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Unimodular Roots of Special Littlewood Polynomials

Idris David Mercer

Abstract. We call $\alpha(z) = a_0 + a_1 z + \cdots + a_{n-1}z^{n-1}$ a Littlewood polynomial if $a_j = \pm 1$ for all j. We call $\alpha(z)$ self-reciprocal if $\alpha(z) = z^{n-1}\alpha(1/z)$, and call $\alpha(z)$ skewsymmetric if $n = 2m + 1$ and $a_{m+j} = (-1)^j a_{m-j}$ for all j. It has been observed that Littlewood polynomials with particularly high minimum modulus on the unit circle in $\mathbb{C}$ tend to be skewsymmetric. In this paper, we prove that a skewsymmetric Littlewood polynomial cannot have any zeros on the unit circle, as well as providing a new proof of the known result that a self-reciprocal Littlewood polynomial must have a zero on the unit circle.

1 Introduction and Statement of Results

We let $\mathcal{L}_n$ denote the set of all 2^n polynomials of the form

$$\alpha(z) = a_0 + a_1 z + \cdots + a_{n-1}z^{n-1}, \quad \text{where } a_j = \pm 1 \text{ for all } j,$$

and we call such a polynomial a *Littlewood polynomial*. Erdős, Littlewood, and others have formulated conjectures about how "flat" a polynomial in $\mathcal{L}_n$ can be on the unit circle

$$\mathbb{S} := \{z \in \mathbb{C} : |z| = 1\}.$$

One conjecture [3, 6] says that for infinitely many n, there exists $\alpha \in \mathcal{L}_n$ that satisfies

$$K_1\sqrt{n} \le |\alpha(z)| \le K_2\sqrt{n} \quad \text{for all } z \in \mathbb{S} \tag{1}$$

where K_1 and K_2 are positive constants independent of n.

No one has shown the existence of an infinite family of Littlewood polynomials satisfying just the lower bound in (1). A family satisfying just the upper bound is given by the Rudin–Shapiro polynomials (see [1, Chapter 4]), which exist when n is a power of 2 and satisfy $|\alpha(z)| \le \sqrt{2}\cdot\sqrt{n}$ on $\mathbb{S}$. Moreover, Spencer [10] used probabilistic methods to show that for sufficiently large fixed K, the number of polynomials $\alpha \in \mathcal{L}_n$ satisfying $|\alpha(z)| \le K\sqrt{n}$ ($z \in \mathbb{S}$) is eventually bounded below by an exponential function of n (so there are many Littlewood polynomials whose modulus on $\mathbb{S}$ is at most $K\sqrt{n}$).

In this paper, we are interested in two special classes of Littlewood polynomials. We call $\alpha \in \mathcal{L}_n$ *self-reciprocal* if $\alpha(z) = z^{n-1}\alpha(1/z)$ (informally, if the coefficient sequence of α is palindromic). If $n = 2m+1$, we call $\alpha \in \mathcal{L}_n$ *skewsymmetric* if $a_{m+j} = (-1)^j a_{m-j}$ for $1 \le j \le m$. Littlewood [7] describes skewsymmetric polynomials as

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having "a central term and two stretches of $n/2$ terms on either side, the end one having the coefficients of the front one written backwards, but affected with signs alternately $-$ and $+$" (but note that his n is our $n-1$).

For modest values of n, an exhaustive search can find the polynomial in $\mathcal{L}_n$ having highest minimum modulus on $\mathbb{S}$. Computations in [9] show that for all odd n from 11 to 25, the polynomial in $\mathcal{L}_n$ with highest minimum modulus happens to be skewsymmetric, and satisfies $|\alpha(z)| \geq 0.6\sqrt{n}$ for $z \in \mathbb{S}$. So roughly speaking, if the modulus of $\alpha(z) \in \mathcal{L}_n$ does not become small for $z \in \mathbb{S}$, then there is a tendency for $\alpha(z)$ to be skewsymmetric. A kind of converse of this tendency is the main result of this paper.

Theorem 1 *A skewsymmetric Littlewood polynomial has no zeros on $\mathbb{S}$ (in other words, no roots of unit modulus, or unimodular roots).*

In contrast, it is known [2] that a self-reciprocal Littlewood polynomial must have a zero on $\mathbb{S}$. We give a new proof of this fact by deriving it, along with one of the results in [5], as corollaries of Theorem 2 below.

