1
General Principles

1.1 Introduction

Magnetohydrodynamics is concerned with the study of the interaction of a fluid with a magnetic field. It takes as its philosophy a *continuum* approach, describing its phenomena in macroscopic terms rather than in terms of particle motions. Thus, it is close in spirit to fluid mechanics, which studies the properties of fluids from such a continuum viewpoint. Magnetohydrodynamics – commonly abbreviated to MHD – may also be viewed from a particle approach, discussing the motions of charged particles (electrons or ions) in the presence of a magnetic field. This is the realm of plasma physics. In our account here we will consider the subject principally from the macroscopic viewpoint.

The description of magnetic effects in magnetohydrodynamics is rooted in the celebrated equations of electromagnetism formulated by James Clerk Maxwell in 1864, though it is generally only the pre-Maxwellian form of the equations of electromagnetism that are used. The displacement current introduced by Maxwell is ignored on the basis that rapidly varying phenomena, such as electromagnetic waves, are best described from an electromagnetic viewpoint. Thus there are no electromagnetic waves or light in magnetohydrodynamics. Instead, the subject is concerned with relatively slow phenomena, such as sound waves or convective flows or field generation by dynamo action. It is to such phenomena that a fluid approach is particularly suited.

By a *fluid* we mean a gas, plasma or liquid that may be treated from a continuum approach. In magnetohydrodynamics the fluid is a conductor of electricity, and motions within the fluid occur in the presence of an applied magnetic field. The current that flows throughout the volume of the fluid is determined by Ampere's law which, when expressed in partial differential equation form, relates the current density to the 'curl' (a vector operator) of the magnetic field. Motions in the fluid are subject to a magnetic force, the Lorentz force (or $\mathbf{j} \times \mathbf{B}$ force), arising from the current density $\mathbf{j}$ and the magnetic (induction) field $\mathbf{B}$; this force, together with any others that may act (such as pressure gradients or gravity), serves to define the motion of the fluid.

Temporal changes in the magnetic field $\mathbf{B}$ are determined by Faraday's law of induction, which links such changes in $\mathbf{B}$ to the 'curl' of the electric field $\mathbf{E}$. The electric field is in turn related to the current density $\mathbf{j}$ through Ohm's law, expressed in a form appropriate for a moving conductor (the fluid). Motions within the fluid are thus inextricably linked to the magnetic field embedded within it, so that movements of the fluid entail movements in the

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field, and *vice versa*. It is this intimate link of fluid and field that gives magnetohydrodynamics its distinctive nature.

Magnetohydrodynamics, then, is the offspring from a marriage of fluid mechanics and electromagnetism. It was an offspring that took its time in developing. The basic physical laws and principles of its parents were well known by the end of the nineteenth century, but it was well into the twentieth century before the stirrings of magnetohydrodynamics took shape and then at first only in a somewhat sporadic fashion. By the 1940s, however, the subject was in full growth and has continued this way ever since. A brief account of the early years of development of magnetohydrodynamics is provided in Cowling (1962).

The initially slow development of magnetohydrodynamics is in some ways surprising, given the pedigree of its parents. But early experiments with laboratory fluids such as mercury or sodium, aimed at investigating magnetohydrodynamic phenomena, were fraught with difficulties, principally connected with the liquids themselves and the maintenance of sufficiently strong magnetic fields. Strong ohmic attenuation of motions made comparison between theory and experiment somewhat qualitative, though reasonable agreement was obtained. However, it was through the application of magnetohydrodynamics to large-scale phenomena, such as exhibited in the magnetic fields of the Earth and its magnetosphere, the Sun and our Galaxy, that a spur to sustained development was provided. That spur has continued to the present day, increasing to ever greater effect as space and ground-based observations of, most notably, the Sun and the Earth's magnetosphere give firm direction to magnetohydrodynamics. Moreover, the laboratory fluid has not been left behind, as detailed studies of fusion plasmas have revealed the utility of a magnetohydrodynamic description of certain phenomena as a valuable addition to a plasma approach.

An early advance in magnetohydrodynamics, though paradoxically for some time it seemed more like a backward step, was made by T. G. Cowling who showed, in 1934, that a dynamo must have a non-symmetric component (Cowling 1934). That a magnetohydrodynamic fluid could support a wave motion, distinct from the familiar electromagnetic and sound waves of the parents, was not however realized until the early 1940s. In a brief half-page letter to the journal *Nature*, H. Alfvén showed that in a perfectly conducting incompressible fluid a transverse wave may propagate along a homogeneous magnetic field; the speed of the wave was proportional to the strength of the applied magnetic field and inversely proportional to the square root of the mass density of the fluid in which the field was embedded (Alfvén 1942a, b). Alfvén termed this wave an 'electromagnetic-hydrodynamic wave', but it later became apparent that the *Alfvén wave* was born! It seems that the term 'Alfvén wave' entered into use with the work of V. C. A. Ferraro and J. W. Dungey (Ferraro 1954; Dungey 1954). For a recent general discussion of Alfvén's contribution to magnetohydrodynamic waves, see Russell (2018). Alfvén was later, in 1950, awarded the Nobel prize for his contributions to magnetohydrodynamics. In his 1942 work Alfvén made the suggestion that the observed latitudinal drift of sunspots on the Sun's surface towards the equator may be a wave phenomenon controlled by wave motions (Alfvén waves) deep below the solar surface (Alfvén, 1942a). This direct linkage of sunspot drift with Alfvén waves is not thought likely now but magnetohydrodynamic waves do arise in sunspots themselves.

Alfvén did not discuss what happens to wave motions in a fluid that is compressible. That extension, particularly important for astrophysical applications, was left to another brief letter to *Nature*, written by N. Herlofson (1950), and a more extensive treatment by H. C. van de Hulst (1951). Herlofson and van de Hulst made the remarkable theoretical discovery that in a magnetohydrodynamic medium there are in fact three modes of propagation open to the system: the Alfvén wave (uninfluenced by the compressibility of the medium) and two compressible waves. There are three distinct wave speeds associated with these modes, and moreover these speeds depend upon the *direction* of propagation of the wave. In other words, magnetohydrodynamic waves are *anisotropic*. During the 1950s and early 60s, the theoretical properties of these waves were further explored; see, for example, Friedrichs and Kranzer (1958) and Lighthill (1960).

More recently, developments in the subject of wave propagation have been motivated on a number of fronts: by the possibility of heating of laboratory plasmas by magnetohydrodynamic waves; by the observations of pulsations in the magnetosphere; by the *direct* observation of waves in the Sun's corona and their use in coronal seismology; and by the realization that astrophysical plasmas generally, but most clearly the solar atmosphere, are likely to be strongly inhomogeneous. Perhaps above all has been the spur provided by the direct observations of magnetohydrodynamic waves in the solar corona, which has undoubtedly been a powerful stimulus in the further development of theoretical aspects. Underpinning much of these theoretical developments is the detailed study of magnetohydrodynamic wave motions in structured magnetic atmospheres, which are significantly different from those of a uniform medium, though an understanding of this simpler case is, of course, the basis for any study of a non-uniform medium. In any case, structured media provide wave guides for magnetohydrodynamic waves.

We end this section with a brief comment about units. We are adopting the mks [metre kilogram second] system of units in our treatment, with electromagnetic quantities expressed in SI [System Internationale, rationalized mks] units. In magnetohydrodynamics it is convenient to regard the magnetic field $\mathbf{B}$ and the velocity $\mathbf{u}$ as the primary variables; other variables, such as the current density $\mathbf{j}$ and electric field $\mathbf{E}$, are then of secondary interest, following from a knowledge of the primary variables if and when required. Generally, then, of the electromagnetic variables, in applications or illustrations of our equations we will only quote values of the magnetic field strength $B\ (= |\mathbf{B}|)$; in SI units, B is expressed in tesla (T). However, it frequently proves convenient to quote B in gauss (G), noting that $1\,\mathrm{T} = 10^4\,\mathrm{G} = 10\,\mathrm{kG}$.

1.2 A Variety of Plasmas

Magnetohydrodynamics has found application to a wide variety of plasmas, extending from the small scale of the laboratory plasma to the vast scale of the galactic medium. The fact that the Earth – and indeed many of the planets – has a magnetic field has prompted the development of magnetohydrodynamic theories aimed at describing its maintenance and temporal variation. This is the magnetohydrodynamic dynamo problem (see, for example, Moffatt 1978; Parker 1979a). In particular, extensive developments have taken place in connection with the Sun's plasma: in its interior (where magnetic fields are both stored and manipulated) through to its surface (where fields are observed and measured in great detail); into the tenuous but enigmatic coronal outer atmosphere; and on into the solar wind that blows past the Earth and the other planets, interacting with their magnetic fields. The magnetospheres that envelop the magnetized planets are ever subject to variations in the solar wind, variations that commonly have their origin in events in the Sun's lower atmosphere. The Earth's magnetosphere, in particular, displays an array of phenomena that may be modelled using MHD and exhibits a variety of oscillations (Walker 2005; Wright and Mann 2006; Southwood, Cowley and Mitton 2015). Indeed, it is profitable to compare and contrast oscillatory phenomena in the Earth's magnetosphere and in the Sun's atmosphere (Nakariakov *et al.* 2016). However, our chief interest here is the solar atmosphere. Accordingly, we turn now to a brief overview of the Sun.

1.2.1 The Sun

The most distinctive property of the Sun as a plasma is its size. With a radius of $R_\odot = 6.96 \times 10^8$ m, the Sun displays a wide variety of plasma conditions ranging from its hot and dense interior, out through its visible and relatively cool surface, and on into its hot but tenuous atmosphere. Gravitational stratification makes for a complicated plasma, doubly compounded by the fact that the Sun possesses a complex and often dynamic magnetic field. Magnetism is the cause of almost all the exotic phenomena displayed by the Sun; for without a magnetic field the Sun would be a very quiet and relatively uninteresting plasma indeed. The possession of a magnetic field is a property it shares with a wide range of stars, many of which must surely display yet more exotic phenomena than we see on the Sun, simply by virtue of their stronger magnetic fields; for to detect that a star has a magnetic field, that field must be about 10^2 times stronger than occurs in the Sun viewed as a star.

