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ON SEQUENCES $\{\xi t_n(\text{mod } 1)\}$

BY
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J. Mycielski has conveyed to me the following problem by P. Erdös. Let $\{t_n\}$ be a sequence of natural numbers such that

$$\frac{t_{n+1}}{t_n} \geq \alpha > 1 \quad \text{for } n = 1, 2, \ldots, .$$

(a) Does there exist an irrational number η such that the sequence $\{t_n\eta(\text{mod } 1)\}$ is not dense in $[0, 1)$?

(b) Does there exist a real number ξ such that 0 (and if possible also 1) are not limit points of the sequence $\{t_n\xi(\text{mod } 1)\}$?

P. Erdös and S. J. Taylor have proved in [1] that the set of numbers ρ such that $\{t_n\rho(\text{mod } 1)\}$ has not the equipartition property in the interval [0, 1] has Hausdorff dimension 1.

In this note we use the methods of [2] to give a partial answer to question (*b*). Namely we shall prove the following

THEOREM. *Let* $\{t_n\}$ *be a sequence of positive* (*not necessarily natural numbers*) *such that*

$$q_n = \frac{t_{n+1}}{t_n} \geq (5)^{1/3} \quad \textit{for } n = 1, 2, \ldots, .$$

Then there exist positive numbers ξ *and* β *such that*

$$t_n\xi(\text{mod } 1) \in [\beta, 1-\beta] \quad \textit{for } n = 1, 2, \ldots, . \tag{1}$$

Let us note that it is sufficient to prove the theorem under an additional restriction that $q_n \leq 3$ for all natural n. In fact, if for some fixed n, $q_n = (t_{n+1}/t_n) > 3$, then assuming that s is a natural number such that

$$3^{s/2}t_n < t_{n+1} \leq 3^{(s+1)/2}t_n,$$

we shall insert between t_n and t_{n+1} new terms

$$3^{1/2}t_n, 3t_n, \ldots, 3^{(s-1)/2}t_n.$$

The new extended sequence $\{t_n'\}$ will still satisfy (1) (since $\sqrt{3} > 1.73$) but we shall have $q_n' \leq 3$ for all natural n. Obviously, if the assertion of the theorem holds for

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some sequence $\{t'_n\}$ it is also true for any subsequence $\{t_n\}$ of $\{t'_n\}$. In other words, to prove the theorem it is sufficient to show that:

If $\{t_n\}$ is a sequence of positive numbers such that

$$(5)^{1/3} \leq q_n = \frac{t_{n+1}}{t_n} \leq 3 \quad \text{for } n = 1, 2, \ldots, \tag{2}$$

then there exist positive numbers ξ and β such that

$$t_n\xi(\text{mod } 1) \in [\beta, 1-\beta] \quad \text{for } n = 1, 2, \ldots.$$

The proof of the theorem will be based on several lemmas. We shall refer often to the following conditions:

Given intervals $\Delta_n=[a_n, b_n]$ and $\Delta_{n+1}=[a_{n+1}, b_{n+1}]$, the interval Δ_{n+1} is said to satisfy condition (A_{n+1}) if

$$q_n a_n \leq a_{n+1} < b_{n+1} \leq q_n b_n. \tag{A_{n+1}}$$

An interval $\Delta=[x, y]$ satisfies condition (B) if

$$[x, y](\text{mod } 1) \subset [0, 1]. \tag{B}$$

In other words an interval Δ satisfies B iff no integer is an interior point of Δ.

An interval $\Delta=[x, y]$ satisfies condition (C) if

$$d = y-x = 1. \tag{C}$$

Given a sequence $\{b_n\}$ we shall say that b_k satisfies condition $(D_{k,m}(\gamma))$, $(\gamma>0)$ if there exists a natural number $m(k)$ (depending on k) such that

$$b_{k+m} \leq \frac{t_{k+m}}{t_k}(b_k-\gamma). \tag{$D_{k,m}(\gamma)$}$$

Definition. Given a sequence of intervals $\{\Delta_n\}$, $\Delta_n=[a_n, b_n]$, $n=1, 2, \ldots$, an interval Δ_{k+m} is said to be a proper mth successor of the interval Δ_k with $\gamma=\gamma_0$, $(\gamma_0>0)$ if

(i) Δ_{k+p} satisfy conditions (A_{n+p}) and (B) for $p=1, 2, \ldots, m$,
(ii) $d_{k+m}=1$,
(iii) b_k satisfies $(D_{k,m}(\gamma_0))$.