Theorem 2 *If $f\colon [0,\pi] \to \mathbb{R}$ is a function of the form*

$$f(\theta) = \cos(n\theta) + a_{n-1}\cos((n-1)\theta) + \cdots + a_1\cos(\theta) \tag{2}$$

where $a_1, \dots, a_{n-1}$ are real, then $f(\theta) = +1$ for some $\theta \in [0,\pi]$ and $f(\theta) = -1$ for some $\theta \in [0,\pi]$.

One can use basic properties of Chebyshev polynomials to prove the weaker version of Theorem 2 obtained by replacing "and" with "or", but such a technique does not seem to immediately yield Theorem 2 as written. However, Theorem 2 does have a quick proof using complex analysis discovered by the referee of this paper.

A trivial corollary of Theorem 2 is that a function of the form

$$f(\theta) = a_n\cos(n\theta) + a_{n-1}\cos((n-1)\theta) + \cdots + a_1\cos(\theta) \quad (a_j \in \mathbb{R}) \tag{3}$$

must attain both of the values a_n and $-a_n$. Expressions of the form (3) are called "zero-mean cosine polynomials" in [4], where the following question is answered: what is the largest β (necessarily $\beta < \pi$) such that there exists a nontrivial function of the form (3) that is nonnegative on $[0,\beta]$?

2 Relevant Facts About Chebyshev Polynomials

Before commencing our proofs of Theorems 1 and 2, we will find it useful to recall some relevant facts about cosine sums and Chebyshev polynomials.

Let θ be a real variable and let $c := \cos\theta$. For nonnegative integers n, each of the expressions

$$T_n := \cos(n\theta),$$
$$U_n := \frac{\sin((n+1)\theta)}{\sin\theta}$$

is a polynomial in c of degree n, called the *Chebyshev polynomials* of the first and second kind respectively. It is easy to check that

$$T_0 = 1, \quad U_0 = 1,$$
$$T_1 = c, \quad U_1 = 2c,$$

and one can use well-known trigonometric identities to show that for $n \geq 1$, we have

$$T_{n+1} = 2cT_n - T_{n-1},$$
$$U_{n+1} = 2cU_n - U_{n-1}.$$

Some facts about Chebyshev polynomials are easy to prove by induction. For instance:

- Both T_n and U_n are odd when n is odd, and even when n is even. (An *odd polynomial* is one containing only odd powers of the variable; an *even polynomial* is defined analogously.)
- For $n \geq 1$, the leading term of T_n is $2^{n-1}c^n$. For $n \geq 0$, the leading term of U_n is $2^n c^n$.

Since T_n has degree n, any polynomial in c of degree n can be written uniquely as a linear combination of $T_0, T_1, \dots, T_n$. In particular, it is natural to ask how to write U_n as a linear combination of $T_0, T_1, \dots, T_n$. The answer to this question is that

$$U_{2m} = T_0 + \sum_{k=1}^{m} 2T_{2k}, \tag{4}$$

$$U_{2m+1} = \sum_{k=0}^{m} 2T_{2k+1} \tag{5}$$

for all $m \geq 0$, which can be proved by induction and which appears as part of Problem 16 in Part VI of [8].

If we define the new variable $x := 2c$, then of course T_n and U_n can be regarded as polynomials in x. It is easy to show by induction that U_n and $2T_n$ have integer coefficients when regarded as polynomials in x, since we have

$$2T_{n+1} = 4cT_n - 2T_{n-1} = x \cdot 2T_n - 2T_{n-1},$$
$$U_{n+1} = 2cU_n - U_{n-1} = xU_n - U_{n-1}.$$

Thus $T_0, 2T_1, 2T_2, \dots$ and $U_0, U_1, U_2, \dots$ belong to $\mathbb{Z}[x]$, where as usual, $\mathbb{Z}[x]$ denotes the ring of polynomials in x with integer coefficients. Notice that when U_n is regarded as a polynomial in $\mathbb{Z}[x]$, its leading term is x^n. The same is true of $2T_n$ if $n \geq 1$. From now on, we will write U_n and $2T_n$ as $\overline{U_n}$ and $\overline{2T_n}$, respectively, if regarding them as polynomials in x, in order to avoid possible ambiguity.