The Sun's size and the variety of phenomena it displays has led us to regard the plasma as made up of separate regions. This is a convenient view to take, though one should not overlook the fact that these different regions are connected to one another. The Sun's interior, the region below the visible surface, is divided into three zones: an inner core, where nuclear reactions maintain the heat supply; a radiative zone, where the generated heat is distributed outwards by radiative transport; and, occupying the immediate layers below the visible surface, a *convection zone*, where heat transport is in the form of convective cells. Blending in with the top of the convection zone is the *photosphere*, the visible layer of the solar surface. The photosphere extends upwards for a height of about 500 km by which the Sun's temperature has fallen to its lowest value, of about 4200 K. This is the *temperature minimum*. Higher still in the atmosphere, the temperature rises, at first slowly in the *chromosphere* but then rapidly as we enter the *corona*. The temperature of the chromosphere ranges from the temperature minimum value to some 5×10^4 K whereafter it rises steeply in a thin layer, known as the *transition region*, to the order of 10^6 K. This hot outer region is known as the *corona*. Gravitational stratification ensures that the plasma density falls off with height, so the chromosphere and more especially the corona are tenuous plasmas in comparison with the photosphere.

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Both the chromosphere and the corona are dominated by magnetism. That the Sun has a magnetic field was not known until 1908, when G. E. Hale used the then newly discovered Zeeman effect to measure the magnetic field of sunspots (Hale 1908), the dark blemishes frequently visible on the solar surface (the photosphere). Hale obtained field strengths of typically 3000 G. The field is sufficiently strong and covers a sufficiently large region – about 10^4 km across – that it locally modifies the convective transport of heat, resulting in a cool patch in the photosphere.

Sunspots are magnetically complex structures. They are frequently observed in groups, where their magnetism tends to blend together to produce complex field patterns in the solar atmosphere that are somewhat similar to the patterns made by iron filings one sees around bar magnets in the school laboratory. But even in an isolated spot the field patterns detectable in the photosphere and chromosphere are complex. There are two distinct regions of a mature spot: its central cool *umbra* where the field is strongest and is predominantly vertical, and a surrounding *penumbra* where the field is weaker and has bent over towards the horizontal. The temperature in the umbra is typically 4000 K, compared with about 5000 K in the penumbra and 6000 K in the photosphere.

The magnetism measured in sunspots at the solar surface is presumed to be generated and manipulated by flows deep within the interior. The solar interior, made up of about 90% hydrogen and 10% helium (there are also small amounts of heavier elements), is believed to consist of an inner core where nuclear reactions keep the plasma exceedingly hot, at some 1.6×10^7 K. The inner core occupies some 25% of the solar radius. The region outside the inner core and extending out to about 70% of the solar radius is the radiative zone, a region where energy transport is predominantly by radiation. The outer 30% of the solar interior is occupied by the convection zone, a region where convective cells carry the heat at the bottom of this zone out to the cooler solar surface some 2×10^8 m above. There are several scales of convection operating. The two most distinctive convective patterns of flow are the supergranules and the granules. Granules have horizontal sizes ranging between 200 km and 2000 km, with 1000 km providing a characteristic scale. The flows in granules are fairly vigorous, at some $1-3\,\mathrm{km\,s^{-1}}$ (about 2000−6000 miles per hour), to be compared with terrestrial wind speeds in hurricanes of perhaps 150 miles per hour and the Earth's record wind speed of 230 miles per hour recorded on the top of Mt Washington in the USA. Supergranules, with a horizontal scale of about 3×10^4 km and so typically 30 times bigger than the granules, have flows of $0.1-0.4\,\mathrm{km\,s^{-1}}$. Both these flow patterns are detectable in the Sun's surface layers, with the granules enveloped by the supergranules. It is at the base of the convection zone and just below that magnetic field lines are believed to be manipulated by Coriolis forces and brought to the solar surface through buoyancy effects.

Sunspots are the obvious locations of magnetism in the solar surface. But even away from sunspots there are smaller-scale concentrations of magnetic field. The smallest of these concentrations are the *intense magnetic flux tubes*, which typically occupy regions about 200 km across wherein magnetic fields of some 1−2 kG strength are confined by external gas pressure forces. The intense tubes are generally located in the regions between convective cells where downdraughts occur. Outside of sunspots, over 90% of the magnetic flux appearing in the Sun's surface layers is in the form of concentrated flux tubes. Sunspots are known to support several types of wave motion.

Above the photosphere the concentrations of magnetic field, be they in the small-scale intense flux tubes or the larger scale sunspots, rapidly spread out to fill the available space. This is simply a consequence of stratification. At the photospheric level the gas pressure is sufficient to confine the magnetic fields, once formed in a concentrated form. But the confining external pressure falls off exponentially fast from the photosphere to the chromosphere, decreasing by a factor of e $(= 2.178\ldots)$, to some 37% of its value, over a distance of about 150 km, and this permits the confined magnetic fields to expand out in immediate response. By the mid-chromosphere the fields have filled the atmosphere and at coronal levels completely dominate the nature of the plasma.

The coronal plasma is characterized by its low density – in terrestrial terms it would be regarded as almost a vacuum – and high temperature. The high temperature of the corona, in excess of 10^6 K, was discovered in the 1930s and it remains one of the great puzzles of solar physics: what effects conspire to reverse the strong decline in temperature from the interior of the Sun to its surface, producing an extremely hot outer atmosphere? The answer to this question is important not only for the Sun but for stellar physics in general, for a wide variety of stars are believed to possess a corona.

Observations of the Sun's corona from space have revealed that in X-ray and EUV wavelengths the corona appears not as an amorphous hot glow, as was commonly thought prior to the Skylab mission in the 1970s, but as a complex and *structured* atmosphere; for an extensive discussion see Aschwanden (2004) and Priest (2014). The basis for this structure is the ubiquitous presence of magnetism in the corona. Despite the overall complexity of the coronal plasma, it would appear that there are fundamentally two different coronae: regions in which the magnetic field lines are curved in the form of loops or arcades with their ends anchored in the dense photosphere, and regions where the field lines emanate from the photosphere but are then carried out into space. The regions with re-entrant magnetic fields – the magnetic loops – are the hottest and most dense parts of the corona; they glow the brightest in X-ray and EUV pictures of the Sun. These are the *active regions*. They are characterized by temperatures of $2{-}3 \times 10^6$ K, plasma densities of 10^{16} particles per m^3 and have magnetic field strengths of about 10^2 G. The magnetic field, with its footpoints tied to the photosphere, confines the coronal plasma and heats it to its high temperature. By contrast, where the field is open the plasma blows out into space; these are the *coronal holes*, the source regions of the high-speed solar wind that blows from the Sun and flows on past the Earth. The plasma density in coronal holes, at 10^{14} particles per m^3, is two orders of magnitude less dense than that in the active regions. The plasma is also cooler, at $1.5{-}2 \times 10^6$ K, and the open magnetic field, with a strength of around 10 G, is a factor of 10 weaker than the field in active regions. These, then, are the two fundamentally different regions of the corona. Of particular interest here is the discovery by the space instrument *TRACE* (Transition Region And Coronal Explorer) that coronal loops support a variety of oscillations. Oscillations carry information about the medium in which they occur; such information may be used to obtain indirectly solar quantities that are otherwise difficult to measure. This is the new subject of *coronal seismology*.

There are, of course, many other structures in the corona in addition to the two fundamental forms. Of particular interest are *quiescent prominences*. In a sense, quiescent prominences are bits of the chromosphere that find themselves in a coronal environment. They are cool, dense structures, sometimes resembling a thin sheet of dense plasma, magically suspended in a tenuous corona. The source for their support is the magnetic field that threads through the prominence. They are generally passive structures, surviving for long periods (perhaps months) but then dramatically erupting, only to reform in much the same location shortly thereafter. In photographs of the chromosphere and corona they show up as thin (perhaps 6000 km across), filament-like, dark curves winding their way (for some 2×10^4 km) through the local magnetic structure; their height is about 5×10^4 km. Prominences have a typical density of $10^{17}\,\mathrm{m}^{-3}$, some two orders of magnitude larger than in their coronal surroundings, and a typical temperature of 7000 K. (There is evidence that the corona may be locally somewhat depleted in density in the neighbourhood of a prominence.) Quiescent prominences are observed to oscillate, a fact which may have important implications for *coronal and prominence seismology*.

1.3 The Magnetohydrodynamic Equations

We have remarked above that magnetohydrodynamics is a combination of fluid mechanics and electromagnetism with Maxwell's displacement current neglected. Here we describe the equations of this subject. We do not provide a derivation of these equations from basic principles; that route has been fully described elsewhere. Instead, we prefer to simply write down each of the relevant equations and to add some explanatory comments to illustrate various features of the equations. Derivations and discussions of the properties of the equations are given in, for example, Alfvén (1950), Cowling (1957, 1976), Kendall and Plumpton (1964), Ferraro & Plumpton (1966), Jeffrey (1966), P. H. Roberts (1967), Boyd and Sanderson (1969, 2003), Parker (1979a, 2007) and Priest (1982, 2014). Solar applications are given special attention in Bray and Loughhead (1974), Parker (1979a), Priest (1982, 2014), Bray *et al.* (1991), Choudhuri (1998), Goossens (2003), Aschwanden (2004), Goedbloed and Poedts (2004), Goedbloed, Keppens and Poedts (2010), Narayanan (2013), Ryutova (2015) and Nakariakov *et al.* (2016).

Consider a fluid with mass density ρ and motions $\mathbf{u}$. Conservation of matter – the statement that matter is neither created nor destroyed within the system (so that there are no sources or sinks of matter) – is described by the equation

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho \mathbf{u} = 0. \tag{1.1}$$

Equation (1.1) is commonly referred to as the equation of continuity.

The equation of momentum is the statement that changes in momentum are a result of forces acting in the fluid; it is Newton's second law applied to a fluid. The momentum equation is

$$\rho \left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \operatorname{grad})\mathbf{u} \right] = -\operatorname{grad} p + \mathbf{j} \times \mathbf{B} + \rho \mathbf{g} + \mathbf{F}. \tag{1.2}$$

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Here p denotes the fluid (or plasma) pressure and $\rho\mathbf{g}$ is the force per unit volume on the fluid because of gravity of vectorial strength $\mathbf{g}$ (we will generally assume gravity to be uniform). $\mathbf{B}$ denotes the magnetic field that threads the fluid and $\mathbf{j}$ is the current density; these two terms produce the magnetic body force $\mathbf{j} \times \mathbf{B}$, perpendicular to both $\mathbf{B}$ and $\mathbf{j}$. Finally, there may also be other forces $\mathbf{F}$ acting, such as the viscous force.