We now start proving the theorem. In the sequel, $\{t_n\}$ denotes a sequence of positive numbers satisfying condition (2).

Lemma 1. *Assume that*

$$b_{k+m} \leq b_k\frac{t_{k+m}}{t_k}-10^{-4},$$

then b_k *satisfies* $(D_{k,m}(3^{-m}\times 10^{-4}))$.

Proof. Since $q_n \leq 3$ for all natural n, we obtain

$$b_{k+m} \leq b_k \frac{t_{k+m}}{t_k} - 10^{-4} = b_k q_k \cdots q_{k+m-1} - 10^{-4}$$

$$= \frac{t_{k+m}}{t_k}\left(b_k - \frac{10^{-4}}{q_k \cdots q_{k+m-1}}\right)$$

$$\leq \frac{t_{k+m}}{t_k}(b_k - 3^{-m} \times 10^{-4}).$$

LEMMA 2. *Assume that* $\Delta_k = [a_k, b_k]$ *satisfies conditions* (B) *and* (C) *but there exists no proper first successor of* Δ_k *with* $\gamma = 3^{-1} \times 10^{-4}$, *then there is an integer* N_1 (*see* Fig. 1; *dots indicate integers*) *such that*

2(i) $$N_1 - 1 < a_k q_k < N_1 < b_k q_k < N_1 + 1 + 10^{-4},$$

2(ii) $$z_{k+1} - a_k q_k > q_k - 10^{-4},$$

where $$z_{k+1} = \min(b_k q_k, N_1 + 1).$$

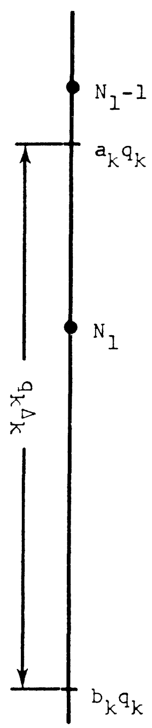

Fig. 1

Proof. Since $d_k=b_k-a_k=1$ and $q_k\geq (5)^{1/3}>1$, we have

$$b_kq_k-a_kq_k = q_k > 1 \tag{3}$$

and so, the interval $q_k\Delta_k=[a_kq_k, b_kq_k]$ contains at least one integer. Denote by N_1 the smallest integer belonging to the interval $q_k\Delta_k$. Then we have

$$N_1-1 < a_kq_k \leq N_1 < b_kq_k.$$

Assuming that $b_kq_k\geq N_1+1+10^{-4}$ we have

$$\Delta_{k+1} = [N_1, N_1+1] \subset [a_kq_k, b_kq_k].$$

Thus putting $a_{k+1}=N_1$, $b_{k+1}=N_1+1$, we obtain an interval $[a_{k+1}, b_{k+1}]$ satisfying (A_{k+1}), B and C, and since $b_{k+1}=N_1+1\leq b_kq_k-10^{-4}$, by Lemma 1, b_k satisfies the condition $D_{k,1}(3^{-1}\times 10^{-4})$. This means that Δ_{k+1} is a proper first successor of Δ_k with $\gamma=3^{-1}\times 10^{-4}$, contrary to the assumptions of Lemma 2. Consequently,

$$b_kq_k < N_1+1+10^{-4}. \tag{4}$$

Now if $a_kq_k=N_1$ then $b_kq_k-a_kq_k=b_kq_k-N_1<1+10^{-4}<q_k$, contrary to (3). Hence the inequality 2(i) is proved completely.

Now, if $b_kq_k\leq N_1+1$, then $z_{k+1}=b_kq_k$ and therefore, by (3)

$$z_{k+1}-a_kq_k = q_k > q_k-10^{-4}.$$

If, on the other hand $b_kq_k>N_1+1$, then $z_{k+1}=N_1+1$ and, by (4) and (3), we obtain

$$z_{k+1}-a_kq_k = N_1+1-a_kq_k > b_kq_k-a_kq_k-10^{-4} = q_k-10^{-4}$$

as required in 2(ii).