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3 Proof of Theorem 1

We define $\mathbb{Z}_2 := \mathbb{Z}/(2\mathbb{Z})$ (the integers mod 2). Let φ be the natural homomorphism from $\mathbb{Z}$ to $\mathbb{Z}_2$, and let Φ be the homomorphism from $\mathbb{Z}[x]$ to $\mathbb{Z}_2[x]$ defined by

$$\Phi(a_0 + a_1x + \cdots + a_nx^n) = \varphi(a_0) + \varphi(a_1)x + \cdots + \varphi(a_n)x^n$$

(that is, Φ simply reduces all coefficients $\bmod 2$). The crucial ingredient in our proof of Theorem 1 is the following lemma.

Lemma 3 *Let n be a nonnegative integer, and let $A(x), B(x)$ be two polynomials in $\mathbb{Z}[x]$ satisfying*

- $\deg A(x) = n + 1$,
- $\deg B(x) = n$,
- $\Phi(A(x)) = \Phi(\overline{U_{n+1}})$,
- $\Phi(B(x)) = \Phi(\overline{U_n})$,
- *one of $A(x), B(x)$ is odd and the other is even.*

Then no complex number is a root of both $A(x)$ and $B(x)$.

Proof If $n = 0$, the hypotheses of the lemma say $B(x)$ is an odd nonzero constant and hence has no roots whatsoever. Assume the lemma is true for n, and let $A(x), B(x) \in \mathbb{Z}[x]$ satisfy

- $\deg A(x) = n + 2$,
- $\deg B(x) = n + 1$,
- $\Phi(A(x)) = \Phi(\overline{U_{n+2}})$,
- $\Phi(B(x)) = \Phi(\overline{U_{n+1}})$,
- one of $A(x), B(x)$ is odd and the other is even.

We wish to show $A(x)$ and $B(x)$ have no common roots. The hypotheses imply that the leading term of $A(x)$ is ax^{n+2} where a is odd, and that the leading term of $B(x)$ is bx^{n+1} where b is odd. Define $r := \operatorname{lcm}(a, b)$, $s := r/a$, and $t := r/b$, so r, s, t are odd integers. Then $C(x) := sA(x) - txB(x)$ is a linear combination of $A(x)$ and $B(x)$ where the x^{n+2} term has been "killed". Notice that $C(x)$ is odd if $A(x)$ is odd, and is even if $A(x)$ is even. Thus $\deg C(x) \le n$. Furthermore, any common root of $A(x)$ and $B(x)$ is also a root of $C(x)$. We now observe that

$$\begin{aligned}
\Phi(C(x)) &= \Phi(sA(x) - txB(x)) \\
&= \Phi(s)\Phi(A(x)) + \Phi(-tx)\Phi(B(x)) \\
&= \Phi(1)\Phi(\overline{U_{n+2}}) + \Phi(-x)\Phi(\overline{U_{n+1}}) \\
&= \Phi(\overline{U_{n+2}} - x\overline{U_{n+1}}) \\
&= \Phi(\overline{-U_n}) = \Phi(\overline{U_n}).
\end{aligned}$$

This means that $B(x)$ and $C(x)$ satisfy the induction hypothesis, so $B(x)$ and $C(x)$ have no common roots. This implies $A(x)$ and $B(x)$ have no common roots, as required. ∎

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Proof of Theorem 1 Let $\alpha(z)$ be a skewsymmetric Littlewood polynomial. Hence $\alpha(z)$ has even degree, so say

$$\alpha(z) = a_0 + a_1 z + \cdots + a_{2m} z^{2m} \quad (a_j = \pm 1)$$

where $a_{m+j} = (-1)^j a_{m-j}$ for $j \in \{1, 2, \ldots, m\}$. We then have

$$\begin{aligned}\frac{\alpha(z)}{z^m} &= a_0 \frac{1}{z^m} + \cdots + a_{m-1}\frac{1}{z} + a_m + a_{m+1} z + \cdots + a_{2m} z^m \\ &= a_m + \sum_{j=1}^{m} \Big(a_{m+j} z^j + a_{m-j} \frac{1}{z^j} \Big) \\ &= a_m + \sum_{j=1}^{m} a_{m+j} \Big(z^j + (-1)^j \frac{1}{z^j} \Big) =: f(z).\end{aligned}$$

