathscr{F}$ *and if* $\oplus \sum_{\alpha \in \mathscr{A}} Rx_\alpha \subseteq I$, *then there exists a collection* $\{I_\alpha\}_{\alpha \in \mathscr{A}}$ *of two-sided ideals satisfying the following conditions*:

(1) $Rx_\alpha \subseteq I_\alpha \subseteq I$ *for each* $\alpha \in \mathscr{A}$;

(2) $R/I_\alpha \in \mathscr{F}$ *for each* $\alpha \in \mathscr{A}$;

(3) $I_\alpha/Rx_\alpha \in \mathscr{T}$ *for each* $\alpha \in \mathscr{A}$;

(4) $\sum_{\alpha \in \mathscr{A}} I_\alpha$ *is direct.*

Proof. By $(*\mathscr{T})$ there exists, for each $\alpha \in \mathscr{A}$, $J_\alpha \subset Rx_\alpha$ such that J_α is a two-sided ideal of R and $Rx_\alpha/J_\alpha \in \mathscr{T}$. Define I_α by $I_\alpha/J_\alpha = \mathscr{T}(R/J_\alpha)$ for each $\alpha \in \mathscr{A}$. Clearly $Rx_\alpha \subseteq I_\alpha$, (2) holds, and (3) holds. Since $(I_\alpha + I)/I \in \mathscr{F}$ and since $(I_\alpha + I)/I \cong I_\alpha/(I \cap I_\alpha)$ is a homomorphic image of $I_\alpha/J_\alpha \in \mathscr{T}$, then $(I_\alpha + I)/I = 0$; hence $I_\alpha \subseteq I$. Since J_α is a two-sided ideal for each α, so is I_α.

If $0 \neq x \in I_\beta \cap \sum_{\alpha \in \mathscr{A} - \{\beta\}} I_\alpha$, then $0 \neq x = a_\beta = \sum_{\alpha \in B} a_\alpha$, where B is a finite subset of $\mathscr{A} - \{\beta\}$. Since $R \in \mathscr{F}$, $(Rx_\alpha : a_\alpha)$ is an essential left ideal for each $\alpha \in B \cup \{\beta\}$. Hence there exists

$$y \in \bigcap_{\alpha \in B \cup \{\beta\}} (Rx_\alpha : a_\alpha)$$

such that $0 \neq yx = ya_\beta = \sum_{\alpha \in B} ya_\alpha \in Rx_\beta \cap \sum_{\alpha \in B} Rx_\alpha = 0$, which is a contradiction. Hence (4) holds.

LEMMA 1.3. *Let* R, $(\mathscr{T}, \mathscr{F})$, *and* $\{I_\alpha\}_{\alpha \in \mathscr{A}}$ *be as in Lemma* 1.2. *If* R *satisfies* (*)

and $(\mathscr{T}, \mathscr{F})$ *has CSP, then, there exists a set of orthogonal idempotents* $\{e_\alpha\}_{\alpha \in \mathscr{A}}$ *such that* $I_\alpha = Re_\alpha$ *for each* $\alpha \in \mathscr{A}$.

Proof. By Lemma 1.1, $I_\alpha = I_\alpha^2$ for each $\alpha \in \mathscr{A}$. By CSP, I_α/J_α is a cyclic module; so from part (1) of Lemma 1.2, it follows that each I_α is generated by two elements. Hence the result follows from (*).

LEMMA 1.4. *Let* $(\mathscr{T}, \mathscr{F})$ *be a hereditary torsion theory for* R-mod *such that* $R \in \mathscr{F}$ *and* R *satisfies* $(^*\mathscr{T})$. *Then every (module) direct summand of* R *is a two-sided ideal of* R.

Proof. Let ${}_RR = A \oplus B$, and suppose that $b \in B$. The map $A \to Ab$ given by right multiplication is an R-epimorphism. By $(^*\mathscr{T})$ there is a two-sided ideal T of R such that $T \subseteq A$ and $A/T \in \mathscr{T}$. But $Tb \subseteq A \cap B = 0$; so the induced epimorphism $A/T \to Ab$ implies that $Ab \in \mathscr{T}$. As $R \in \mathscr{F}$, then $AB = 0$.

We now can state our first main result, which characterizes CSP for generalizations of $\mathscr{S}$.

THEOREM 1.5. *Let* $\mathscr{T}$ *be a generalization of* $\mathscr{S}$, *and let* $(\mathscr{T}, \mathscr{F})$ *be a hereditary torsion theory for* R-mod. *If* $R \in \mathscr{F}$ *and* R *satisfies* (*) *and* $(^*\mathscr{T})$, *then the following statements are equivalent.*
(1) $(\mathscr{T}, \mathscr{F})$ *has CSP.*
(2) *If* $R/K \in \mathscr{F}$, *then* K *is a ring direct summand of* R.
(3) *Every cyclic in* $\mathscr{F}$ *is projective.*

Proof. (2) $\Rightarrow$ (3) $\Rightarrow$ (1) is trivial.

Assume (1) holds. Let $R/K \in \mathscr{F}$, and let $\oplus \sum_{\alpha \in \mathscr{A}} Rx_\alpha$ be a direct sum of cyclic modules such that $\oplus \sum_{\alpha \in \mathscr{A}} Rx$ is an essential submodule of K. By Lemma 1.3, there exists a family $\{e_\alpha\}_{\alpha \in \mathscr{A}}$ of orthogonal idempotents such that $Re_\alpha = I_\alpha$, where I_α is as in Lemma 1.2. Let $I/\oplus \sum_{\alpha \in \mathscr{A}} I_\alpha = \mathscr{T}(R/\sum_{\alpha \in \mathscr{A}} I_\alpha)$ define the two-sided ideal I. Since $I/\sum_{\alpha \in \mathscr{A}} I_\alpha \in \mathscr{T}$ and $(I + K)/K \in \mathscr{F}$, it follows from the existence of an epimorphism

$$I/\sum_{\alpha \in \mathscr{A}} I_\alpha \to I/(I \cap K) \cong (I + K)/K$$

that $I \subseteq K$. By (1), $I/\sum_{\alpha \in \mathscr{A}} I_\alpha$ has an idempotent generator $g + \sum_{\alpha \in \mathscr{A}} I_\alpha$ in $R/\sum_{\alpha \in \mathscr{A}} I_\alpha$.

Case 1. If $Rg = I$, then by (*) $I = Re$ for some $e = e^2$. Since I is essential in K it follows that $I = K$; so (2) follows from Lemma 1.4.

Case 2. If $Rg = 0$, then $I = \oplus \sum_{\alpha \in \mathscr{A}} I_\alpha$. Let M_α be a maximal submodule of $I_\alpha = Re_\alpha$ for each $\alpha \in \mathscr{A}$. Then

$$\mathscr{T}\Big(R/\oplus \sum_{\alpha \in \mathscr{A}} M_\alpha\Big) = I/\oplus \sum_{\alpha \in \mathscr{A}} M_\alpha \cong \oplus \sum_{\alpha \in \mathscr{A}} I_\alpha/M_\alpha$$

must be finitely generated by (1). Hence $\mathscr{A}$ is a finite set; so I is finitely generated and a summand of R. Consequently (2) follows from Lemma 1.4.