LEMMA 3. *Assume that in addition to the conditions of Lemma* 2, Δ_k *has also no proper second successor with* $\gamma=3^{-2}\times 10^{-4}$, *then there exists an integer* N_2 (*see* Fig. 2) *such that*

3(i) $N_2-1 < a_kq_kq_{k+1} < N_2 < N_1q_{k+1} < N_2+1 < z_{k+1}q_{k+1} < N_2+2+10^{-4}$,

3(ii) $$z_{k+2}-a_kq_kq_{k+1} > q_kq_{k+1}-4\times 10^{-4},$$

where

$$z_{k+2} = \min(z_{k+1}q_{k+1}, N_2+2).$$

Proof. As in the proof of Lemma 2, denote by N_2 the smallest integer such that $N_2\geq a_kq_kq_{k+1}$. The interval $[a_kq_kq_{k+1}, N_1q_{k+1}]$ cannot contain two integers since they would be the endpoints of a proper second successor of Δ_k. It is easy to show that the length of this interval is greater than 1, and so N_2 cannot coincide with any of its endpoints. So,

$$N_2-1 < a_kq_kq_{k+1} < N_2 < N_1q_{k+1} < N_2+1.$$

Assuming that $z_{k+1}q_{k+1}<N_2+1$, we obtain

$$z_{k+1}q_{k+1}-a_kq_kq_{k+1} < 2,$$

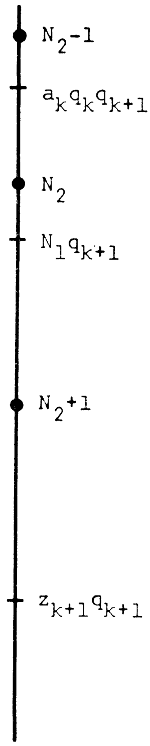

Fig. 2

but, by 2(ii) and (3), we have

$$z_{k+1}q_{k+1}-a_kq_kq_{k+1} > (q_k-10^{-4})q_{k+1} > 2.8.$$

Again, as in Lemma 2, the inequality $z_{k+1}q_{k+1} \geq N_2+2+10^{-4}$ implies the existence of a second proper successor with $\gamma=3^{-2}\times 10^{-4}$. This remark concludes the proof of inequality 3(i).

Now, if $z_{k+1}q_{k+1} \leq N_2+2$ then, $z_{k+2}=z_{k+1}q_{k+1}$ and thus, by 2(ii) and 2,

$$(5) \quad z_{k+2}-a_kq_kq_{k+1} = (z_{k+1}-a_kq_k)q_{k+1} > (q_k-10^{-4})q_{k+1} \geq q_kq_{k+1}-3\times 10^{-4}.$$

If on the other hand, $z_{k+1}q_{k+1} > N_2+2$, then $z_{k+2}=N_2+2$ and by (2), 2(ii), 3(i) and (5)

$$z_{k+2}-a_kq_kq_{k+1} = N_2+2-a_kq_kq_{k+1} > z_{k+1}q_{k+1}-10^{-4}-a_kq_kq_{k+1} > q_kq_{k+1}-4\times 10^{-4},$$

LEMMA 4. *Assume that Δ_k satisfies conditions of Lemma* 3 *and that at least one of the ratios q_k, q_{k+1}, q_{k+2} is not less than* 1.73.

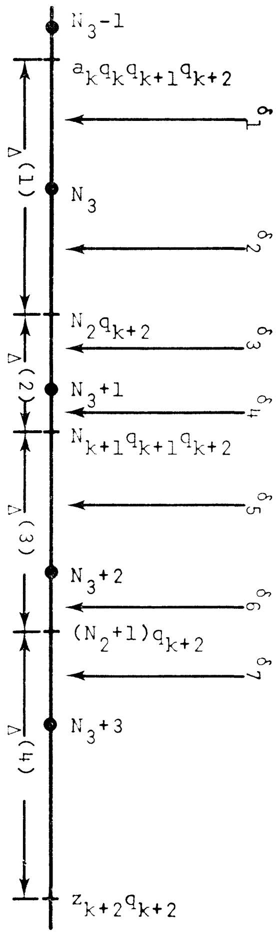

Fig. 3

Denote (*see* Figs. 2 *and* 3)

$$\Delta^{(1)} = q_{k+2}[a_k q_k q_{k+1}, N_2],$$
$$\Delta^{(2)} = q_{k+2}[N_2, N_1 q_{k+1}],$$
$$\Delta^{(3)} = q_{k+2}[N_1 q_{k+1}, N_2+1],$$
$$\Delta^{(4)} = q_{k+2}[N_2+1, z_{k+2}].$$

At least one of the intervals $\Delta^{(i)}$ ($i=1, 2, 3, 4$) *contains a proper third successor of* Δ_k *with* $\gamma=3^{-3}\times 10^{-4}$.