Showing $\alpha(z)$ has no zeros on $\mathbb{S}$ is equivalent to showing $f(z)$ has no zeros on $\mathbb{S}$, which in turn is equivalent to showing $f(iz)$ has no zeros on $\mathbb{S}$. Observe that

$$\begin{aligned} f(iz) &= a_m + \sum_{j=1}^{m} a_{m+j} \Big((iz)^j + (-1)^j \frac{1}{(iz)^j} \Big) \\ &= a_m + \sum_{j=1}^{m} a_{m+j} \Big(i^j z^j + \Big(\frac{-1}{i}\Big)^j \frac{1}{z^j} \Big) \\ &= a_m + \sum_{j=1}^{m} a_{m+j} i^j \Big(z^j + \frac{1}{z^j} \Big) \\ &= a_m + \sum_{j=1}^{m} a_{m+j} i^j \cdot 2\cos(j\theta) \quad (\text{where } z = e^{i\theta}).\end{aligned}$$

To show $f(iz)$ is never 0 on $\mathbb{S}$, it suffices to show that $\operatorname{Re} f(iz)$ and $\operatorname{Im} f(iz)$ cannot both be 0. Recalling that each a_j is ± 1, and defining $r := \lfloor m/2 \rfloor$, we see that

$$\begin{aligned}\operatorname{Re} f(iz) &= \pm 1 \pm 2\cos(2\theta) \pm 2\cos(4\theta) \pm \cdots \pm 2\cos(2r\theta), \\ \operatorname{Im} f(iz) &= \pm 2\cos(\theta) \pm 2\cos(3\theta) \pm 2\cos(5\theta) \pm \cdots \pm 2\cos((2r \pm 1)\theta),\end{aligned}$$

which, using the notation defined earlier, can be rewritten as

$$\begin{aligned}\operatorname{Re} f(iz) &= \pm 1 \pm 2T_2 \pm 2T_4 \pm \cdots \pm 2T_{2r}, \\ \operatorname{Im} f(iz) &= \pm 2T_1 \pm 2T_3 \pm 2T_5 \pm \cdots \pm 2T_{2r\pm 1}.\end{aligned}$$

Now let $A := \operatorname{Re} f(iz)$ and let $B := \operatorname{Im} f(iz)$. Both A and B can be regarded as polynomials in x with integer coefficients, where as before, $x := 2c := 2\cos\theta$. Notice

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that one of A, B is odd and the other is even, and that $\deg A$ and $\deg B$ differ by 1. We now observe that

$$\begin{aligned}
\Phi(A) &= \Phi(\pm 1 \pm \overline{2T_2} \pm \overline{2T_4} \pm \cdots \pm \overline{2T_{2r}}) \\
&= \Phi(\pm 1) + \Phi(\pm\overline{2T_2}) + \Phi(\pm\overline{2T_4}) + \cdots + \Phi(\pm\overline{T_{2r}}) \\
&= \Phi(1) + \Phi(\overline{2T_2}) + \Phi(\overline{2T_4}) + \cdots + \Phi(\overline{T_{2r}}) \\
&= \Phi(1 + \overline{2T_2} + \overline{2T_4} + \cdots + \overline{2T_{2r}}) \\
&= \Phi(\overline{U_{2r}}) \quad \text{by (4)}
\end{aligned}$$

and by similar reasoning, we have $\Phi(B) = \Phi\big(\overline{U_{2r\pm 1}}\big)$. Thus A and B (in some order) satisfy the hypotheses of Lemma 3. Hence $\operatorname{Re} f(iz)$ and $\operatorname{Im} f(iz)$ are never both zero, and the theorem is proved. ■

4 Proof of Theorem 2

The following short proof using complex analysis is essentially due to the referee. As in the statement of Theorem 2, we have

$$f(\theta) = \cos(n\theta) + a_{n-1}\cos((n-1)\theta) + \cdots + a_1\cos(\theta)$$

where $a_0, \ldots, a_{n-1}$ are real.