Case 3. If $0 \neq Rg \neq I$, then by $(*\mathscr{T})$ there exists a two-sided ideal G such that $G \subseteq Rg$ and $0 \neq Rg/G \in \mathscr{T}$. As in the proof of Lemma 1.3 there exists an idempotent e such that $Re/G = \mathscr{T}(R/G)$ and $Re \subseteq I$.

Write $R = Re \oplus F$, where $F = R(1 - e)$. Note that $I = Re + \sum_{\alpha \in \mathscr{A}} Re_\alpha$. By the modular law $I = Re \oplus (I \cap F)$. But

$$I \cap F = (I \cap F)(1 - e) \subseteq I(1 - e) \cap F \subseteq \Big(\sum_{\alpha \in \mathscr{A}} Re_\alpha \Big) \cap F$$

by Lemma 1.4. Hence $I = Re \oplus [(\sum_{\alpha \in \mathscr{A}} Re_\alpha) \cap F]$.

By the modular law,

$$\sum_{\alpha \in \mathscr{A}} Re_\alpha = \Big[\Big(\sum_{\alpha \in \mathscr{A}} Re_\alpha \Big) \cap Re\Big] \oplus \Big[\Big(\sum_{\alpha \in \mathscr{A}} Re_\alpha \Big) \cap F\Big].$$

By [**11**, Theorem 1], $(\sum_{\alpha \in \mathscr{A}} Re_\alpha) \cap F$ is a direct sum of countably generated modules. If $(\sum_{\alpha \in \mathscr{A}} Re_\alpha) \cap F$ is not finitely generated, then $(\sum_{\alpha \in \mathscr{A}} Re_\alpha) \cap F$ has a countably generated direct summand W which is not finitely generated. By closure under extensions and by $R \in \mathscr{F}$, $R/W \in \mathscr{F}$. Let W be generated by the set $\{x_i\}_{i=1}^{\infty}$. For each positive integer n, there exists a least positive integer $k(n)$ such that $x_{k(n)} \notin Rx_1 + Rx_2 + \ldots + Rx_n$. By Zorn's Lemma, choose K_n maximal with respect to $x_{k(n)} \notin K_n$ and $\sum_{i=1}^{n} Rx_n \subseteq K_n \subseteq W$. Then $(Rx_{k(n)} + K_n)/K_n$ is an essential simple submodule of R/K_n; so $(Rx_{k(n)} + K_n)/K_n \in \mathscr{T}$. Since $(\mathscr{T}, \mathscr{F})$ has (CSP), then $(\mathscr{T}, \mathscr{F})$ is stable; hence $R/K_n \in \mathscr{T}$. Define

$$\varphi: W \to \oplus \sum_{n=1}^{\infty} W/K_n : w \to \sum_{n} \varphi_n(w),$$

where $\varphi_n : W \to W/K_n$ is the canonical epimorphism given by $w \to w + K_n$. If $H = \ker \varphi$, then $W/H \cong$ image of φ. Since $R/W \in \mathscr{F}$, then $\mathscr{T}(R/H) \subseteq W/H$, which is a direct summand of R/H by (1). Hence $\mathscr{T}(R/H)$ is finitely generated. But $\mathscr{T}(R/H) = \mathscr{T}(W/H)$ cannot be finitely generated; for otherwise the isomorphic copy of $\mathscr{T}(W/H)$ in the image of φ would have non-zero coordinates in finitely many W/K_n. This contradiction shows that $(\sum_{\alpha \in \mathscr{A}} Re_\alpha) \cap F$ must be finitely generated.

Therefore, $I = Re \oplus [(\sum_{\alpha \in \mathscr{A}} Re_\alpha) \cap F]$ is finitely generated. By Lemma 1.3 and (*), I is generated by an idempotent element and hence is a direct summand of R. Since I is an essential submodule of K, $I = K$; so (2) follows from Lemma 1.4.

One widely studied torsion theory is Goldie's torsion theory $(\mathscr{G}, \mathscr{N})$; e.g., see [**6**; **8**; **15**] and their references. $\mathscr{G}$ is the smallest torsion class containing the singular modules; $\mathscr{N}$ is precisely the class of nonsingular modules. If R is a commutative integral domain, then $\mathscr{G}$ coincides with the class of modules which are torsion in the classical sense.

By Theorem 1.5 and an argument of [**15**, p. 459], we have the following result.

COROLLARY 1.6. *Let* $(\mathscr{T}, \mathscr{F})$ *and* R *be as in the hypotheses of Theorem* 1.5. *If* $(\mathscr{T}, \mathscr{F})$ *has CSP, then* $\mathscr{T} = \mathscr{G}$.

As a consequence of the corollary, when R is a commutative integral domain and $\mathscr{T}$ is a generalization of $\mathscr{S}$, $(\mathscr{T}, \mathscr{F})$ has *CSP* if and only if $\mathscr{T}$ contains the usual torsion class. In particular, if R has Krull dimension α, then $(\mathscr{T}_\beta, \mathscr{F}_\beta)$ has *CSP* if and only if $\beta \geqq \alpha$.

But, for the ring of Example 0.1, $(\mathscr{T}_{\alpha+2}, \mathscr{F}_{\alpha+2})$ has *CSP* by Theorem 1.5 and part (9) of Example 0.1. However, $\mathscr{S} \neq \mathscr{T}_{\alpha+2} \neq R$-mod.

LEMMA 1.7. *Let* $\mathscr{T}$ *be a hereditary torsion class for* R-mod. *If* R *satisfies* $(*\mathscr{T})$, *then every left ideal* I *contains a two-sided ideal* I' *such that* $I/I' \in \mathscr{T}$; *moreover, if* $\mathscr{T}(R) = 0$, *then* I' *is essential in* I.

Proof. For each $x \in I$, we use $(*\mathscr{T})$ to find a two-sided ideal I_x such that $Rx/I_x \in \mathscr{T}$. Set $I' = \sum_{x \in I} I_x$. It is easy to see that I' has the desired properties.

We now can use Corollary 1.6 and Lemma 1.7 to apply results of [**6**] and [**8**] in order to obtain results about *FGSP* and *BSP* for generalizations of $\mathscr{S}$.

THEOREM 1.8. *Let* $\mathscr{T}$ *be a generalization of* $\mathscr{S}$, *and let* $(\mathscr{T}, \mathscr{F})$ *be a torsion theory for* R-mod. *If* $R \in \mathscr{F}$ *and if* R *satisfies* $(*)$ *and* $(*\mathscr{T})$, *then the following statements are equivalent.*

(1) $(\mathscr{T}, \mathscr{F})$ *has FGSP.*

(2) $\mathscr{T} = \mathscr{G}$, *and each finitely generated module* $F \in \mathscr{F}$ *has the following properties*: (a) *F is finitely related*; (b) *hd* $F \leqq 1$; (c) $\mathrm{Tor}_1^R(\mathrm{Hom}_z(A, D), F) = 0$ *for any* $A \in \mathscr{T}$ *and any divisible Abelian group* D.