Proof. Assume that the assertion of Lemma 4 is not true. By arguments similar to those used in the proof of Lemmas 2 and 3, we infer that (see fig. 3) the intervals $\Delta^{(1)}$, $\Delta^{(2)}$, $\Delta^{(3)}$ contain not more than one integer each. The interval $\Delta^{(4)}$ contains not more than two integers, and in case $\Delta^{(4)}$ contains two integers and M is the larger of them, $z_{k+2}q_{k+2}<M-10^{-4}$. Consequently, the interval $\Delta=[a_kq_kq_{k+1}q_{k+2}, z_{k+2}q_{k+2}]$ contains not more than five integers. In case Δ contains four integers only, its length $l<5$. In the remaining case, the right endpoint of Δ exceeds the largest integer $M\in\Delta$ by less than 10^{-4}. So, in this case $l<5+10^{-4}$. In other words, the assumption that the assertion of Lemma 4 is incorrect implies that

$$l = z_{k+2}q_{k+2}-a_kq_kq_{k+1}q_{k+2} < 5+10^{-4}. \tag{6}$$

On the other hand, by 3(ii) and (2) we have

$$\begin{aligned} l &= (z_{k+2}-a_kq_kq_{k+1})q_{k+2} > (q_kq_{k+1}-4\times 10^{-4})q_{k+2} \\ &\geq q_kq_{k+1}q_{k+2}-1\cdot 2\times 10^{-3}. \end{aligned} \tag{7}$$

Since $q_n \geq (5)^{1/3}$ for each natural n and, by assumption, at least one of q_k, q_{k+1}, q_{k+2} is not less than 1.73, we obtain

$$q_kq_{k+1}q_{k+2} \geq (25)^{1/3}\times 1.73 > 5.01.$$

Thus (7) implies that

$$l > 5+10\times 10^{-3}-1.2\times 10^{-3} > 5+10^{-4},$$

which contradicts inequality (6). This concludes the proof of Lemma 4.

LEMMA 5. *Assume that Δ_k satisfies conditions of Lemma* 3 *and none of the intervals* $q_k\Delta_k$ (*see* Fig. 1), $[N_1q_{k+1}, z_{k+1}q_{k+1}]$ (*see* Fig. 2), $\Delta^{(4)}$ (*see* Fig. 3) *contains more than one integer. Then there is a third proper successor of Δ_k with $\gamma=3^{-3}\times 10^{-4}$.*

Proof. If $q_k\Delta_k$ contains one integer only, this means that $b_kq_k<N_1+1$ (see Fig. 1) and so $z_{k+1}=b_kq_k$. Similarly, conditions of Lemma 5 imply that

$$z_{k+2} = z_{k+1}q_{k+1} = b_kq_kq_{k+1}. \tag{8}$$

Assume that Δ_k has no third proper successor. It has been shown in Lemma 4 that in this case if $\Delta^{(4)}$ contains one integer only then

$$l = z_{k+2}q_{k+2}-a_kq_kq_{k+1}q_{k+2} < 5.$$

In view of (8), we obtain

$$(b_k-a_k)q_kq_{k+1}q_{k+2} < 5.$$

But $b_k-a_k=1$, so

$$q_kq_{k+1}q_{k+2} < 5,$$

which is impossible because $q_n\geq (5)^{1/3}$ for all natural n. The contradiction proves Lemma 5.

LEMMA 6. *Assume that Δ_k satisfies conditions of Lemma* 3 *and there is no third proper successor of Δ_k with $\gamma=3^{-3}\times 10^{-4}$. Then each of the intervals $\Delta^{(1)}$, $\Delta^{(2)}$, $\Delta^{(3)}$ defined in Lemma* 4 *contains exactly one integer.*

Proof. As we know already from the proof of Lemma 4, the conditions of Lemma 6 imply that each of the above intervals contains not more than one integer. Assuming that at least one of them does not contain an integer, we would obtain that

$$l = z_{k+2}q_{k+2} - a_k q_k q_{k+1} q_{k+2} < 4+10^{-4}.$$

On the other hand, since $q_n \geq (5)^{1/3}$ for each natural n, by (7), we obtain

$$l > q_k q_{k+1} q_{k+2} - 1.2\times 10^{-3} \geq 5 - 1.2\times 10^{-3} > 4+10^{-4}.$$