Proof of Theorem 2 Note that $f(\theta) \pm 1 = \operatorname{Re}(p(e^{i\theta}))$, where

$$p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z \pm 1.$$

Since the product of all roots of p is ± 1, we conclude p has at least one root inside or on the unit circle. Let Γ denote the closed curve formed by $p(e^{i\theta})$ for $\theta \in [0, 2\pi]$. If p has a root on the unit circle, then certainly Γ passes through the origin and thus intersects the line $\operatorname{Re} z = 0$. If p has no roots on the unit circle, then p has at least one root inside the unit circle. By the Argument Principle, we then conclude Γ goes around the origin at least once and thus intersects the line $\operatorname{Re} z = 0$. In either case, $f(\theta) \pm 1 = \operatorname{Re}(p(e^{i\theta}))$ must have at least one real zero. ■

This yields quick proofs of Corollaries 5 and 6 of this paper, which appear in the next section. Corollary 4, by contrast, is a corollary of the author's original proof of Theorem 2, as opposed to a corollary of the statement of Theorem 2. We therefore now give a sketch of the author's original proof of Theorem 2.

Sketch of alternate proof of Theorem 2 We define

$$g(\theta) = a_{n-1}\cos((n-1)\theta) + \cdots + a_1\cos(\theta), \tag{6}$$

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so we have $f(\theta) = \cos(n\theta) + g(\theta)$. Observe that since f has average value 0 on $[0, \pi]$, it suffices to show that $f(\theta) \geq +1$ for some θ and that $f(\theta) \leq -1$ for some θ. We now consider two cases.

Case 1 Suppose n is even; say $n = 2m$. Then $\cos(n\theta) = +1$ at each of the $m+1$ points

$$\theta = 0, \frac{2\pi}{n}, \frac{4\pi}{n}, \dots, \pi$$

and $\cos(n\theta) = -1$ at each of the m points

$$\theta = \frac{\pi}{n}, \frac{3\pi}{n}, \frac{5\pi}{n}, \dots, \frac{(n-1)\pi}{n}.$$

We show that the $m+1$ values

$$g(0), g\Big(\frac{2\pi}{n}\Big), g\Big(\frac{4\pi}{n}\Big), \dots, g(\pi)$$

cannot all be negative by showing they cannot all have the same sign, and we show that the m values

$$g\Big(\frac{\pi}{n}\Big), g\Big(\frac{3\pi}{n}\Big), g\Big(\frac{5\pi}{n}\Big), \dots, g\Big(\frac{(n-1)\pi}{n}\Big)$$

cannot all be positive by showing they cannot all have the same sign. We can accomplish this by proving that the identities

$$g(0) + 2g\Big(\frac{2\pi}{n}\Big) + 2g\Big(\frac{4\pi}{n}\Big) + \cdots + 2g\Big(\frac{(n-2)\pi}{n}\Big) + g(\pi) = 0 \tag{7}$$

and

$$g\Big(\frac{\pi}{n}\Big) + g\Big(\frac{3\pi}{n}\Big) + g\Big(\frac{5\pi}{n}\Big) + \cdots + g\Big(\frac{(n-1)\pi}{n}\Big) = 0 \tag{8}$$

are true independently of the values of $a_1, \dots, a_{n-1}$.

Case 2 Suppose n is odd; say $n = 2m - 1$. Then $\cos(n\theta) = +1$ at each of the m points

$$\theta = 0, \frac{2\pi}{n}, \frac{4\pi}{n}, \dots, \frac{(n-1)\pi}{n}$$

and $\cos(n\theta) = -1$ at each of the m points

$$\theta = \frac{\pi}{n}, \frac{3\pi}{n}, \frac{5\pi}{n}, \dots, \pi.$$

Analogously to Case 1, we show that the m values

$$g(0), g\Big(\frac{2\pi}{n}\Big), g\Big(\frac{4\pi}{n}\Big), \dots, g\Big(\frac{(n-1)\pi}{n}\Big)$$

cannot all be negative by showing they cannot all have the same sign, and we show that the m values

$$g\left(\frac{\pi}{n}\right), g\left(\frac{3\pi}{n}\right), g\left(\frac{5\pi}{n}\right), \ldots, g(\pi)$$

cannot all be positive by showing they cannot all have the same sign. We can accomplish this by proving that the identities

$$g(0) + 2g\left(\frac{2\pi}{n}\right) + 2g\left(\frac{4\pi}{n}\right) + \cdots + 2g\left(\frac{(n-1)\pi}{n}\right) = 0 \tag{9}$$

and

$$2g\left(\frac{\pi}{n}\right) + 2g\left(\frac{3\pi}{n}\right) + \cdots + 2g\left(\frac{(n-2)\pi}{n}\right) + g(\pi) = 0 \tag{10}$$

are true independently of the values of $a_1, \ldots, a_{n-1}$.