The magnetic field $\mathbf{B}$ is related to the current density $\mathbf{j}$ by Ampere's law, namely

$$\mu\mathbf{j} = \operatorname{curl}\mathbf{B}, \tag{1.3}$$

where μ is the magnetic permeability of the fluid; generally it is assumed that $\mu = 4\pi \times 10^{-7}$ henry m^{-1}, its value in free space.

Temporal changes in the magnetic field $\mathbf{B}$ are related to spatial changes in the electric field $\mathbf{E}$ through Faraday's law of induction:

$$\frac{\partial\mathbf{B}}{\partial t} = -\operatorname{curl}\mathbf{E}. \tag{1.4}$$

There is a constraint on the magnetic field: it must be solenoidal,

$$\operatorname{div}\mathbf{B} = 0. \tag{1.5}$$

This constraint is the statement that there are no magnetic monopoles: magnetic field lines have no ends, but either close upon themselves or are infinite in extent (which we may view as closing at infinity). There is thus a sharp contrast between magnetic field lines and electric field lines, for the latter originate in concentrations of charge and so may be viewed as emanating from a point.

In view of the vector identity

$$\operatorname{div}\operatorname{curl} \equiv 0,$$

we see that equation (1.4) implies that $\partial(\operatorname{div}\mathbf{B})/\partial t = 0$ and so, as a consequence of Faraday's law of induction, $\operatorname{div}\mathbf{B}$ is time independent (and thus is zero for all times if zero at any instant). The constraint (1.5) is stronger, though, insisting that the divergence of $\mathbf{B}$ is necessarily zero always. There is also an implied constraint on lines of current $\mathbf{j}$, for the above vector identity taken with Ampere's law (1.3) implies that $\operatorname{div}\mathbf{j} = 0$, and so lines of current density $\mathbf{j}$ (like lines of magnetic field) also have no ends.

The electric field $\mathbf{E}$ is related to the current density $\mathbf{j}$ by Ohm's law, as applied to a moving conductor – the fluid moving with an internal velocity $\mathbf{u}$:

$$\mathbf{j} = \sigma(\mathbf{E} + \mathbf{u} \times \mathbf{B}), \tag{1.6}$$

where σ is the electrical conductivity of the fluid. Here $\mathbf{E} + \mathbf{u} \times \mathbf{B}$ is the total electric field in the fluid, allowing for the induced electric field arising from the component of motion $\mathbf{u}$ across the field $\mathbf{B}$.

By combining equations (1.4) and (1.6) we may eliminate the electric field $\mathbf{E}$:

$$\frac{\partial\mathbf{B}}{\partial t} = \operatorname{curl}(\mathbf{u} \times \mathbf{B}) - \operatorname{curl}\left(\frac{1}{\sigma}\mathbf{j}\right). \tag{1.7}$$

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Using Ampere's law (1.3), we may eliminate **j** to obtain

$$\frac{\partial \mathbf{B}}{\partial t} = \mathrm{curl}(\mathbf{u} \times \mathbf{B}) - \mathrm{curl}(\eta\, \mathrm{curl}\, \mathbf{B}), \tag{1.8}$$

where we have written $\eta = 1/(\mu\sigma)$; η is referred to as the magnetic diffusivity of the fluid, and has units $\mathrm{m}^2\,\mathrm{s}^{-1}$. Equation (1.8) is the general form of the magnetohydrodynamic induction equation.

Changes in the fluid are generally considered to proceed according to an energy balance equation of the form

$$\frac{\partial p}{\partial t} + \mathbf{u} \cdot \mathrm{grad}\, p = \frac{\gamma p}{\rho}\left(\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \mathrm{grad}\, \rho\right) - (\gamma - 1)\mathcal{L}, \tag{1.9}$$

where $\mathcal{L}$ is the gain or loss function (energy per unit volume) and γ is the ratio of specific heats at constant pressure and constant volume. The term $\mathcal{L}$ includes contributions from thermal conduction and radiation. Mechanical heating from external sources as well as the *Joule* (or *ohmic*) *heating* may also be added to the right-hand side of (1.9). Joule heating, arising from the dissipation of current within the fluid, amounts to j^2/σ watts m^{-3}, for a current density of strength $j\ (= |\mathbf{j}|)$. Frequently all these heat losses or gains are considered to be negligible, and then isentropic (or adiabatic) conditions pertain:

$$\frac{\partial p}{\partial t} + \mathbf{u} \cdot \mathrm{grad}\, p = \frac{\gamma p}{\rho}\left(\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \mathrm{grad}\, \rho\right). \tag{1.10}$$

The ratio of specific heats, γ, is generally assumed to be constant. In numerical illustrations we take $\gamma = 5/3$, the value appropriate for a fully ionized gas. Heat losses are discussed in Chapter 12.

The fluid we are considering will be treated as a perfect gas, for which the ideal gas law is

$$p = \frac{k_B}{\hat{m}}\rho T, \tag{1.11}$$

where $k_B\ (= 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}})$ is the Boltzmann constant, T is the absolute temperature of the fluid in degrees kelvin (K), equal to the temperature in degrees celsius (°C) plus 273, and $\hat{m}$ is its mean particle mass.

1.4 Some Properties of the MHD Equations

The above system of equations forms the basis for a description of waves in a magnetohydrodynamic fluid. However, as the physicist E. N. Parker says in the Preface of his 1979 monograph *Cosmical Magnetic Fields*, treating the physics of large-scale magnetic fields in fluids, 'The fundamental equations of physics may contain all knowledge, but they are close-mouthed and do not volunteer that knowledge' (Parker 1979a). Thus, in particular, the nature of wave propagation in a magnetic fluid, as described by the equations introduced earlier, is not transparent and indeed serves as the topic for this book. Certain basic features of the equations can, however, be immediately uncovered and these act as points of illumination in our general discourse. We set out these aspects here, as a preliminary to our more detailed discussion of the nature of wave propagation.

1.4.1 Fundamental Speeds

The magnetohydrodynamic equations have embedded within them the usual equations of acoustics, which follow by taking $\mathbf{B} \equiv 0$. Consequently, the magnetohydrodynamic equations must contain the familiar sound speed c_s, defined by

$$c_s = \left(\frac{\gamma p_0}{\rho_0}\right)^{1/2}. \tag{1.12}$$

Here ρ_0 and p_0 refer to the fluid density and pressure in the unperturbed state of the medium. The occurrence of such a speed as (1.12) is evident on general dimensional grounds. For, by balancing in the momentum equation (1.2) the acceleration term $\rho(\partial\mathbf{u}/\partial t)$ with the pressure force grad p we obtain the dimensional combination

$$V\tau^{-1} \sim \frac{p}{L\rho}.$$

Here V denotes a characteristic speed, L a characteristic length, τ a characteristic timescale, with p and ρ denoting a characteristic pressure and density; the use of '$\sim$' here denotes a dimensional balance. Writing $V \sim L\tau^{-1}$ then gives $V^2 \sim p/\rho$, which leads to the combination (1.12) for a characteristic speed (though without the important factor of γ) when we take the equilibrium pressure p_0 and density ρ_0 as representative values.

A similar argument for a magnetic speed can be made by equating, in dimensional terms, the acceleration term $\rho(\partial\mathbf{u}/\partial t)$ with the magnetic force $\mathbf{j} \times \mathbf{B}$. We obtain $\rho V\tau^{-1} \sim JB$, where J and B are characteristic values of the current density and magnetic field. But from Ampere's law (1.3) we have $\mu J \sim BL^{-1}$, which allows us to eliminate J. Setting $V \sim L\tau^{-1}$ then gives $V^2 \sim B^2/(\mu\rho)$. We thus obtain a characteristic magnetic speed, a speed that arises in phenomena for which the magnetic force plays a role. We take this speed[1] as

$$c_A = \left(\frac{B_0^2}{\mu\rho_0}\right)^{1/2}, \tag{1.13}$$

choosing an equilibrium field strength B_0 and a plasma density ρ_0 as representative values of the magnetic field and fluid density. The speed c_A defined by equation (1.13) is the *Alfvén speed*, the speed obtained by Alfvén in his short letter to *Nature* in 1942 (Alfvén 1942a). It is central to all magnetohydrodynamic wave phenomena, just as the sound speed is central to all acoustic phenomena.

The sound speed and the Alfvén speed underpin all wave phenomena described by the magnetohydrodynamic equations. Other speeds also play an important role, but these are always constructed in terms of c_s and c_A. Accordingly, we consider the sound and Alfvén speeds in a little more detail.

Plasma Pressure and Density

To begin with suppose our medium is a fully ionized hydrogen plasma, consisting of n_e electrons and n_p protons (ions) in each unit volume of space. The total pressure is

$$p = n_e k_B T_e + n_p k_B T_p$$

[1] In the Gaussian cgs system of units, the Alfvén speed is define as $c_A = B_0/(4\pi\rho_0)^{1/2}$ where the magnetic field strength B_0 is in gauss (G) and the plasma density ρ_0 is in grams per cubic centimetre (g cm^{-3}).

where T_e denotes the electron temperature and T_p the proton temperature. We will assume that the electron temperature and proton temperature are equal, so that $T_e = T_p = T$, where T denotes the temperature of the medium (the plasma or fluid temperature). Then,

$$p = (n_e + n_p)k_B T.$$

Now charge neutrality requires that the number of electrons be the same as the number of protons, so that $n_e = n_p$; then total number of particles is $n = n_e + n_p = 2n_e$ and the pressure is

$$p = 2n_e k_B T = nk_B T.$$

The density ρ is determined by the number n_e of electrons of mass m_e together with the number n_p of protons of mass m_p in a unit volume, so that

$$\rho = n_e m_e + n_p m_p.$$

However, whilst the number of electrons equals the number of protons, the mass m_p ($= 1.673 \times 10^{-27}$ kg) of a proton is much larger than the mass m_e ($= 9.109 \times 10^{-31}$ kg) of an electron, so we can neglect the electron mass and take

$$\rho = n_e m_p = \frac{1}{2} n m_p.$$

Accordingly, we can write

$$p = \frac{k_B}{\hat{\mu} m_p}\rho T = \frac{R}{\hat{\mu}}\rho T, \qquad \rho = \hat{\mu} m_p n = \hat{m} n, \qquad R = \frac{k_B}{m_p} \tag{1.14}$$

as a description of the plasma pressure p, density ρ and temperature T. Here $\hat{m} = \hat{\mu} m_p$ denotes the mean particle mass of the plasma; for a hydrogen plasma, $\hat{\mu} = 1/2$ and $\hat{m} = m_p/2$.