(3) (a) $(A : x)$ *is finitely generated for every* $x \in E(R)$ *and every finitely generated* $A \subseteq E(R)$, *where* $E(R)$ *denotes the injective hull of* R;

(b) *if* I *is any right ideal which contains a two-sided, essential left ideal of* R, *then* I_R *is flat and* $\mathrm{Tor}_1^R(R/I, E(R)) = 0$; *and*

(c) *if* L *is an essential left ideal of* R, *then* $R/L \in \mathscr{T}$.

Proof. By Corollary 1.6, $\mathscr{T} = \mathscr{G}$; so the equivalence of (1) and (2) follows from [**6**, Corollary 2]. By $R \in \mathscr{F}$ and Lemma 1.7, every essential left ideal of R contains an essential two-sided ideal of R; so the equivalence of (1) and (3) follows from [**8**, Theorem 4.9].

COROLLARY 1.9. *Let* R *be a commutative ring, let* $\mathscr{T}$ *be a generalization of* $\mathscr{S}$, *and let* $(\mathscr{T}, \mathscr{F})$ *be a torsion theory for* R-mod. *If* $R \in \mathscr{F}$, *then the following statements are equivalent.*

(1) $(\mathscr{T}, \mathscr{F})$ *has FGSP.*

(2) *R is semihereditary*, $R \cap A$ *is finitely generated for every finitely generated* $A \subseteq E(R)$, *and* $R/L \in \mathscr{T}$ *for every essential left ideal* L *of* R.

Proof. Combine Corollary 1.6 and [**8**, Corollary 4.10] to obtain this result.

Before we can deal with the *BSP* for $(\mathscr{T}, \mathscr{F})$, we must introduce a Loewy-type construction and prove a technical homological lemma.

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Suppose that T is a two-sided ideal of R and that R/T is a right semiartinian ring (which occurs whenever R/T is left perfect). Define the two-sided ideals T_α of R inductively as follows: $T_0 = T$; if α is not a limit ordinal, then $T_\alpha/T_{\alpha-1}$ is the right socle of $R/T_{\alpha-1}$; if α is a limit ordinal, then $T_\alpha = \bigcup_{\beta<\alpha} T_\beta$. Hence the set $\{T_\alpha/T\}_{\alpha\in\mathscr{A}}$ forms a right Loewy series for R/T, where $\mathscr{A}$ is an index set of ordinals such that $T_\beta/T = R/T$ for some $\beta \in \mathscr{A}$.

LEMMA. 1.10. *Suppose that R/T is a left perfect ring and that K is a right ideal of R satisfying $T_\alpha \subseteq K \subseteq T_{\alpha+1}$ for some ordinal α. If T_α and $T_{\alpha+1}$ are flat as right R-modules, so is K.*

Proof. Since $T_{\alpha+1}$ is flat, we have the exact sequence

$$\mathrm{Tor}_2{}^R(T_{\alpha+1}/K, \_) \rightarrow \mathrm{Tor}_1{}^R(K, \_) \rightarrow \mathrm{Tor}_1{}^R(T_{\alpha+1}, \_) = 0;$$

so it suffices to show that $\mathrm{Tor}_2{}^R(T_{\alpha+1}/K, \_) = 0$. Since $T_{\alpha+1}/T_\alpha$ is semisimple, then as a right R-module

$$T_{\alpha+1}/T_\alpha \cong (K/T_\alpha) \oplus (T_{\alpha+1}/K),$$

and hence

$$\mathrm{Tor}_2{}^R(T_{\alpha+1}/T_\alpha, \_) \cong \mathrm{Tor}_2{}^R(K/T_\alpha, \_) \oplus \mathrm{Tor}_2{}^R(T_{\alpha+1}/K, \_).$$

Consequently, it is sufficient to show that $\mathrm{Tor}_2{}^R(T_{\alpha+1}/T_\alpha, \_) = 0$. But this follows from the flatness of $T_{\alpha+1}$ and T_α and the exact sequence

$$\mathrm{Tor}_2{}^R(T_{\alpha+1}, \_) \rightarrow \mathrm{Tor}_2{}^R(T_{\alpha+1}/T_\alpha, \_) \rightarrow \mathrm{Tor}_1{}^R(T_\alpha, \_).$$

THEOREM 1.11. *Let $\mathscr{T}$ be a generalization of $\mathscr{S}$, and let $(\mathscr{T}, \mathscr{F})$ be a torsion theory. If $R \in \mathscr{F}$ and if R satisfies (*) and (*$\mathscr{T}$), then the following statements are equivalent.*

(1) *$(\mathscr{T}, \mathscr{F})$ has BSP and is stable.*

(2) *R is a finite direct sum of left Öre domains $D_i (i = 1, 2, \ldots, n)$, each of which has the following properties:*

(a) *for each two-sided ideal I of D_i, D_i/I is a left perfect ring and $D_i/I \in \mathscr{T}$;*

(b) *if H is any right ideal of D_i which contains a two-sided ideal, then H is flat and* $\mathrm{Tor}_1{}^{D_i}(D_i/H, E(D_i)) = 0$.

(3) *R is a finite direct sum of left Öre domains $D_i (i = 1, 2, \ldots, n)$, each of which satisfies the following properties:*

(i) *for each two-sided ideal I of D_i, D_i/I is a left perfect ring and $D_i/I \in \mathscr{T}$;*

(ii) *each two-sided ideal of D_i is flat as a right module;*

(iii) *if M is a maximal right ideal of D_i which contains a two-sided ideal, then* $\mathrm{Tor}_1{}^{D_i}(D_i/M, E(D_i)) = 0$.

Proof. (1) $\Rightarrow$ (2). Let M be a finitely generated module. By (1) and [**17**, Lemma 3.2], there exists a left ideal I of R such that every element of $\mathscr{T}(M)$ has an annihilator of the form $\bigcap_{i=1}^n (I : r_i)$, where $r_1, r_2, \ldots, r_n \in R$ and $R/I \in$

$\mathscr{T}$. By ($*\mathscr{T}$) and Lemma 1.7, I contains a two-sided ideal I' such that $I/I' \in \mathscr{T}$. Hence $I'\mathscr{T}(M) = 0$; so $\mathscr{T}(M)$ has bounded order. By (1) M splits. Therefore, $(\mathscr{T}, \mathscr{F})$ has *FGSP* and hence *CSP*.

Thus $\mathscr{T} = \mathscr{G}$ by Corollary 1.6. Moreover, since ($*\mathscr{T}$) holds, it follows from Lemma 1.7 that our definition of *BSP* for $\mathscr{G}$ coincides with the definition for *BSP* given in [**8**] (in this case). Whenever R is a direct sum of rings, then $(\mathscr{G}, \mathscr{N})$ has *BSP* if and only if the Goldie torsion theory for each direct summand has *BSP* (see [**15**, p. 452]). It now follows from [**8**, Theorem 5.3] that it is sufficient to show that R is a (ring) direct sum of finitely many left Öre domains.