LEMMA 7. *Assume that Δ_k satisfies conditions of Lemma 6 and $q_{k+3} < 1.73$. Then Δ_k contains a fourth proper successor with $\gamma = 3^{-4}\times 10^{-4}$.*

Proof. Denote by $z_{k+3} = \min(z_{k+2}q_{k+2}, N_3+3)$ (see Fig. 3). Let us note that according to Lemma 6 the endpoints of intervals $\Delta^{(1)}$, $\Delta^{(2)}$, $\Delta^{(3)}$ and the corresponding integers are distributed as shown on Fig. 3. Moreover $N_3+3 < z_{k+3}$. Assume that none of the intervals $q_{k+3}\delta_7 = q_{k+3}[(N_2+1)q_{k+2}, N_3+3]$ and $q_{k+3}\delta_8 = q_{k+3}[N_3+3, z_{k+3}]$ contains a proper fourth successor of Δ_k (see Fig. 3 and 4).

In this case $\delta_7 q_{k+3}$ contains at most one integer, $\delta_8 q_{k+3}$ contains at most two integers. In case $\delta_8 q_{k+3}$ contains two integers and M' is the larger one, by Lemma 1,

$$z_{k+3}q_{k+3} < M'+10^{-4}.$$

Denote by N the largest integer satisfying the inequality

$$N < (N_2+1)q_{k+2}q_{k+3}. \tag{9}$$

It follows (see Fig. 4) that

$$z_{k+3}q_{k+3} - N < 3+10^{-4}. \tag{10}$$

By Lemma 4, $q_{k+2} < 1.73$. By assumption of Lemma 7, $q_{k+3} < 1.73$. Thus $q_{k+2}q_{k+3} < 3$. Consequently,

$$(N_2+1)q_{k+2}q_{k+3} - N_2 q_{k+2}q_{k+3} = q_{k+2}q_{k+3} < 3. \tag{11}$$

Inequalities (9) and (11) imply that

$$N-3 < N_2 q_{k+2}q_{k+3}, \tag{12}$$

and from (10) we obtain

$$z_{k+3}q_{k+3} - (N-3) < 6+10^{-4}. \tag{13}$$

Assume now that each of the intervals $\delta_1 q_{k+3}$ and $\delta_2 q_{k+3}$ (see Fig. 4) contains not more than one integer. In view of (12), it follows that there is no more than one integer between $a_k q_k \cdots q_{k+3}$ and $N-3$. Thus

$$N-3-a_k q_k \cdots q_{k+3} < 2. \tag{14}$$

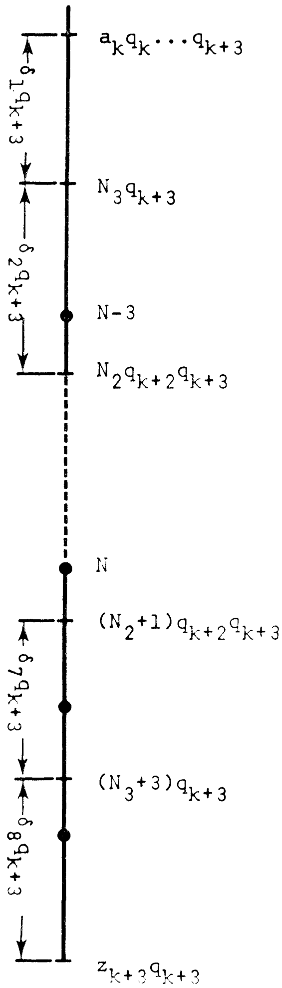

Fig. 4 (dots denote integers).

From (13) and (14) we obtain

$$z_{k+3}q_{k+3} - a_kq_k \cdots q_{k+3} < 8 + 10^{-4}. \tag{15}$$

Similarly as it has been done before we can prove that

$$z_{k+3}q_{k+3} - a_kq_k \cdots q_{k+3} > q_k \cdots q_{k+3} - 4\times 10^{-3} \geq 5(5)^{1/3} - 4\times 10^{-3} > 8.5. \tag{16}$$

Since inequalities (15) and (16) are incompatible, at least one of the intervals $\delta_1 q_{k+3}$ and $\delta_2 q_{k+3}$ contains two integers, thus yielding a proper fourth successor for Δ_k.