If the plasma consists of a more complicated mixture of hydrogen and helium (and other elements too) then we can take relations (1.14) as still standing but now the factor $\hat{\mu}$ is no longer $1/2$; in the solar corona, to allow for the contributions from those other elements that make up the plasma, it is common to take $\hat{\mu} \approx 0.6$ (see, for example, Aschwanden 2004).

Sound Speed

The ideal gas law (1.14) allows us to express the sound speed in terms of the square root of the temperature T_0 of the medium:

$$c_s = \left(\frac{\gamma p_0}{\rho_0}\right)^{1/2} = \left(\frac{\gamma k_B T_0}{\hat{m}}\right)^{1/2} = \left(\frac{\gamma k_B T_0}{\hat{\mu} m_p}\right)^{1/2}. \tag{1.15}$$

The mean particle mass $\hat{m}$ depends upon the nature of the plasma. With Boltzmann constant $k_B = 1.38 \times 10^{-23}\ \mathrm{J\,K^{-1}}$, the proton mass $m_p = 1.673 \times 10^{-27}$ kg (so that $R = k_B/m_p = 8.25 \times 10^3\ \mathrm{m^2\,s^{-2}\,K^{-1}}$) and an adiabatic index $\gamma = 5/3$, equation (1.15) gives a sound speed

$$c_s = 1.17 \times 10^2\,(T_0/\hat{\mu})^{1/2}\ \mathrm{m\,s^{-1}}. \tag{1.16}$$

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In the solar corona, the plasma is mainly made up of hydrogen and helium, which produces a mean atomic weight of $\hat{\mu} \approx 0.6$. Thus, in the corona the sound speed is

$$c_s = 151\, T_0{}^{1/2}\,\mathrm{m\,s^{-1}}. \tag{1.17}$$

A coronal temperature of, say, $T_0 = 10^6$ K then yields a sound speed of $c_s = 151\,\mathrm{km\,s^{-1}}$. The high temperature of the coronal plasma thus leads to a correspondingly high sound speed, far in access of the $340\,\mathrm{m\,s^{-1}}$ sound speed in the Earth's atmosphere or the $1400\,\mathrm{m\,s^{-1}}$ sound speed in water.

Lower in the solar atmosphere, a photospheric density of $\rho_0 = 3 \times 10^{-4}\,\mathrm{kg\,m^{-3}}$ $(= 3 \times 10^{-7}\,\mathrm{g\,cm^{-3}})$ and pressure of $p_0 = 2 \times 10^4\,\mathrm{N\,m^{-2}}$ $(= 2 \times 10^5\,\mathrm{dynes\,cm^{-2}})$ (see, for example, Parker 1979a, p. 212) produces (for $\gamma = 5/3$) a sound speed of $c_s = 10.5\,\mathrm{km\,s^{-1}}$.

Alfvén Speed

Turning to the Alfvén speed, we have

$$c_A = \left(\frac{B_0^2}{\mu\rho_0}\right)^{1/2} = 2.18 \times 10^{16} \times \frac{B_0}{(\hat{\mu}n)^{1/2}}\,\mathrm{m\,s^{-1}}. \tag{1.18}$$

We have taken $\mu = 4\pi \times 10^{-7}$ henry $\mathrm{m^{-1}}$ and the plasma density as $\rho_0 = \hat{\mu}m_p n$, for total number density n (in particles per cubic metre), and the magnetic field strength B_0 is in tesla (T). Thus, with $\hat{\mu} \approx 0.6$ appropriate for the corona we obtain

$$c_A = 2.816 \times 10^{16} \times \frac{B_0}{n^{1/2}}\,\mathrm{m\,s^{-1}}. \tag{1.19}$$

Thus, with a number density of say $n = 10^{15}$ particles per $\mathrm{m^3}$ typical of the corona we obtain $\rho_0 = 2 \times 10^{-12}\,\mathrm{kg\,m^{-3}}$ (or $2 \times 10^{-15}\,\mathrm{g\,cm^{-3}}$) and an Alfvén speed of $c_A = 0.890 \times 10^9\, B_0\,\mathrm{m\,s^{-1}}$.

It is common to quote the magnetic field strength in gauss (G), noting that $1\,\mathrm{T} = 10^4\,\mathrm{G}$. Then the expression for the Alfvén speed reads

$$c_A = 2.816 \times 10^{12} \times \frac{B_0\,(\mathrm{G})}{n^{1/2}}\,\mathrm{m\,s^{-1}}, \tag{1.20}$$

with B_0 in gauss. For example, with a coronal field strength of $B_0 = 10^{-3}\,\mathrm{T}\,(= 10\,\mathrm{G})$, we obtain an Alfvén speed of some $c_A = 890\,\mathrm{km\,s^{-1}}$. In an active region the field is stronger; for example, with $B_0 = 10^{-2}\,\mathrm{T}\,(= 10^2\,\mathrm{G})$ in a medium with an electron number density $n_e = 10^{16}$ particles per $\mathrm{m^3}$ (10^{10} particles per $\mathrm{cm^3}$) and a total number density $n = 2 \times 10^{16}$ particles per $\mathrm{m^3}$, we obtain $c_A \approx 2000\,\mathrm{km\,s^{-1}}$. Thus, despite the corona's high temperature and correspondingly high sound speed, the tenuous nature of the coronal plasma acts to produce a yet higher Alfvén speed.

In the photosphere, it is usual to quote values of fluid density ρ_0 directly, basing these values on model computations of the convection zone and atmosphere above. A density of $\rho_0 = 3 \times 10^{-4}\,\mathrm{kg\,m^{-3}}$ $(= 3 \times 10^{-7}\,\mathrm{g\,cm^{-3}})$ is typical of the surface layers of the Sun (see, for example, Parker 1979a, p. 149). In a magnetic field of $B_0 = 0.15\,\mathrm{T}\,(= 1500\,\mathrm{G})$ the corresponding Alfvén speed is $c_A = 0.077 \times 10^5\,\mathrm{m\,s^{-1}}$. Thus in regions of strong

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photospheric magnetic field the Alfvén speed is some 7.7 $\mathrm{km\,s^{-1}}$ and sound and Alfvén speeds are typically comparable.

1.4.2 Magnetic Flux Tubes

It is convenient to introduce the notion of a magnetic flux tube. The concept of a magnetic flux tube goes back to the writings of Faraday and Maxwell. Faraday, with his non-mathematical approach to electromagnetic phenomena, pictured the movement and distortion of magnetic field lines. Maxwell later added the mathematical detail to give such an intuitive approach a more rigorous basis. Consider a curve drawn arbitrarily in a magnetic field. All the field lines passing through this curve are considered to be related, forming a single entity called a *magnetic flux tube*. Since the choice of the curve that relates the various field lines is entirely arbitrary, the flux tube thus formed is also an arbitrary collection of magnetic field lines. However, in Nature it is often found that a certain collection of field lines is of particular interest and form, giving special definition to those field lines. In the solar photosphere, for example, we have seen that isolated magnetic flux tubes occur, corresponding to concentrations of magnetic field surrounded by a field-free environment. In such objects the magnetic flux tube is given definition by the field itself. In the corona, magnetic loops are flux tubes given definition not so much by their field strength – the field may indeed be essentially uniform – but by the fact that certain field lines are loaded with more plasma or are at a higher temperature than other field lines; the flux tube is thus here given definition by an enhancement in the plasma density, or by temperature differences between one region or another. Certain types of prominence structures are also examples of this kind. The magnetospheres of planets also provide examples of flux tubes which are given definition by their magnetic field structure, being commonly twisted.

One property of a magnetic flux tube follows immediately from the solenoidal constraint on a magnetic field. For with $\mathbf{B}$ satisfying equation (1.5), application of the divergence theorem in an arbitrary volume $\mathcal{V}$ yields

$$\int_{\mathcal{V}} \operatorname{div} \mathbf{B}\, dV = \int_{S_{\mathcal{V}}} \mathbf{B} \cdot \mathbf{dS} = 0, \tag{1.21}$$

where the volume $\mathcal{V}$ is enclosed by the surface $S_{\mathcal{V}}$ and $\mathbf{dS}$ is the surface element pointing (by convention) *out* of the volume $\mathcal{V}$.

For a magnetic flux tube we may choose that surface to be the curved surface of the flux tube together with a 'top' surface S_{t} and a 'bottom' surface S_{b} to make a closed volume. Then, since no magnetic flux leaves the curved surface of the tube (on which $\mathbf{B} \cdot \mathbf{dS} = 0$), equation (1.21) implies that

$$\int_{S_{\mathrm{t}}} \mathbf{B} \cdot \mathbf{dS} = \int_{S_{\mathrm{b}}} \mathbf{B} \cdot \mathbf{dS}, \tag{1.22}$$

where the cross-sectional surface element $\mathbf{dS}$ points in the same sense as $\mathbf{B}$. In other words, the magnetic flux across any cross-section S of the tube is the same at all locations along the tube; it is thus an invariant of the motion. Since this property follows directly from the solenoidal constraint it is independent of whether we are considering dissipative effects (such as viscosity or diffusivity) or not. Stated loosely, the product of the field strength B of a tube and its normal cross-sectional area S is a constant. Thus, where a tube narrows its field strength is large, whereas in the expanded regions of the tube the field strength is correspondingly reduced.