Let $\sum_{\alpha \in \mathscr{A}} Rx_\alpha$ be any essential submodule of R with $x_\alpha \neq 0$ for each $\alpha \in \mathscr{A}$. By the proof of Lemma 1.3 and by Lemma 1.4, there exist two-sided ideals $\{J_\alpha\}_{\alpha \in \mathscr{A}}$ and orthogonal idempotents $\{e_\alpha\}_{\alpha \in \mathscr{A}}$ such that $J_\alpha \subsetneq Rx_\alpha \subseteq Re_\alpha$. By [**8**, Theorem 5.3], $R/\sum_{\alpha \in \mathscr{A}} J_\alpha$ is a left perfect ring. Since $\{e_\beta + \sum_{\alpha \in \mathscr{A}} J_\alpha\}_{\beta \in \mathscr{A}}$ is a set of non-zero orthogonal idempotents in $R/\sum_{\alpha \in \mathscr{A}} J_\alpha$, then $\mathscr{A}$ must be a finite set by [**1**, Theorem P]. Hence ${}_RR$ is finite dimensional.

Let $\oplus \sum_{i=1}^n Ry_i$ be a maximal direct sum of nonzero uniform left ideals of R. By Lemma 1.3, there exists a set $\{e_i\}_{i=1}^n$ of orthogonal idempotents such that

$$\oplus \sum_{i=1}^n Ry_i \subseteq \oplus \sum_{i=1}^n Re_i;$$

thus $R = \oplus \sum_{i=1}^n Re_i$ is a ring direct sum by Lemma 1.4. Set $D_i = Re_i$. Since $R \in \mathscr{F}$, then Ry_i is an essential uniform submodule of $Re_i = D_i$; thus D_i must be an integral domain (as $\mathscr{T} = \mathscr{G}$).

(2) $\Rightarrow$ (1). Condition ($*\mathscr{T}$) and $R \in \mathscr{F}$ imply that the set of essential left ideals has a cofinal subset of two-sided ideals (by Lemma 1.7). Hence our definition of *BSP* for $(\mathscr{G}, \mathscr{N})$ coincides with that of [**8**] in this case. From (2) and [**8**, Theorem 5.3] it follows that the Goldie theory for each D_i has *BSP*, and hence $(\mathscr{G}, \mathscr{N})$ must have *BSP* (as $R = D_i \dotplus D_2 \dotplus \ldots \dotplus D_n$). Also $(\mathscr{G}, \mathscr{N})$ is stable. But condition 2(a) and Lemma 1.7 imply that a cyclic module is in $\mathscr{G}$ if and only if it is in $\mathscr{T}$; hence $\mathscr{T} = \mathscr{G}$.

(2) $\Rightarrow$ (3). This is trivial.

(3) $\Rightarrow$ (2). First, we let H be a right ideal of D, where D is any D_i. Assuming that H contains a two-sided ideal T, we wish to show that (i) and (ii) imply that H is a flat right D-module. This will be done by transfinite induction. By (i) D/T is left perfect; so we define $K_\alpha = H \cap T_\beta$ for all $\beta \in \mathscr{A}$. (T_β is defined just prior to Lemma 1.10.)

Since $T = T_0 \subseteq K_1 \subseteq T_1$, then Lemma 1.10 and (ii) imply that K_1 is a flat right D-module.

Suppose that $\beta = \alpha + 1$ is not a limit ordinal, and suppose $K_{\beta-1}$ is a flat right D-module. By Lemma 1.10 and (ii), K is a flat right D-module whenever K is a right ideal such that $T_\alpha \subseteq K \subseteq T_{\alpha+1}$. Set $K = K_\beta + T_\alpha = (H \cap T_{\alpha+1}) + T_\alpha$. Then K is a flat right D-module, and the exact sequence

$$0 \to K_\beta \to K \to K/K_\beta \to 0$$

yields the exact sequence

$$\mathrm{Tor}_2^D(K/K_\beta, \_) \to \mathrm{Tor}_1^D(K_\beta, \_) \to \mathrm{Tor}_1^D(K, \_) = 0.$$

Thus it suffices to show that $\mathrm{Tor}_2^R(K/K_\beta, \_) = 0$. Now $K/K_\beta \cong ((\mathrm{H} \cap T_\beta) + T_{\beta-1})/(H \cap T_\beta) \cong T_{\beta-1}/(H \cap T_{\beta-1} \cap T_\beta) = T_{\beta-1}/(H \cap T_{\beta-1})$. Since $T_{\beta-1}$ is flat by (ii) and since $H \cap T_{\beta-1}$ is flat by our induction hypothesis, then there is an exact sequence

$$0 = \mathrm{Tor}_2^D(T_{\beta-1}, \_) \to \mathrm{Tor}_2^D(T_{\beta-1}/(H \cap T_{\beta-1}), \_) \to \mathrm{Tor}_1^D(H \cap T_{\beta-1}, \_) = 0.$$

Hence $\mathrm{Tor}_2^D(K/K_\beta, \_) \cong \mathrm{Tor}_2^D(T_{\beta-1}/(H \cap T_{\beta-1}), \_) = 0$.

Let β be a limit ordinal, and assume that K_α is a flat right D-module for all $\alpha < \beta$. Since $K_\beta = H \cap T_\beta = \bigcup_{\alpha<\beta} (H \cap T_\alpha) = \varinjlim H \cap T_\alpha$, then

$$\mathrm{Tor}_1^D(K_\beta, \_) = \mathrm{Tor}_1^D(\varinjlim K_\alpha, \_) = \varinjlim \mathrm{Tor}_1^D(K_\alpha, \_) = 0.$$

Hence K_β is a flat right D-module.

Since D/T is left perfect, $H = H \cap T_\beta$ for some ordinal β; hence H must be a flat right D-module.

Next, we wish to show that if H is a right ideal of D which contains a two-sided ideal T, then (i) and (iii) imply that $\mathrm{Tor}_1^D(D/H, E(D)) = 0$, where D is any D_i. By (i), D/T is left perfect; so every nonzero homomorphic image of the right D-module D/H has nonzero (right) socle. Moreover each simple right module which appears in the (right) Loewy series for D/H (see [**4**; **13**; **16**]) must be annihilated by T; i.e. the annihilator of any element of a simple right module which appears in the Loewy series for D/H must be a maximal right ideal which contains a two-sided ideal. Hence if D/M is a simple right module which appears in the Loewy series for D/H, then $\mathrm{Tor}_1^D(D/M, E(D)) = 0$ by (iii). Since Tor_1^D commutes with direct sums and direct limits, an easy transfinite induction on the (right) Loewy series of D/H shows that $\mathrm{Tor}_1^D(D/H, E(D)) = 0$.

Remarks. (1) The proof of Theorem 1.11 is actually the first time that we needed to use the property that I is proper in Rx in condition $(*\mathscr{T})$. In particular, we needed the "proper" hypothesis to insure that the idempotents $\{e_\beta + \sum_{\alpha\in\mathscr{A}} J_\alpha\}_{\beta\in\mathscr{A}}$ were all nonzero.

(ii) If $R = D[[x]]$ is the ring of all power series with coefficients in a division ring D, then $(\mathscr{S}, \mathscr{F})$ has BSP and is stable by Theorem 1.11.