This completes the proof of Lemma 7.

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LEMMA 8. *Assume that Δ_k satisfies conditions of Lemma* 6 *and* $q_{k+3}\geq 1.73$. *If there is no* 4th *proper successor of* Δ_k *with* $\gamma=3^{-4}\times 10^{-4}$ *and* 5th *proper successor of* Δ_k *with* $\gamma=3^{-5}\times 10^{-4}$ *then there is a* 6th *proper successor of* Δ_k *with* $\gamma=3^{-6}\times 10^{-4}$.

Proof. Since Δ_k has no first, second or third proper successors, by Lemma 5, at least one of the intervals $q_k\Delta_k$ (see Fig. 1), $[N_1q_{k+1}, z_{k+1}q_{k+1}]$ (see Fig. 2) or Δ_k (see Fig. 3) contains two integers. We shall assume that $\Delta^{(4)}$ contains two integers: N_3+3 and N_3+4 (see Fig. 3). Remaining cases can be considered in a similar manner. Put

$$a_{k+1} = N_1, \qquad b_{k+1} = z_{k+1} \qquad \text{(Notations of Lemma 2 and Fig. 1)},$$

$$a_{k+2} = N_2+1, \qquad b_{k+2} = z_{k+2} \qquad \text{(Notations of Lemma 3 and Fig. 2)},$$

$$a_{k+3} = N_3+3, \qquad b_{k+3} = N_3+4 \qquad \text{(Notations of Lemma 4 and Fig. 3)}.$$

The intervals $\Delta_{k+i}=[a_{k+i}, b_{k+i}]$ satisfy conditions (A_{k+i}) $i=1, 2, 3$, (B) and $d_{k+3}=b_{k+3}-a_{k+3}=1$. Since the interval Δ_{k+3} satisfies conditions (B) and (C) and $q_{k+3}\geq 1.73$, it has either first or second proper successor or, by Lemma 4, Δ_{k+3} has a third proper successor with $\gamma=3^{-3}\times 10^{-4}$. Consider for example the case when Δ_{k+3} has a third proper successor. This means there exist intervals Δ_{k+4}, Δ_{k+5} satisfying (A_{k+4}), (A_{k+5}) and (B) and an interval $A_{k+6}=[a_{k+6}, b_{k+6}]$ satisfying (A_{k+6}), (B), (C) and $(D_{k+3,k+6}(3^{-3}\times 10^{-4}))$. The last condition means that

$$b_{k+6} \leq \frac{t_{k+6}}{t_{k+3}}(b_{k+3}-3^{-3}\times 10^{-4}).$$

But $b_{k+3}\leq b_{k+2}q_{k+2}\leq b_{k+1}q_{k+1}q_{k+2}\leq b_k(t_{k+3}/t_k)$. So, we obtain

$$\begin{aligned} b_{k+6} &\leq \frac{t_{k+6}}{t_{k+3}}\left(\frac{t_{k+3}}{t_k}\,b_k-3^{-3}\times 10^{-4}\right) \\ &= \frac{t_{k+6}}{t_k}\left(b_k-\frac{3^{-3}\times 10^{-4}}{q_kq_{k+1}q_{k+2}}\right) \\ &\leq \frac{t_{k+6}}{t_k}(b_k-3^{-6}\times 10^{-4}), \end{aligned}$$

thus Δ_{k+6} is a proper successor of Δ_k with $\gamma=3^{-6}\times 10^{-4}$.

We have exhausted all possibilities, so we may state the following

COROLLARY 1. *If an interval Δ_k satisfies conditions* (B) *and* (C), *then there is a sequence $\Delta_{k+1}, \ldots, \Delta_{k+m}$ $(m\leq 6)$ of at most six intervals such that the last one is a proper successor of Δ_k with $\gamma=3^{-6}\times 10^{-4}$. (We are using here the following property of condition $(D_{k,m}(\gamma))$: if $0<\gamma_1<\gamma_2$ and $(D_{k,m}(\gamma_2))$ holds then $(D_{k,m}(\gamma_1))$ is also satisfied).*

Analogously to the condition $(D_{k,m}(\gamma))$ we may introduce condition $(L_{k,m}(\gamma))$ as follows: Given a sequence $\{a_n\}$ we shall say that a_k satisfies condition $(L_{k,m}(\gamma))$