1.4.3 The Induction Equation

The induction equation (1.8) makes clear that there is a direct link in magnetohydrodynamics between fluid motions and the magnetic field, and that this link is independent of other properties of the fluid, such as its density and pressure, unless they enter indirectly through the magnetic diffusivity. In fact, in a plasma the electrical conductivity σ and magnetic diffusivity η are principally determined by the temperature of the medium. For a fully ionized hydrogen plasma at a temperature of T K, we have an electrical conductivity of $\sigma \approx 8 \times 10^{-4} T^{3/2}\,\mathrm{mho\,m^{-1}}$, leading to a magnetic diffusivity of (Parker 1979a, sect. 4.6 and 7.6; Priest 2014, sect. 2.1.5)

$$\eta \approx 10^9\, T^{-3/2}\, \mathrm{m^2\, s^{-1}}. \tag{1.23}$$

Thus the diffusivity varies quite strongly with the temperature of the plasma, being low in high temperature plasmas. Complications in the description of diffusivity arise when the level of ionization in the medium is very low or when the presence of the applied magnetic field is properly allowed for. Low ionization leads to a significant reduction in the value of η. The presence of a magnetic field renders the diffusivity as a tensor, with a different value along the field from that across the field. However, such complications will not be discussed here.

To illustrate the link between the flow and the magnetic field embedded in the fluid, as described by the induction equation, it is convenient to discuss the nature of the induction equation for the case of constant magnetic diffusivity, ignoring the temperature dependence given in equation (1.23), except in so far as it provides an appropriate numerical value for η. In fact the complications introduced by a variable diffusivity η do not introduce anything of general significance as regards the nature of the link between the flow and the magnetic field. Accordingly, we consider the induction equation under the assumption that η is a constant. We may then simplify equation (1.8) by use of the vector identity

$$\mathrm{curl\,curl} \equiv \mathrm{grad\,div} - \nabla^2.$$

Coupled with the solenoidal constraint (1.5) on the magnetic field, the above vector identity allows us to write equation (1.8) in the form

$$\frac{\partial \mathbf{B}}{\partial t} = \mathrm{curl}(\mathbf{u} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}. \tag{1.24}$$

This is the induction equation for a medium with uniform magnetic diffusivity. The magnetic field $\mathbf{B}$ grows or diminishes in time according to the combined influences of a velocity-dependent *advective* term (the first term on the right-hand side of equation (1.24)) and a velocity-independent *diffusive* term (the term involving η). These terms have quite distinct effects and, as we shall see, are generally of quite different magnitudes.

There is an interesting analogy between the evolution of magnetic field, as described by the induction equation (1.24), and the evolution of vorticity $\omega \equiv \text{curl}\,\mathbf{u}$ in a viscous non-magnetic liquid with uniform density. Consider the momentum equation (1.2) in the absence of magnetism ($\mathbf{B} = 0$) and gravity ($\mathbf{g} = 0$), for a liquid with constant density ρ. The viscous force $\mathbf{F}$ in such a liquid is given by

$$\mathbf{F} = \rho\nu\nabla^2\mathbf{u}, \tag{1.25}$$

where ν is the kinematic viscosity (assumed constant) of the liquid. The momentum equation is thus

$$\rho\left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\text{grad})\mathbf{u}\right] = -\text{grad}\,p + \rho\nu\nabla^2\mathbf{u}. \tag{1.26}$$

Taking the ‘curl’ of the above equation, and noting the vector identity

$$\text{curl}\,\text{grad} \equiv 0,$$

we obtain

$$\frac{\partial \omega}{\partial t} + \text{curl}[(\mathbf{u}\cdot\text{grad})\mathbf{u}] = \text{curl}\,(\nu\nabla^2\mathbf{u}).$$

Then, making use of the above vector identities together with

$$\frac{1}{2}\text{grad}(\mathbf{u}\cdot\mathbf{u}) \equiv \mathbf{u}\times\text{curl}\,\mathbf{u} + (\mathbf{u}\cdot\text{grad})\mathbf{u},$$

we obtain

$$\frac{\partial \omega}{\partial t} = \text{curl}\,(\mathbf{u}\times\omega) + \nu\nabla^2\omega. \tag{1.27}$$

Thus we see that the vorticity ω in a uniform non-magnetic liquid evolves in time in much the same way as the magnetic field evolves in a magnetized fluid. There is a difference, however, in that whereas the vorticity is directly related to the motion ($\omega = \text{curl}\,\mathbf{u}$), the magnetic field $\mathbf{B}$ and motion $\mathbf{u}$ are not so related. This means that whereas the vorticity equation is *nonlinear*, involving a product of $\mathbf{u}$ with $\text{curl}\,\mathbf{u}$, the induction equation is *linear* in $\mathbf{B}$ for a given motion $\mathbf{u}$. Moreover, the kinematic diffusivity ν in a liquid is generally much smaller than the magnetic diffusivity η of a fluid. The kinematic viscosity of water, for example, is $\nu \approx 10^{-6}\,\text{m}^2\,\text{s}^{-1}$, some two orders of magnitude smaller than that of olive oil; the electrically conducting fluid mercury has a kinematic viscosity of $\nu \approx 10^{-7}\,\text{m}^2\,\text{s}^{-1}$, about a tenth that of water. These kinematic viscosities ν are several orders of magnitude smaller than the corresponding magnetic diffusivities η; in the case of liquid mercury, there is a difference of seven orders of magnitude. There are thus distinctive differences between the two systems. Nonetheless, the analogy can prove useful.

The magnitudes of the two terms that make up the right-hand side of the induction equation (1.24) are easily estimated using dimensional considerations. Their ratio forms a number R_m, known as the magnetic Reynolds number (by analogy with the Reynolds number of a viscous fluid). We have

$$R_\text{m} \sim |\text{curl}\,(\mathbf{u}\times\mathbf{B})|/|\eta\nabla^2\mathbf{B}|.$$

This leads to $R_m \sim L^{-1}VB/(\eta L^{-2}B)$, on noting that 'curl' involves a single division by a spatial scale L whereas '∇^2' involves division by L^2; as before, V and B denote characteristic values of the motion and magnetic field strength, each assumed to vary on the same scale L. Thus

$$R_m \sim \frac{LV}{\eta}. \tag{1.28}$$

This may be compared with the Reynolds number R of a viscous flow: $R = LV/\nu$.

The dimensionless number R_m determines which term on the right-hand side of the induction equation is dominant. If the magnetic Reynolds number is large ($R_m \gg 1$) then the right-hand side of equation (1.24) is dominated by the advection term and diffusive effects are essentially negligible. Only if one follows the advected field for a long time would one detect the small diffusion of field that takes place when $R_m \gg 1$. Precisely the opposite conclusion is reached when the magnetic Reynolds number is small. For with $R_m \ll 1$ diffusive effects are dominant and the advection term is essentially negligible. Only if one follows the evolution of the magnetic field **B** for a long time would one see the influence of advection on the overall diffusion of the field. In the intermediate case, when R_m is of order unity, diffusion and advection play comparable roles in the evolution of the magnetic field.

Now the size of R_m is determined not so much by whether a fluid is electrically a good or a poor conductor, but by the size L of the fluid in which the magnetic field is entrained and the flow V that is present. For these quantities can change by orders of magnitude, from circumstance to circumstance, over-shadowing the possible variations in η in all but the most severe cases (when large temperature differences can bring about correspondingly large changes in R_m, through changes in η). Generally, then, we find that for a liquid (such as mercury or molten iron) in laboratory circumstances R_m is small or of order unity, simply because L and V are relatively small. In astrophysical circumstances, however, R_m is large, simply because L (and perhaps V) are large. Thus laboratory systems tend to be dominated by diffusive effects, whereas astrophysical plasmas are largely free from the influences of diffusion. An exception in astrophysical systems occurs in regions where **B** may undergo a sudden reversal in direction or rapid variation; then the scale of variation of **B** is no longer necessarily the same as that of the flow and a local diffusion of magnetic field or magnetic reconnection may occur (see, for example, Priest and Forbes 2000; Priest 2014).

To illustrate specifically the magnitude of R_m, consider a laboratory fluid with spatial extent $L = 1$ m and a flow of order $V = 0.1\,\text{m}\,\text{s}^{-1}$; for molten iron (with $\eta = 0.06\ \text{m}^2\,\text{s}^{-1}$), this produces a magnetic Reynolds number of order unity. For the Earth's liquid core, with a scale of $L = 3.5 \times 10^6$ m and a diffusivity of $\eta = 3\,\text{m}^2\,\text{s}^{-1}$, fluid motions of say $0.1\,\text{mm}\,\text{s}^{-1}$ ($= 10^{-4}\,\text{m}\,\text{s}^{-1}$) produce a magnetic Reynolds number of order 10^2. Turning to the Sun, there are flows of order $1\,\text{km}\,\text{s}^{-1}$ ($= 10^3\,\text{m}\,\text{s}^{-1}$) on a scale of 10^3 km observed at its surface (in granules); with $\eta = 10^5\,\text{m}^2\,\text{s}^{-1}$, we obtain an R_m of 10^4.

We turn now to a brief examination of the behaviour of the induction equation in the two extremes of small and large magnetic Reynolds number, corresponding to diffusive effects being either important or negligible.

1.4.4 Diffusion of Magnetic Field ($R_m \ll 1$)

Consider the extreme of the induction equation that arises for $R_m \ll 1$, when the diffusion term $\eta\nabla^2\mathbf{B}$ dominates over advection. Then the induction equation reduces to

$$\frac{\partial \mathbf{B}}{\partial t} = \eta\nabla^2\mathbf{B}. \tag{1.29}$$

This reduction of the induction equation is exact when no flows are present ($\mathbf{u} = 0$). Generally, though, we may regard equation (1.29) as giving the dominant short-term behaviour of the system when $R_m \ll 1$.

We recognize equation (1.29) as the *vector* form of the *diffusion equation* (or heat conduction equation). In a Cartesian coordinate system each component of the magnetic field satisfies the familiar (scalar) diffusion equation. The diffusion equation acts so as to smooth out irregularities or steep gradients in $\mathbf{B}$.