(iii) If R is the ring of Example 0.1, then $\mathscr{T}_{\alpha+2} = \mathscr{G}$ by Corollary 1.6 and its subsequent comments. Therefore $\mathscr{T}_{\alpha+2}$ is stable. Since R has no nontrivial idempotent elements and since $R/M \cong T$ is not a left perfect ring, then $(\mathscr{T}_{\alpha+2}, \mathscr{F}_{\alpha+2})$ does not have BSP by Theorem 1.11.

The following corollaries of Theorem 1.11 show that it is very difficult for $(\mathscr{T}_\alpha, \mathscr{F}_\alpha)$ to have (BSP) unless $\mathscr{T}_\alpha = \mathscr{S}$ or R-mod.

COROLLARY 1.12. *Let R be a left and right duo ring satisfying* (*). *Let $\alpha \geqq 1$ be an ordinal, and suppose that $\mathscr{T}_\alpha(R) = 0$. If $(\mathscr{T}_\alpha, \mathscr{F}_\alpha)$ has BSP and is stable, then $\mathscr{T}_\alpha = \mathscr{S} = \mathscr{G}$.*

Proof. Since R is a left duo ring, then R satisfies $(^*\mathscr{S})$ and hence $(^*\mathscr{T}_\alpha)$. By Theorem 1.11, $R = D_i \dotplus D_2 \dotplus \ldots \dotplus D_n$ (ring direct sum) such that, for each $i = 1, 2, \ldots, n$ and each two-sided ideal K_i of D_i, D_i/K_i is a left perfect ring. Since D_i/K_i is left perfect and R is left and right duo, then D_i/K_i is right perfect; hence $D_i/K_i \in \mathscr{S}$ by [**1**, Theorem P].

From Corollary 1.6, $\mathscr{T}_\alpha = \mathscr{G}$. Let I be an essential left ideal of R. By Lemma 1.7 there exists a two-sided ideal $I' \subseteq I$ such that $I/I' \in \mathscr{T}_\alpha = \mathscr{G}$. Then $I' = \sum_{i=1}^n (D_i \cap I')$. Set $K_i = D_i \cap I'$. It follows that

$$R/I' \cong \left(\oplus \sum_{i=1}^n D_i\right) \Big/ \left(\oplus \sum_{i=1}^n K_i\right) \cong \oplus \sum_{i=1}^n D_i/K_i \in \mathscr{S},$$

hence $R/I \in \mathscr{S}$. Therefore, every cyclic module in $\mathscr{T}_\alpha$ is $\mathscr{S}$, so it follows that $\mathscr{T}_\alpha = \mathscr{S}$.

COROLLARY 1.13. *Let R be a commutative ring. Let $\alpha \geqq 1$ be an ordinal, and suppose that $\mathscr{T}_\alpha(R) = 0$. If $(\mathscr{T}_\alpha, \mathscr{F}_\alpha)$ has BSP and is stable, then $\mathscr{T}_\alpha = \mathscr{S} = \mathscr{G}$.*

COROLLARY 1.14. *Let R be a commutative Noetherian ring. Let $\alpha \geqq 1$ be an ordinal, and suppose that $\mathscr{T}_\alpha(R) = 0$. If $(\mathscr{T}_\alpha, \mathscr{F}_\alpha)$ has BSP, then $\mathscr{T}_\alpha = \mathscr{S} = \mathscr{G}$.*

Proof. Since R is commutative and Noetherian, every hereditary torsion theory is stable [**14**]; so the result follows from Corollary 1.13.

COROLLARY 1.15. *Let R be a commutative ring. Let $\alpha \geqq 1$ be an ordinal. Then $(\mathscr{T}_\alpha, \mathscr{F}_\alpha)$ has SP if and only if R is a semiartinian ring.*

Proof. This corollary is immediate from Lemma 0.2, [**14**, Proposition 4.2], Corollary 1.13, and [**17**, Theorem 5.1].

COROLLARY 1.16. *Let R be a ring which has Krull dimension as a right R-module. Suppose that $\alpha \geqq 1$ is an ordinal, $\mathscr{T}_\alpha(R) = 0$, and R satisfies* (*) *and* $(^*\mathscr{T}_\alpha)$. *If $(\mathscr{T}_\alpha, \mathscr{F}_\alpha)$ has BSP and is stable, then $\mathscr{T}_\alpha = \mathscr{S} = \mathscr{G}$.*

Proof. By Theorem 1.11, $R = D_i \dotplus D_2 \dotplus \ldots \dotplus D_n$ (ring direct sum) such that, for each $i = 1, 2, \ldots, n$ and each two-sided ideal K of D_i, D_i/K is a left perfect ring. Since R_R has Krull dimension; so does $(D_i/K)_R$. But a semi-artinian (right) R-module with Krull dimension is artinian. Therefore, rad (D_i/K) is nilpotent, and hence D_i/K is also right perfect. Thus $D_i/K \in \mathscr{S}$. The Corollary now follows from the same argument used in the second paragraph of the proof of Corollary 1.12.

2. The case $\mathscr{T} = \mathscr{S}$. We begin section two with the following generalization of [**15**, Theorem 4.3].

THEOREM 2.1. *If* $\mathscr{S}(R) = 0$ *and if* R *satisfies* (*) *and* (*$\mathscr{S}$), *then the following statements are equivalent.*

(1) $(\mathscr{S}, \mathscr{F})$ *has CSP.*

(2) R *is a* (ring) *direct sum of finitely many left Öre domains and* $\mathscr{S} = \mathscr{G}$.

Proof. (2) $\Rightarrow$ (1). Let $D = D_1 \dot{+} D_2 \dot{+} \ldots \dot{+} D_n$ be a ring direct sum. It is known that $\mathscr{G} = \mathscr{S}$ has *CSP* if and only if, for each $i = 1, 2, \ldots, n$, the torsion theory induced on D_i-mod by $\mathscr{G} = \mathscr{S}$ has *CSP* (see [**15**, p. 452]). Since each D_i is a left Öre domain, then $\mathscr{G}$ induces the classical torsion theory on D_i-mod for each $i = 1, 2, \ldots, n$. But then each induced torsion theory has *CSP*.

(1) $\Rightarrow$ (2). We have $\mathscr{S} = \mathscr{G}$ by Corollary 1.6. Temporarily assume that R is left finite dimensional. Then let $\sum_{i=1}^{n} Rx_i$ be a maximal direct sum of cyclic submodules of R. By Lemma 1.3, there exist orthogonal idempotents $e_1, e_2, \ldots, e_n$, such that

$$\oplus \sum_{i=1}^{n} Rx_i \subseteq \oplus \sum_{i=1}^{n} Re_i \subseteq R.$$

Since $\oplus \sum_{i=1}^{n} Re_i$ is a direct summand of R, then $\sum_{i=1}^{n} Re_i = R$. Since $R \in \mathscr{F}$ and $\mathscr{S} = \mathscr{G}$, then $(0 : x) = \sum_{i \neq j} Re_j$ for any nonzero $x \in Re_i$. Hence each Re_i is an integral domain.