($\gamma>0$) if there exists a natural number $m(k)$ (depending on k) such that

$$(L_{k,m}(\gamma)) \qquad a_{k+m} \geq \frac{t_{k+m}}{t_k}(a_k+\gamma).$$

Also similarly, we define a concept of a left proper successor:

Definition. Given a sequence of intervals $\{\Delta_n\}$, an interval Δ_{k+m} is said to be a left proper successor of Δ_k with $\gamma=\gamma_0$, ($\gamma_0>0$) if

(i) Δ_{k+p} satisfy conditions (A_{n+p}) and (B) for $p=1, 2, \ldots, m$,
(ii) $d_{k+m}=1$,
(iii) a_k satisfies $(L_{k,m}(\gamma_0))$.

In a similar manner we may prove that given an interval Δ_k satisfying (B) and (C), we may construct a sequence of at most six intervals $\Delta_{k+1}, \ldots, \Delta_{k+m}$ $(m\leq 6)$ such that the last one is a left proper successor of Δ_k with $\gamma=3^{-6}\times 10^{-4}$.

We are now in a position to prove the Theorem stated in the paper.

Proof of Theorem. Take $a_1=1$, $b_1=2$. Construct intervals $\Delta_2, \ldots, \Delta_k$ $(k\leq 7)$ such that Δ_k is a (right) proper successor of $\Delta_1=[a_1, b_1]$ with $\gamma_0=3^{-6}\times 10^{-4}$. Then choose intervals $\Delta_{k+1}, \ldots, \Delta_{k+m}$ $(m\leq 6)$ such that Δ_{k+m} is a left proper successor of Δ_k, with $\gamma=\gamma_0$. Then we are getting intervals $\Delta_{k+m+1}, \ldots, \Delta_{k+m+p}$, the last one being a (right) proper successor of Δ_{k+m} etc.

Denote $\Delta_n=[a_n, b_n]$ for $n=1, 2, \ldots$. The intervals satisfy condition (B) for all natural n, conditions (A_n) for $n=2, \ldots,$. Moreover, it is easy to show that if Δ_k is a proper right (left) successor of Δ_{j+m} $(k>j+m, m>0)$ with $\gamma=\gamma_0$ then Δ_k is also a proper right (left) successor of Δ_j with $\gamma=3^{-m}\gamma_0$. It follows that for each of n, there exist m and p $(m, p\leq 17)$ such that conditions $(D_{n,m}(\beta))$ and $(L_{n,p}(\beta))$ are satisfied with $\beta=3^{-17}\times 10^{-4}$.

Consider the sequence of intervals $\{[a_n/t_n, b_n/t_n]\}$. Replacing in (A_{n+1}) q_n by t_{n+1}/t_n we obtain

$$a_n\frac{t_{n+1}}{t_n} \leq a_{n+1} < b_{n+1} \leq b_n\frac{t_{n+1}}{t_n},$$

or since $t_n>0$,

$$\frac{a_n}{t_n} \leq \frac{a_{n+1}}{t_{n+1}} < \frac{b_{n+1}}{t_{n+1}} \leq \frac{b_n}{t_n}.$$

Thus the sequence $\{[a_n/t_n, b_n/t_n]\}$ is a nested sequence of closed intervals. Consequently there exists a number ξ belonging to all intervals of this sequence. So,

$$\frac{a_n}{t_n} \leq \xi \leq \frac{b_n}{t_n} \quad \text{for } n=1, 2, \ldots.$$

Taking into account that for each n conditions $(L_{n,p}(\beta))$ and $(D_{n,m}(\beta))$ are satisfied, we obtain

$$(a_n+\beta)\frac{t_{n+p}}{t_n}\frac{1}{t_{n+p}} \leq \frac{a_{n+p}}{t_{n+m}} \leq \xi \leq \frac{b_{n+m}}{t_{n+m}} \leq \frac{1}{t_{n+m}}\frac{t_{n+m}}{t_n}(b_n-\beta),$$

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or

$$a_n+\beta \leq \xi t_n \leq b_n-\beta.$$

Since for each n, $[a_n, b_n]$ (mod 1) $\subset$ [0, 1] this means that

$$\xi t_n (\text{mod } 1) \in [\beta, 1-\beta]$$

as required in the Theorem.

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