We note that the decay or leakage of magnetic field through the fluid is not an entirely passive process, for it is accompanied by ohmic heating j^2/σ ($= \mu\eta j^2$), which is of order $(\eta/\mu)B^2L^{-2}$ in a field of characteristic strength B and spatial scale L. The process of field decay operates on a timescale τ, readily estimated from dimensional considerations. In dimensional terms, from equation (1.29) we have $B\tau^{-1} \sim \eta BL^{-2}$. Solving for the decay time $\tau = \tau^{\rm decay}$ we obtain

$$\tau^{\rm decay} \sim L^2/\eta. \tag{1.30}$$

Hence the magnetic field in a motionless conductor decays away on a timescale that is proportional to the electrical conductivity σ of the medium and proportional to the *square* of the spatial scale L: $\tau^{\rm decay} \sim \mu\sigma L^2$.

The time $\tau^{\rm decay}$ gives an estimate of how long it takes a concentration of magnetic field to leak away by a factor of $1/e$ (with $e = 2.718\ldots$ being Euler's constant), reducing to some 37% of its initial value. Of course, the estimate provided by equation (1.30) is rather rough, as it ignores such factors as the geometry of the object and the precise choice for L. Only by solving the diffusion equation in a specific case can one determine $\tau^{\rm decay}$ more precisely. In the case of a sphere of radius a it turns out that $\tau^{\rm decay} = a^2/(\eta\pi^2)$, so that for the sphere we have an effective L of a/π. Such factors of π and the like are in fact significant if, as occurs here, they are squared or raised to a higher power. For the sphere the factor of π reduces the simple estimate of a^2/η for the decay time by an order of magnitude.

The presence in $\tau^{\rm decay}$ of the *square* of L – an immediate consequence of the Laplacian operator ∇^2 in the diffusion equation – makes for very short timescales in laboratory situations but very long ones in astrophysical circumstances. We may readily illustrate this wide range in decay (diffusion) times. For example, in a copper sphere (with $\eta = 0.01\,{\rm m^2\,s^{-1}}$) of radius $a = 1$ m, the leakage time of the magnetic field through the sphere is of order $\tau^{\rm decay} = 10$ s, and so rather short. By contrast, for a sphere the size of the Sun as a whole, with a radius of 6.96×10^8 m and a diffusivity of $\eta = 1\,{\rm m^2\,s^{-1}}$, the value given by equation (1.23) for a temperature of 10^6 K (roughly representative of the wide range in temperature in the solar interior from some 10^7 K in the core to 6000 K at the surface), we obtain a decay time of around 10^9 years (Cowling 1946, 1976), comparable with the determined age of the Sun (as 5×10^9 years). Thus, a primordial magnetic field entrapped within the Sun at its formation would still largely be entrapped today. Of course, we *observe* magnetic fields on the Sun changing on relatively short timescales (ranging from hours to years), so magnetic fields must be manipulated by forces that are more effective than the simple passive diffusion of the field.

The decay time for the field in an individual object within the Sun is, of course, much shorter than the time for the Sun as a whole. For example, a sunspot with radius $L = 10^7$ m and diffusivity $\eta = 10^3\ \mathrm{m^2\,s^{-1}}$ (corresponding to a temperature of 10^4 K, representative of the layers below the surface) we obtain a decay time of over 300 years, as originally estimated by Cowling. In fact, sunspots change their magnetism on much shorter times than this and so flows must play a significant part in determining their magnetic history.

1.4.5 Advection of Magnetic Field ($R_{\mathrm{m}} \gg 1$)

Consider now the extreme of the induction equation arising when we neglect diffusivity. A fluid in which $\eta = 0$ is said to be a *perfect* (or *ideal*) conductor, for which the induction equation is simply

$$\frac{\partial \mathbf{B}}{\partial t} = \mathrm{curl}\,(\mathbf{u} \times \mathbf{B}). \tag{1.31}$$

This equation is exact if $\eta = 0$ but applies more generally to the case of high magnetic Reynolds number, describing the short-term behaviour when $R_{\mathrm{m}} \gg 1$.

The analogy between the induction equation and the vorticity equation for a liquid may be exploited here. We may immediately invoke Kelvin's circulation theorem, which tells us that the vorticity ω in a flow is advected with the fluid. Accordingly, we may conclude that the magnetic field in a conducting fluid is advected with the flow. Alfvén (1943) expressed this picturesquely, saying that the magnetic field lines are *frozen* into the flow: motions along the field lines leave them unchanged, whereas motions perpendicular to the field lines transport the lines with the flow.

That the magnetic field is frozen into the flow in ideal magnetohydrodynamics has an immediate consequence for a magnetic flux tube. Since each magnetic field line moves with the fluid, it follows that a magnetic flux tube moves with the motion: the fluid entrained within a tube at a given time remains entrained within that tube during the subsequent motion of the fluid, though the tube itself may be distorted by the flow (though, as noted earlier, its magnetic flux remains invariant).

Consider, then, an elemental flux tube with field strength B and normal cross-section dS. Then the flux BdS is a constant of the motion. If we consider a section of the flux tube of length ds measured between two cross-sections of the tube, then in the subsequent motion the mass $\rho ds dS$ contained between the two cross-sections is conserved during the motion. Thus,

$$BdS = \text{constant}, \qquad \rho ds dS = \text{constant}. \tag{1.32}$$

Eliminating dS between these two invariants shows that B/ρ is proportional to the distance ds between neighbouring cross-sections. In other words, B/ρ *increases* during the motion if that motion *stretches* the field line, whereas it decreases if the motion reduces the length of the field line between the two cross-sections.

We can view the above discussion directly from the induction equation under ideal conditions and the equation of continuity. First rewrite the induction equation (1.31) by using the vector identity

$$\text{curl}\,(\mathbf{u} \times \mathbf{B}) \equiv \mathbf{u}(\text{div}\,\mathbf{B}) - \mathbf{B}(\text{div}\,\mathbf{u}) + (\mathbf{B} \cdot \text{grad})\mathbf{u} - (\mathbf{u} \cdot \text{grad})\mathbf{B}$$

and setting $\text{div}\,\mathbf{B} = 0$. Then, when combined with the equation of continuity (1.1), we obtain

$$\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \text{grad}\right)\left(\frac{\mathbf{B}}{\rho}\right) = \left(\frac{\mathbf{B}}{\rho} \cdot \text{grad}\right)\mathbf{u}. \tag{1.33}$$

Thus, if the right-hand side of the above equation is ignored we see that the quantity $\mathbf{B}/\rho$ is *conserved* during the motion. The effect then of a non-zero right-hand side is to cause an *increase* (decrease) in $\mathbf{B}/\rho$ if the motion *stretches* (compresses) the field lines, corresponding to

$$\frac{B}{\rho}\frac{\partial \mathbf{u}}{\partial s}$$

being *positive* (negative).

1.4.6 The $\mathbf{j} \times \mathbf{B}$ Force

Magnetic effects in the momentum equation (1.2) are determined by the $\mathbf{j} \times \mathbf{B}$ force, which acts perpendicular to the magnetic field $\mathbf{B}$ and the current $\mathbf{j}$. Now, using Ampere's law (1.3) and the vector identity

$$\frac{1}{2}\,\text{grad}\,(\mathbf{B} \cdot \mathbf{B}) \equiv \mathbf{B} \times \text{curl}\,\mathbf{B} + (\mathbf{B} \cdot \text{grad})\mathbf{B},$$

we obtain

$$\mathbf{j} \times \mathbf{B} = -\text{grad}\left(\frac{B^2}{2\mu}\right) + \frac{1}{\mu}(\mathbf{B} \cdot \text{grad})\mathbf{B}, \tag{1.34}$$

where now $B\,(= |\mathbf{B}|)$ denotes the magnitude of the field $\mathbf{B}$. Comparing this form of the magnetic force with the right-hand side of the momentum equation, we see immediately that we may regard the contribution

$$-\,\text{grad}\,(B^2/2\mu) \tag{1.35}$$

from the $\mathbf{j} \times \mathbf{B}$ force as acting as a *pressure* term, much the same as the fluid pressure force $-\text{grad}\,p$. Accordingly, we may introduce the *magnetic pressure* p_{m},

$$p_{\text{m}} = \frac{1}{2\mu}\mathbf{B} \cdot \mathbf{B} = \frac{B^2}{2\mu}. \tag{1.36}$$

The magnetic pressure p_{m} acts isotropically throughout the fluid, just as the plasma pressure p does.

A magnetic field, then, has a magnetic pressure p_m associated with it, augmenting the plasma pressure p; the magnetic pressure force increases in proportion to the *square* of the magnetic field strength. In illustrations we will commonly quote magnetic field strength in gauss; a field of B gauss produces a magnetic pressure of

$$B^2/(80\pi)\,\text{Pa}. \tag{1.37}$$

This is the magnetic pressure in pascals (Pa), the SI unit of pressure ($1\,\text{Pa} = 1\,\text{N}\,\text{m}^{-2}$). (In cgs units with B in gauss, the magnetic pressure is $B^2/8\pi$ dynes cm^{-2}.)

For example, the Earth's magnetic field of order $1/2\,\text{G}$ produces a magnetic pressure of some 10^{-3} Pa ($= 10^{-3}\,\text{N}\,\text{m}^{-2}$), eight orders of magnitude smaller than a typical atmospheric pressure of 1 bar ($= 10^5$ Pa). By contrast, a field of 3000 G, typical of a sunspot, produces a magnetic pressure of 3.6×10^4 Pa, which (at about 1/3 bar) is roughly comparable with the plasma pressure of 1.6×10^4 Pa at the solar surface. Equilibrium in the spot is achieved because the plasma within the magnetic field sinks to a level below the photosphere where the confining external plasma pressure is higher.

The identification of magnetic pressure, acting in addition to any dynamical pressure, raises the question of their relative importance. This is decided upon by their ratio, commonly referred to as the *plasma beta* and defined by

$$\beta \equiv \frac{p}{p_m} = \frac{p}{(B^2/(2\mu)}. \tag{1.38}$$

If we take p and B as represented by the equilibrium values p_0 and B_0, we see that the plasma β is directly related to the ratio of the sound and Alfvén speeds:

$$\beta = \frac{2}{\gamma}\frac{c_s^2}{c_A^2}. \tag{1.39}$$

Thus a medium with high Alfvén speed ($c_A \gg c_s$) is a low-β plasma, and one with a low Alfvén speed ($c_A \ll c_s$) is a high-β plasma.