Consequently, it is sufficient to show that R is left finite dimensional. Let $\oplus \sum_{\alpha \in \mathscr{A}} Rx_\alpha$ be a direct sum of principal left ideals which is essential in R. By Lemma 1.3, we obtain an infinite set of orthogonal idempotents $\{e_\alpha\}_{\alpha \in \mathscr{A}}$ such that $L = \sum_{\alpha \in \mathscr{A}} Re_\alpha$ is an essential submodule of R. By Corollary 1.6, $A = R/L \in \mathscr{S}$. Hence rad A is right T-nilpotent; so in A idempotents can be lifted modulo rad A. Let S be a simple module in Soc $(A/\text{rad } A)$. Then S is generated by an idempotent element f' of $A/\text{rad } A$, which can be lifted to an idempotent f'' of A. Let $f \in R$ such that $f'' = f + L$. Hence for some finite subset $\mathscr{C}$ of $\mathscr{A}$, $f^2 - f \in \sum_{i \in \mathscr{C}} Re_i$. Let $R' = R/\sum_{i \in \mathscr{C}} Re_i$, and let $\mathscr{S}'$ be the torsion theory for R'-mod which is induced by $\mathscr{S}$. Then $\mathscr{S}'(R') = 0$, and by Lemma 0.2, R' satisfies (*$\mathscr{S}'$). Hence $(Rf + \sum_{i \in \mathscr{C}} Re)/\sum_{i \in \mathscr{C}} Re_i$ is a two-sided ideal of R' by Lemma 1.4. Thus $Rf + \sum_{i \in \mathscr{C}} Re_i$ is an idempotent two-sided ideal of R; so by (*) there exists an idempotent e such that $Re = Rf + \sum_{i \in \mathscr{C}} Re_i$. Hence $(Re + L)/L = (Rf + L)/L$ and

$$\left[\frac{Rf + L}{L}\right] \Big/ \left[\text{Rad } A \cap \frac{Rf + L}{L}\right] \cong \frac{Af'' + \text{rad } A}{\text{rad } A} = Af' = S.$$

Consequently, $Re/(Re \cap L) \cong (Re + L)/L$ has a unique maximal ideal.

Since $R/Re \in \mathscr{F}$ and $R/Re_\alpha \in \mathscr{F}$ for $\alpha \in \mathscr{A}$, then $R/(Re \cap Re_\alpha) \in \mathscr{F}$ for $\alpha \in \mathscr{A}$. Hence by Theorem 1.5, there exist orthogonal idempotents $\{h_\alpha\}_{\alpha \in \mathscr{A}}$ such that $Rh_\alpha = Re \cap Re_\alpha$. If $B = \{\alpha \in \mathscr{A} | h_\alpha \neq 0\}$ were finite, then

$$0 \neq Re/(Re \cap L) = Re/\sum_{\alpha \in \mathscr{A}} Rh_\alpha$$

is a direct summand of $Re \in \mathscr{F}$. But $Re/(Re \cap L) \in \mathscr{S}$, which gives a contradiction. Hence B must be infinite. We partition B into disjoint infinite sets Δ and Γ, each with infinite cardinality. Choose $M \subseteq Re$ maximal with respect to

$$M \cap \sum_{\alpha \in \Delta} Rh_\alpha = 0$$

and

$$M \supseteq \sum_{\alpha \in \Gamma} Rh_\alpha.$$

Then $R/M \in \mathscr{F}$; so M is a (ring) direct summand of R (and Re) by Theorem 1.5. Let $Re = M \oplus N$. Since M and N are finitely generated left R-modules, then $M \nsubseteq L$ and $N \nsubseteq L$. Since M and N are generated by orthogonal idempotents, we obtain the non-trivial ring direct sum

$$(Re + L)/L = ((M + L)/L) \oplus ((N + L)/L).$$

This direct sum forces a contradiction to the fact that $(Re + L)/L$ has a unique maximal (left) ideal.

Example 2.2. Let F be a field, and let ϕ be an automorphism of F. Extend ϕ to $F[x]$ by $\phi(x) = x$. Let D be the quotient field of $F[x]$. Let

$$R = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \middle| \, a \in F[x], b \in D \right\}$$

with addition given coordinatewise (in the usual way) and multiplication defined by the rule

$$\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \begin{bmatrix} c & d \\ 0 & c \end{bmatrix} = \begin{bmatrix} ac & \phi(a) \cdot d + bc \\ 0 & ac \end{bmatrix}.$$

Then R is a ring. (We note that R is commutative if and only if ϕ is the identity map if and only if the multiplication above is the usual matrix multiplication!) We observe that

$$\text{rad } R = \left\{ \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \middle| \, b \in D \right\}, \quad \text{and} \quad (\text{rad } R)^2 = 0.$$

The reader can now verify the following statements.

(1) If $y \in R$ and $y \notin \text{rad } R$, then $Ry \supseteq \text{rad } R$.

(2) If $I^2 = I \neq 0$, then $I \nsubseteq \text{rad } R$; so $I/\text{rad } R$ is an idempotent ideal of $R/\text{rad } R$. Since $R/\text{rad } R$ is (ring) isomorphic to $F[x]$, then $I/\text{rad } R = R/\text{rad } R$; so R satisfies (*).

(3) If $y \in R - \text{rad } R$, then by (1) we may choose a two-sided ideal K maximal with respect to $\text{rad } R \subseteq K$ and $y \notin K$. Then $Ry/K \in \mathscr{S}$ (as $R/\text{rad } R$ is isomorphic to $F[x]$).

(4) Any left ideal contained in rad R is two-sided. Thus if $y \in \text{rad } R$ and

if K is a left ideal chosen such that $K \subseteq Ry$ and K is maximal with respect to $y \notin K$, then K is a two-sided ideal and $Ry/K \in \mathscr{S}$.

(5) Combining (3) and (4), we see that R satisfies $(*\mathscr{T})$.

(6) $\mathscr{S}(R/\text{rad } R) = 0$, and rad R is not a direct summand of R.

(7) Since $\mathscr{S}(R) = 0$, Theorem 2.1 implies that R does not have *CSP*.

Example 2.3. Let D' be a subdivision ring of the division ring D. Let R be the subring of the power series ring $D[[x]]$ consisting of those series whose constant term is in D'; i.e.

$$R = \left\{ d' + \sum_{i=1}^{\infty} d_i x^i | d' \in D', d_i \in D \right\}.$$

The reader can verify the following statements.

(1) R is an integral domain; so $\mathscr{S}(R) = 0$.

(2) If $d' + \sum_{i=1}^{\infty} d_i x^i \in R$ and $d' \neq 0$, then, by solving coefficient equations of x^i in the usual way, $d' + \sum_{i=1}^{\infty} d_i x^i$ has an inverse in R.

(3) The left ideal M generated by $\{dx | d \in D\}$ is the unique maximal left ideal of R. M is a two-sided ideal.

(4) Every principal left ideal contains a power of M; hence $(*\mathscr{S})$ holds for R.

(5) R contains no nontrivial idempotent ideals by a "least degree" argument; so (*) holds for R.

(6) Therefore Theorem 2.1 applies to show that $(\mathscr{S}, \mathscr{F})$ has *CSP*.

(7) We also note that D is (left) Noetherian and has Krull dimension if and only if D is a finite dimensional vector space over D'.

In view of [**1**, Theorem P], Theorem 1.11 becomes the following generalization of [**15**, Corollary 4.5] whenever $\mathscr{T} = \mathscr{S}$.

THEOREM 2.4. *Let $\mathscr{S}(R) = 0$, and suppose that R satisfies (*) and $(*\mathscr{S})$. Then the following statements are equivalent.*

(1) *$(\mathscr{S}, \mathscr{F})$ has BSP and is stable.*

(2) *R is a finite direct sum of left Öre domains $D_i (i = 1, 2, \ldots, n)$, each of which satisfies the following properties*:

(i) *for each two-sided ideal I of D_i, D_i/I is a left and right perfect ring.*

(ii) *each two-sided ideal of D_i is flat as a right module.*

(iii) *if M is a maximal right ideal of D_i which contains a two-sided ideal, then* $\mathrm{Tor}_1^{D_i}(D_i/M, E(D_i)) = 0$.

COROLLARY 2.5. *Let R be a left duo von Neumann regular ring. Then $(\mathscr{S}, \mathscr{F})$ has BSP and is stable if and only if R is a left semiartinian ring.*

Proof. The "if" part is trivial. The "only if" part follows from Lemma 0.2 and Theorem 2.4 (as a regular integral domain is a division ring).

In order to prove our main result on *SP* for $(\mathscr{S}, \mathscr{F})$, we need the following result of Gorbachuk.

PROPOSITION 2.6 [**9**, Theorem 2]. *Let $(\mathscr{T}, \mathscr{F})$ be a hereditary torsion theory.*

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Then $(\mathcal{T}, \mathcal{F})$ *does not have SP provided that there exists a sequence* $P_1, P_2, \ldots$ *of left ideals of* R *satisfying the following properties*:

(i) $R/P_n \in \mathcal{T}$ *for* $n = 1, 2, \ldots$;

(ii) $R/(\bigcap_{n=1}^{\infty} P_n) \notin \mathcal{T}$;

(iii) *for each* $n = 1, 2, \ldots$, *there exists an integer* $m(n)$ *and a* $p_n \in P_n$ *such that* p_n *has zero as its left annihilator and*

$$P_{n+1}p_1p_2 \ldots p_n \supseteq p_1p_2 \ldots p_{m(n)}R.$$

We now can state a generalization of the main results of [**3**; **5**] and the characterization of (SP) for $(\mathcal{S}, \mathcal{F})$ given for commutative rings in [**15**] and [**17**].

THEOREM 2.7. *Suppose that* R *satisfies* (*) *and* $(^*\mathcal{S})$. *Then* $(\mathcal{S}, \mathcal{F})$ *has SP if and only if* R *is a left semiartinian ring.*

Proof. The "if" part is trivial. "Only if": Since R has (*) and $(^*\mathcal{S})$ and $(\mathcal{S}, \mathcal{F})$ has SP, then, by passing to the ring $R/\mathcal{S}(R)$ and applying Lemma 0.2, we may assume that $\mathcal{S}(R) = 0$. By Theorem 2.4 and Lemma 0.2, we may assume that R is a left Öre domain such that (a) R/I is a left and right perfect ring for all nonzero two-sided ideals I of R and (b) R satisfies (*) and $(^*\mathcal{S})$.

Suppose that d is a nonzero element of R and that d does not have a left inverse. Then

$$Rd \supsetneq Rd^2 \supsetneq Rd^3 \supsetneq \ldots.$$

Let $K = \bigcap_{n=1}^{\infty} Rd^n$. If $K \neq 0$, there exists a nonzero, two-sided ideal $H \subseteq K$ by $(^*\mathcal{S})$. Thus the set $\{Rd^n/H\}$ is an infinite descending chain of principal left ideals of R/H. But R/H is right perfect, and hence R/H can have no infinite descending chain of principal left ideals by [**1**, Theorem P]. This contradiction forces K to be 0.

Consequently, Gorbachuk's result (Proposition 2.6) will imply a contradiction to the hypothesis, $(\mathcal{S}, \mathcal{F})$ has SP, provided that we can construct a sequence of (left) ideals P_n and a sequence of nonzero elements p_n such that

(i) $R/P_n \in \mathcal{S}$,

(ii) $\bigcap_{n=1}^{\infty} P_n = 0$,

(iii) $p_n \in P_n$, and

(iv) $P_{n+1}p_1p_2 \ldots p_n \supseteq p_1p_2 \ldots p_{n+2}R$.

To do this, we proceed inductively to define $p_n \in P_n \subseteq Rd^n$.

By $(^*\mathcal{T})$ there exists a two sided ideal $T \subsetneq Rd$. Since $R \in \mathcal{F}$, $T \neq 0$. Since R/T is right perfect, $R/T \in \mathcal{S}$. Set $P_1 = T$, and let p_1 be any nonzero element of P_1.

Now suppose that $p_{k-1} \in P_{k-1} \subseteq Rd^{k-1}$ has been defined appropriately. Let

$$0 \neq x \in Rd^k \cap P_{k-1}p_1p_2 \ldots p_{k-2},$$

which is possible since R is a left Ore domain. (In case $k = 2$, $p_0 = 1$.) By $(^*\mathcal{T})$ there exists a two sided ideal $T' \subsetneq Rx$. Since R/T' is right perfect,

$R/T' \in \mathscr{S}$. Since $R \in \mathscr{F}$, $T' \neq 0$. Set $P_k = T'$, and let p_k be any nonzero element of P_k. Since P_k is two-sided, then $p_1p_2 \ldots p_kR \subseteq P_k \subset Rx \subseteq P_{k-1}\, p_1p_2 \ldots p_{k-2}$. Moreover,

$$\bigcap_{n=1}^{\infty} P_n \subseteq \bigcap_{n=1}^{\infty} Rd^n = K = 0;$$

so we have constructed the desired sequence.

The following corollary may be viewed as a generalization of [**5**, Theorem 3.9].

COROLLARY 2.8. *Let R be a von Neumann regular, left duo ring. Then $(\mathscr{S}, \mathscr{T})$ has SP if and only if R is a left semiartinian ring.*

We now give an example of a ring R such that $(\mathscr{S}, \mathscr{T})$ can be tested for *SP* by Theorem 2.7, and R does not satisfy the hypothesis of any other theorem on *SP*.

Example 2.9. For each integer $m \geqq 2$, let m^* denote the least prime factor of m. Let A_m be the algebraic closure of the field $Z/(m^*)$, where Z denotes the integers. Then A_m has an automorphism ϕ_m defined by $\phi_m(a) = a^{m^*}$ for each $a \in A_m$. Set

$$P_m = \left\{ \sum_{i=1}^{\infty} a_{mi} x_m{}^i | a_{mi} \in A_m \right\}.$$

Then elements of P_m can be added in the obvious way and multiplied as power series subject to the twisting rule, $xa = \phi_m(a)\, x$ for all $a \in A_m$, and its consequences. Now define

$$R = \{c + \pi_{m_1} + \pi_{m_2} + \ldots + \pi_{m_n} | c \in Z, \pi_{m_i} \in P_{m_i}\},$$

where the m_i range over the integers $\geqq 2$. Again elements of R can be added in the obvious way. Define multiplication for R by the following rules and their consequences:

(i) $\pi_{m_i}\pi_{m_j} = 0$ for $\pi_{m_i} \in P_{m_i}$, $\pi_{m_j} \in P_{m_j}$, $m_i \neq m_j$;

(ii) for $c \in Z$ and $\pi_{m_i} \in P_{m_i}$, $c\pi_{m_i} = (c + (m_i{}^*))\, \pi_{m_i}$ and $\pi_{m_i}c = \pi_{m_i}(c + (m_i{}^*))$, where the multiplication on the right side of each equation is the multiplication of P_{m_i};

(iii) any two elements of R in P_{m_i} multiply as elements of P_{m_i}.