A fluid with low plasma β has a correspondingly strong magnetic field, and generally speaking it is mechanically dominated by the magnetic forces. This is the circumstance in much of the Earth's magnetosphere and in the upper atmosphere of the Sun. By contrast, in a high-β plasma magnetic forces are weak compared with the dynamical pressure force and magnetic effects are correspondingly less important; this is the circumstance pertaining in the Earth's interior and below the Sun's photosphere. Of course, in some situations β is of order unity, indicating that both magnetic and dynamical effects are of comparable importance. This situation typically pertains in magnetic concentrations in the surface layers of the Sun.

The magnetic pressure term comprises only one part of the $\mathbf{j} \times \mathbf{B}$ force; there remains the term

$$\frac{1}{\mu}(\mathbf{B} \cdot \text{grad})\mathbf{B}. \tag{1.40}$$

This term may be interpreted as a *magnetic tension*. The magnetic field behaves much as an elastic band, the tension in the band being B^2/μ per unit area, acting along the magnetic field. Accordingly, a distortion or bend introduced into a magnetic field line sets up a tension force – just as with an elastic band – that acts so as to try to straighten out the field line.

To see this in detail, consider a magnetic field line with arc distance s measured along it from a fixed point. Denote by $\hat{\mathbf{s}}$ a unit vector pointing along the magnetic field $\mathbf{B}$. Then $\mathbf{B} = B(s)\hat{\mathbf{s}}$, where $B(s)$ is the field strength at location s. The magnetic tension force is accordingly

$$\frac{1}{\mu}(\mathbf{B}\cdot \text{grad})\mathbf{B} = \frac{1}{\mu}B(s)\frac{\partial}{\partial s}[B(s)\hat{\mathbf{s}}] = \hat{\mathbf{s}}\frac{\partial}{\partial s}\left(\frac{B^2}{2\mu}\right) + \frac{B^2}{\mu}\frac{\partial \hat{\mathbf{s}}}{\partial s}. \tag{1.41}$$

Now it is shown in discussions of vector calculus that

$$\frac{\partial \hat{\mathbf{s}}}{\partial s} = \frac{1}{R_c}\hat{\mathbf{n}},$$

where $R_c(s)$ is the radius of curvature of the field line at the location s and $\hat{\mathbf{n}}$ is the principal unit vector perpendicular to the field line at that location and pointing towards the centre of curvature. Hence

$$\frac{1}{\mu}(\mathbf{B}\cdot \text{grad})\mathbf{B} = \hat{\mathbf{s}}\frac{\partial}{\partial s}\left(\frac{B^2}{2\mu}\right) + \frac{B^2}{\mu R_c}\hat{\mathbf{n}}, \tag{1.42}$$

and so

$$\mathbf{j}\times\mathbf{B} = -\text{grad}\left(\frac{B^2}{2\mu}\right) + \hat{\mathbf{s}}\frac{\partial}{\partial s}\left(\frac{B^2}{2\mu}\right) + \frac{B^2}{\mu R_c}\hat{\mathbf{n}}. \tag{1.43}$$

The contribution from the term acting in the direction of $\hat{\mathbf{s}}$ may be grouped with the contribution from the pressure term, these respective contributions cancelling out in the direction of the magnetic field (as they must do, since the $\mathbf{j}\times\mathbf{B}$ force is perpendicular to $\mathbf{B}$). The term perpendicular to the magnetic field gives a tension force of magnitude $B^2/\mu R_c$, which acts so as to straighten out any bends in the field. The sharper the bend in the field, the smaller is the radius of curvature R_c and so the larger is the tension force acting to straighten out the field.

1.4.7 Energetics

As well as providing a magnetic pressure, the expression $B^2/2\mu$ gives the magnetic energy density (per unit volume) in the plasma. The total magnetic energy in a volume $\mathcal{V}$ is therefore

$$W = \int_{\mathcal{V}} \frac{B^2}{2\mu} dV. \tag{1.44}$$

It is of interest to determine how W varies in time for a fixed volume $\mathcal{V}$. Noting that $B^2 = \mathbf{B}\cdot\mathbf{B}$ and $j^2 = \mathbf{j}\cdot\mathbf{j}$, we may invoke the induction equation (1.24) and Ampere's relation (1.3) to obtain

$$\frac{dW}{dt} = -\int_{\mathcal{V}}\left[\frac{1}{\sigma}j^2 + \mathbf{u}\cdot\mathbf{j}\times\mathbf{B}\right]dV + \frac{1}{\mu}\int_{S_{\mathcal{V}}}\left[(\mathbf{u}\times\mathbf{B})\times\mathbf{B} - \frac{1}{\sigma}(\mathbf{j}\times\mathbf{B})\right]\cdot d\mathbf{S} \tag{1.45}$$

for surface $S_{\mathcal{V}}$ (with vector area element $\mathbf{dS}$) enclosing the volume $\mathcal{V}$. Consider the case when $\mathcal{V}$ is the whole of space, with $S_{\mathcal{V}} = S_\infty$ being the *sphere at infinity*. Then the contribution from the surface integral is negligible if B declines sufficiently fast as $r \to \infty$. For example, if B declines to zero faster than $r^{-3/2}$, a rate which ensures that W is finite (it diverges logarithmically if $B \sim r^{-3/2}$), then the surface integral of the $\mathbf{j} \times \mathbf{B}$ force is negligible. The result is

$$\frac{dW}{dt} = -\int_{S_\infty} \frac{1}{\sigma} j^2 dV - \int_{S_\infty} [\mathbf{u} \cdot \mathbf{j} \times \mathbf{B}] dV, \tag{1.46}$$

where the integrations are now over the whole of space. Thus, the fate of the magnetic energy in the system – whether it decays or grows – depends upon the two contributions on the right-hand side of equation (1.46). The contribution from Joule heating (the first term on the right-hand side of (1.46)) leads to an inexorable decline, to be balanced against the effect of the second term on the right-hand side of (1.46), the contribution from the fluid motions doing work against the opposing magnetic forces. When those motions are sufficiently vigorous and complicated so as to reverse the sign of the second term in equation (1.46), then W may grow in time: *dynamo action* is said to have occurred. To judge from the ubiquitous occurrence of magnetic fields in astrophysical objects, ranging from the planets to the stars and galaxies, Nature seems particularly adept at bringing about such an arrangement.

1.5 Aspects of Wave Propagation

1.5.1 Linearization

Waves generated in a system are often of very small amplitude, a state of affairs that permits one to examine *linear* equations describing the temporal and spatial behaviour of the perturbations (disturbances) about an equilibrium state. The process of obtaining such equations for the perturbations is referred to as *linearization*. We may illustrate the process of linearization by considering the equation of continuity (1.1). Denote by ρ_0 the value of the density in the basic state; suppose that there is no flow ($\mathbf{u} = 0$) in the basic state. Then in the disturbed state we write

$$\text{density} = \rho_0 + \rho, \qquad \text{motion} = \mathbf{u}, \tag{1.47}$$

where ρ and $\mathbf{u}$ now denote the values of the *perturbations* in density and motion. Thus equation (1.1) becomes

$$\frac{\partial \rho}{\partial t} + \text{div}\,(\rho_0 + \rho)\,\mathbf{u} = 0.$$

So far no approximation has been made, given that ρ_0 is independent of time. But if we now suppose that the perturbation ρ is small, so that $|\rho| \ll |\rho_0|$, then the equation of continuity reduces to

$$\frac{\partial \rho}{\partial t} + \text{div}\,\rho_0 \mathbf{u} = 0. \tag{1.48}$$

This represents a considerable simplification since it has resulted in a *linear* equation, and linear equations are relatively easy to solve whereas nonlinear equations (such as the original form of the continuity equation) are difficult. A further simplification occurs if the equilibrium state is a uniform one. For if the density ρ_0 is a constant, then the linearized equation of continuity becomes

$$\frac{\partial \rho}{\partial t} + \rho_0 \mathrm{div}\, \mathbf{u} = 0. \tag{1.49}$$

The equations describing the other perturbations may be treated in a similar fashion. For example, the linearized equation of motion (with $\mathbf{F} = 0$) becomes

$$\rho_0 \frac{\partial \mathbf{u}}{\partial t} = -\mathrm{grad}\, p + \mathbf{j} \times \mathbf{B_0} + \mathbf{j_0} \times \mathbf{B} + \rho \mathbf{g}, \tag{1.50}$$

where p, $\mathbf{B}$ and $\mathbf{j}$ now denote the *perturbations* in fluid pressure, magnetic field and current density about an equilibrium with magnetic field $\mathbf{B_0}$ and current density $\mathbf{j_0}$ (with $\mu \mathbf{j_0} = \mathrm{curl}\, \mathbf{B_0}$). If the equilibrium magnetic field is a uniform one, with $\mathbf{B_0}$ a constant vector, then the current density $\mathbf{j_0}$ in the equilibrium is zero.

1.5.2 Fourier Representation

In general, when the equilibrium is a uniform one, with constant density, pressure and magnetic field, then the linear equations describing the perturbations will have coefficients that are constants. This permits us to construct solutions with a sinusoidal or exponential behaviour. This is most conveniently done using a representation in terms of the complex exponential function. For an unbounded and uniform equilibrium it frequently proves convenient to consider a *plane wave* representation for the perturbations. By a plane wave representation we mean that at time t each perturbation may be expressed in the form $f(\omega t - \mathbf{k} \cdot \mathbf{r})$, where $\mathbf{r}$ denotes the position vector of the point (x, y, z) in a Cartesian coordinate system O, x, y, z. Here ω denotes the angular frequency of the perturbation, $\mathbf{k} = (k_x, k_y, k_z)$ is the wave vector, and the function f describes the shape or profile of the disturbance. The function

$$\mathcal{P} \equiv \omega t - \mathbf{k} \cdot \mathbf{r} \tag{1.51}$$

is known as the *phase*, and the equation $\mathcal{P} =$ constant describes a *plane* with normal vector $\mathbf{k}$. The phase plane moves with a speed c, where

$$c = \frac{\omega}{k} \tag{1.52}$$

with $k = |\mathbf{k}| = (k_x^2 + k_y^2 + k_z^2)^{\frac{1}{2}}$ denoting the magnitude of the wave vector. The disturbance moves in the direction of the vector $\mathbf{k}$. Accordingly, the plane wave moves with a velocity

$$\mathbf{c} = \frac{\omega}{k} \mathbf{e_k} = c\, \mathbf{e_k}, \tag{1.53}$$

where $\mathbf{e_k}$ denotes a unit vector in the direction of propagation, $\mathbf{k}$. The vector $\mathbf{c}$ is referred to as the *phase velocity* of the disturbance, with c being the *phase speed*.