Then R is a ring.

Let X be the ideal of R defined by $X = P_2 + P_3 + P_4 + \ldots$. If I is an ideal of R contained in X, it follows easily (by considering the term of least degree that can appear a member of I) that $I = I^2$ implies $I = 0$. If $J^2 = J$ and $J \not\subseteq X$, then $(J + X)/X$ is an idempotent (left) ideal of R/X. Since R/X is (ring) isomorphic to Z, then $J + X = R$. Hence J contains an element of the form

$$\pm 1 + \pi_{m_1} + \pi_{m_2} + \ldots + \pi_{m_n},$$

where $\pi_{m_i} \in P_i$. But by the method of comparing coefficients of twisted power series, it is straightforward to construct an inverse of this element. Thus $1 \in J$; so $J = R$. Therefore, R satisfies (*).

Since R/X is ring isomorphic to Z, then R is not local; so the results of [**17**, Section 3] cannot be applied to R. Also, if $c \neq 1$ and $c + \pi_{m_1} + \pi_{m_2} + \ldots + \pi_{m_n} \in R$, then choose a prime number p and positive integer k such that p divides c and $p^k > \max \{m_1, m_2, \ldots, m_n\}$. Hence

$$x_{p^k}(c + \pi_{m_1} + \ldots + \pi_{m_n}) = 0 = (c + \pi_{m_1} + \ldots + \pi_{m_n})x_{p^k}.$$

From this fact and an argument in the previous paragraph, it follows that every element of R with zero left or right annihilator is invertible. Thus R does not satisfy the hypotheses of [**9**, Theorem 2 or 3].

If $0 \neq r \in R$, then there exists an integer m such that $0 \neq x_m r \in P_m$. Hence $Rx_m{}^2 r$ is a proper nonzero submodule of Rr. Therefore, soc $R = 0$. But then, if $R/I \in \mathscr{S}$, $I \cap Rx_n \neq 0$ for every $n \geqq 2$. Let $k \in Z$ such that $(Rk + X)/X = (I + X)/X$. If $I \neq R$, then we may assume $k \geqq 2$; so, for each $j = 1, 2, \ldots$, there must be a generator of I of the form

$$sk + \pi_{m_1} + \ldots + \pi_{m_e} + \pi_{(k^*)^j} + \pi_{m_f} + \ldots + \pi_{m_n},$$

where $s \in Z$, $\pi_{m_i} \in P_{m_i}$, and $0 \neq \pi_k \in P_{(k^*)^j}$. Hence I cannot be finitely generated as a left ideal. Thus R does not satisfy the hypothesis of [**16**, Theorem 3.5].

Since soc $R = 0$, we can apply Theorem 2.7 to show that R does not split, provided that R satisfies $(*\mathscr{S})$. Let

$$r = a + \sum_{i=1}^{\infty} a_{m_1 i} x_{m_1}{}^i + \sum_{i=1}^{\infty} a_{m_2 i} x_{m_2}{}^i + \ldots + \sum_{i=1}^{\infty} a_{m_n i} x_{m_n}{}^i \in R,$$

where $a \in Z$ and $\sum_{i=1}^{\infty} a_{m_j i} x_{m_j}{}^i \in P_j$ for $j = 2, 3, \ldots, n$. We wish to find a two-sided ideal T properly contained in Rr such that $Rr/T \in \mathscr{S}$. If $a = \pm 1$, r is invertible; so we assume that $a \neq \pm 1$. If $a \neq 0$, set

$$H = \sum \{P_k | k^* \text{ relatively prime to } a\};$$

if $a = 0$, set $H = 0$. For each $m_j{}^*$ which divides a, let t_j be the least positive integer t such that $a_{m_j t} \neq 0$. Then some laborious computation shows that $T = Ra^2 + H + P_{m_1}{}^{t_1} + P_{m_2}{}^{t_2} + \ldots + P_{m_n}{}^{t_n}$ is the desired two-sided ideal of R.

In fact, since 1 and 0 are the only idempotent elements of R, $(\mathscr{S}, \mathscr{F})$ does not even have *CSP* for R-mod by Theorem 2.1.

References

1. H. Bass, *Finitistic homological dimension and a homological generalization of semi-primary rings*, Trans. Amer. Math. Soc. *95* (1960), 466–488.

2. S. E. Dickson, *A torsion theory for abelian categories*, Trans. Amer. Math. Soc. *121* (1966), 223–235.
3. ——— *Noetherian splitting rings are Artinian*, J. London Math. Soc. *42* (1967), 732–736.
4. L. Fuchs, *Torsion preradicals and ascending Loewy series of modules*, J. Reine Angew. Math. *239/240* (1969), 169–179.
5. J. D. Fuelberth, *On commutative splitting rings*, Proc. London Math. Soc. *20* (1970), 393–408.
6. J. D. Fuelberth and M. L. Teply, *The singular submodule of a finitely generated module splits off*, Pacific J. Math. *40* (1972), 73–82.
7. J. S. Golan, *On the torsion-theoretic spectrum of a non-commutative ring* (preprint, 1973).
8. K. R. Goodearl, *Singular torsion and the splitting properties*, Mem. Amer. Math. Soc. *124* (1972).
9. E. L. Gorbachuk, *Splitting torsion and pretorsion in the category of right* Λ*-modules*, Mat. Zametki *2* (1967), 681–688.
10. R. Gordon and J. C. Robson, *Krull dimension*, Mem. Amer. Math. Soc. *133* (1973).
11. I. Kaplansky, *Projective modules*, Ann. of Math. *68* (1958), 372–377.
12. J. Lambek, *Torsion theories, additive semantics, and rings of quotients*, Lecture Notes in Mathematics *177* (Springer-Verlag, Berlin, 1971).
13. T. S. Shores, *The Structure of Loewy modules*, J. Reine Agnew. Math. *254* (1972), 204–220.
14. B. Stenström, *Rings and modules of quotients*, Lecture Notes in Mathematics *237* (Springer-Verlag, Berlin, 1971).
15. M. L. Teply, *The torsion submodule of a cyclic module splits off*, Can. J. Math *24* (1972), 450–464.
16. ——— *On non-commutative splitting rings*, J. London Math. Soc. *4* (1971), 157–164. (See also Corrigendum in J. London Math. Soc. *6* (1973), 267–268.)
17. M. L. Teply and J. D. Fuelberth, *The torsion submodule splits off*, Math. Ann. *188* (1970), 270–284.

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