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It is convenient to use a complex variable representation of a perturbation, such as the fluid motion $\mathbf{u}$ and the perturbed plasma density ρ, by writing

$$\mathbf{u}(x, y, z, t) = \mathbf{u_1} \exp i(\omega t - \mathbf{k} \cdot \mathbf{r}), \qquad \rho(x, y, z, t) = \rho_1 \exp i(\omega t - \mathbf{k} \cdot \mathbf{r}), \tag{1.54}$$

where $i^2 = -1$. Here $\mathbf{u_1}$ is a complex constant vector (i.e., a vector with components that are complex constants) and ρ_1 is a complex constant. The modulus of these quantities, $|\mathbf{u_1}|$ and $|\rho_1|$, gives information about the *amplitude* of the motion $\mathbf{u}$ and the associated density variations ρ. Other perturbations (such as the pressure or magnetic field) may be expressed in a similar complex exponential form. The actual physical perturbations may be obtained by taking the real parts of the above complex representations, once the relationships between the various complex constants and vectors, such as ρ_1 and $\mathbf{u_1}$, are determined.

The value of using a complex exponential representation is that it converts the various differential operators arising in the linear equations for the perturbations into simple *algebraic* scalar and vector forms. Thus, the relations

$$\frac{\partial}{\partial t} \equiv i\omega, \qquad \frac{\partial}{\partial z} \equiv -ik_z, \qquad \nabla^2 \equiv -k^2, \tag{1.55}$$

$$\text{div} \equiv -i\mathbf{k}\cdot, \qquad \text{grad} \equiv -i\mathbf{k}, \qquad \text{curl} \equiv -i\mathbf{k}\times \tag{1.56}$$

provide algebraic scalar and vector representations (involving the scalar product $(\cdot)$ for div and the vector product $(\times)$ for curl) of the partial differential operators, converting such operators into relatively simple forms. Thus, for example, $\partial\rho/\partial t = i\omega\rho$ and $\text{div}\, \mathbf{u} = -i\mathbf{k} \cdot \mathbf{u}$ and so the linearized equation of continuity (1.49) becomes

$$\omega\rho = \rho_0 \mathbf{k} \cdot \mathbf{u} = \rho_0(k_x u_x + k_y u_y + k_z u_z), \tag{1.57}$$

where $\mathbf{u} = (u_x, u_y, u_z)$ and we have cancelled a common factor i. Thus, the density perturbation is related to the scalar product of $\mathbf{k}$ with $\mathbf{u}$. Similar algebraic relationships arise from the other partial differential equations.

1.5.3 Dispersion Relations, Phase Speed and Group Velocity

The system of linear algebraic equations that arise from a Fourier representation of the perturbations leads, in general, to a *dispersion relation*. This is a relationship between the frequency ω and the wave vector $\mathbf{k}$. When the phase speed $c = \omega/k$ is independent of k, we say that the system is *non-dispersive*: all waves travel with the same speed c whatever their wavelength, $2\pi/k$. However, in magnetohydrodynamics it turns out (see Chapter 2) that while the waves are non-dispersive the phase speed c nonetheless varies with *direction*: a wave propagating in the direction of the equilibrium field $\mathbf{B_0}$, say, has a different speed from one propagating at an angle to the field. We say the medium is *anisotropic*. This anisotropy arises simply because the presence of an applied magnetic field in the equilibrium state introduces *a preferred direction* in the magnetohydrodynamic system, and this directionality is reflected in the behaviour of the phase speed c and phase velocity $\mathbf{c}$.

In addition to the phase velocity $\mathbf{c}$, there is particular interest in the *group velocity* $\mathbf{c}_g$ defined as (see, for example, Whitham 1974; Lighthill 1978)

$$\mathbf{c}_g \equiv \frac{\partial \omega}{\partial \mathbf{k}} \equiv \left(\frac{\partial \omega}{\partial k_x}, \frac{\partial \omega}{\partial k_y}, \frac{\partial \omega}{\partial k_z} \right). \tag{1.58}$$

Thus the group velocity is the gradient in $\mathbf{k}$-space of $\omega = \omega(\mathbf{k})$ and may be determined once the dispersion relation $\omega = \omega(\mathbf{k})$ is known. In general, $\mathbf{c}_g$ is different from $\mathbf{c}$, in both magnitude and direction, though in some systems – most notably for acoustic waves – the two speeds are equal. Since $\omega = kc$, we may write

$$\mathbf{c}_g = \frac{\partial}{\partial \mathbf{k}}(kc) = \frac{\partial k}{\partial \mathbf{k}} c + k \frac{\partial c}{\partial \mathbf{k}} = \mathbf{c} + k \frac{\partial c}{\partial \mathbf{k}}, \tag{1.59}$$

on noting that $\partial k / \partial k_x = k_x / k$, etc., and so $\partial k / \partial \mathbf{k} = \mathbf{k}/k$.

Now, as noted earlier, magnetohydrodynamic waves are not dispersive but are anisotropic, with a phase speed that is independent of k but a function of the angle of propagation of the wave relative to the direction of the applied magnetic field. Introduce the angle θ that the direction of propagation of a plane wave makes with a fixed direction, taken to be that of the applied magnetic field, and with it a unit vector $\mathbf{e}_\theta$ that is perpendicular to the wave vector $\mathbf{k}$ and points in the direction of increasing θ (see Figure 1.1). Magnetohydrodynamic waves are such that $c = c(\theta)$. It is convenient to align our Cartesian coordinate system so that the plane wave vector $\mathbf{k}$ lies entirely in the xz-plane. Then we may write $\mathbf{k} = (k \sin\theta, 0, k\cos\theta)$ and $\mathbf{e}_\theta = (\cos\theta, 0, -\sin\theta)$, and so

$$\frac{\partial c}{\partial \mathbf{k}} = \frac{1}{k} \frac{dc}{d\theta} \mathbf{e}_\theta. \tag{1.60}$$

Hence

$$\mathbf{c}_g = \mathbf{c} + \frac{dc}{d\theta} \mathbf{e}_\theta. \tag{1.61}$$

Thus the group velocity is the vector sum of the phase velocity and a component perpendicular to the direction of propagation. The component of the group velocity in the direction of propagation is simply the phase speed: $\mathbf{c}_g \cdot \mathbf{e}_\mathbf{k} = c$. The component perpendicular to the direction of propagation may be positive, zero or negative; all three cases arise in magnetohydrodynamics and are discussed in Chapter 2.

The difference between phase and group velocities is perhaps most readily illustrated for the special case of one-dimensional propagation of a dispersive wave, where $\mathbf{k} = (0, 0, k)$ with $k_z = k$. Consider a travelling wave of the form $A_0 \cos(\omega t - kz)$. Then the sum of two such travelling waves with the *same amplitude* A_0 but different frequencies ω_1, ω_2 and wavenumbers k_1, k_2 is

$$A_0 \cos(\omega_1 t - k_1 z) + A_0 \cos(\omega_2 t - k_2 z) = 2\mathcal{A}(z,t) \cos\left[\tfrac{1}{2}(\omega_2 + \omega_1)t - \tfrac{1}{2}(k_2 + k_1)z \right], \tag{1.62}$$

where

$$\mathcal{A}(z,t) = A_0 \cos\left[\tfrac{1}{2}(\omega_2 - \omega_1)t - \tfrac{1}{2}(k_2 - k_1)z \right]. \tag{1.63}$$

This simple addition underlies the well-known acoustic phenomenon of *beats*.

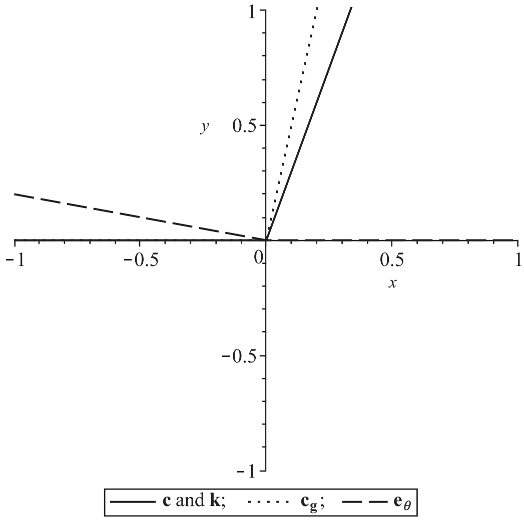

Figure 1.1 The geometry of the phase and group velocities of a wave propagating at an angle θ to a fixed direction. The phase velocity **c** here makes an angle θ measured anti-clockwise from the horizontal axis (representing the direction of the applied magnetic field); $\theta = 0$ corresponds to the horizontal axis (and alignment with the applied magnetic field). The group velocity $\mathbf{c}_{\mathrm{g}}$ in general makes a different angle, ϕ, with the applied magnetic field. The unit vector $\mathbf{e}_\theta$, which acts perpendicular to **c**, is also indicated.

Now suppose that k_1 and k_2 are almost equal: $k_1 \approx k_2 \approx k$. Then we see that the addition of two cosine waves of equal amplitude gives a cosine wave travel with almost the same speed,

$$\frac{\omega_2 + \omega_1}{k_2 + k_1} \approx \frac{\omega}{k}, \tag{1.64}$$

as the original pair of waves. But the effective *amplitude* $\mathcal{A}(z, t)$ of the resulting disturbance is a *slowly varying* function of both time and space, and in particular is of much *greater wavelength* than the original pair. The slowly varying, long wavelength, amplitude moves with the speed

$$\frac{\omega_2 - \omega_1}{k_2 - k_1}. \tag{1.65}$$

In the limit $k_1 \to k, k_2 \to k, \omega_1 \to \omega, \omega_2 \to \omega$, this becomes $d\omega/dk$. Thus the *wave packet* $\mathcal{A}(z, t)$ moves with the group speed $c_{\mathrm{g}} = d\omega/dk$.

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