Since we already gave a short proof of Theorem 2, we omit the details of how (7)–(10) are proved, and content ourselves with the following outline. The left side of any of the equations (7)–(10) can be rewritten as a linear combination of $a_1, \ldots, a_{n-1}$. We can then use well-known trigonometric identities to show that the coefficient of a_k is 0. ■

Note that the following is a corollary of the alternate proof of Theorem 2, as opposed to a corollary of the statement of Theorem 2.

Corollary 4 *Suppose g is of the form* (6) *(where n may be even or odd). Then g cannot maintain the same sign throughout the interval* $[0, (n-1)\pi/n]$, *and g cannot maintain the same sign throughout the interval* $[\pi/n, \pi]$.

Our Corollary 4 constitutes the nonexistence portion of [4, Corollaries 1 and 3]. It is further shown in [4] that the intervals in our Corollary 4 are best possible.

5 Further Comments on Theorem 2

Connection with Chebyshev Polynomials

Any function of the form

$$f = \cos(n\theta) + a_{n-1}\cos((n-1)\theta) + \cdots + a_1\cos(\theta) \tag{11}$$

can of course be rewritten as

$$f = T_n + a_{n-1}T_{n-1} + \cdots + a_1 T_1. \tag{12}$$

Notice that (12) is a polynomial in c of degree n whose leading coefficient is the same as that of T_n. Recalling that the Chebyshev polynomials have minimum supnorm on

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$[-1,1]$ among polynomials of prescribed degree and prescribed leading coefficient, we conclude

$$\max_{0\le\theta\le\pi} |f| = \max_{-1\le c\le 1} |f| \ge \max_{-1\le c\le 1} |T_n| = 1,$$

so we have either $f \ge +1$ somewhere or $f \le -1$ somewhere. By continuity and the fact that the average value of (11) is 0, we conclude that either $f(\theta) = +1$ for some $\theta \in [0,\pi]$ or $f(\theta) = -1$ for some $\theta \in [0,\pi]$. Theorem 2 is the stronger statement that both of these possibilities must occur (and does not seem to follow immediately from basic properties of Chebyshev polynomials).

Roots of Self-Reciprocal Polynomials

As a corollary of Theorem 2, we obtain a new proof of the next result (which also appears as [5, Corollary 2]).

Corollary 5 *Suppose $\alpha(z)$ is a self-reciprocal polynomial of even degree, say*

$$\alpha(z) = a_0 + \cdots + a_{m-1}z^{m-1} + a_m z^m + a_{m-1}z^{m+1} + \cdots + a_0 z^{2m}$$

where $a_0, \dots, a_m$ are real. Suppose $|a_m| \le 2\,|a_0|$ (informally, the middle coefficient is no more than twice as big as the end coefficients). Then $\alpha(z)$ has at least one root on the unit circle

$$\mathbb{S} := \{z \in \mathbb{C} : |z| = 1\}.$$

Proof of Corollary 5 For $\alpha(z)$ as above and $z = e^{i\theta} \in \mathbb{S}$, we have

$$\begin{aligned} \frac{\alpha(z)}{z^m} &= a_0\frac{1}{z^m} + \cdots + a_{m-1}\frac{1}{z} + a_m + a_{m-1}z + \cdots + a_0 z^m \\ &= a_m + 2(a_{m-1}\operatorname{Re} z + \cdots + a_0 \operatorname{Re} z^m) \\ &= a_m + 2\big(a_{m-1}\cos(\theta) + \cdots + a_0\cos(m\theta)\big)\,. \end{aligned} \tag{13}$$

By Theorem 2, the expression (13) attains both of the values $a_m + 2a_0$ and $a_m - 2a_0$ on the interval $[0,\pi]$. Suppose $a_0 \ge 0$ (the other case is similar). Then the condition $|a_m| \le 2\,|a_0|$ gives us

$$-2a_0 \le a_m \le +2a_0,$$

which implies that the interval $[a_m - 2a_0,\ a_m + 2a_0]$ contains 0. By continuity, and the fact that (13) is real-valued, we conclude that $\alpha(z)/z^m$ and hence also $\alpha(z)$ is equal to 0 for some $z \in \mathbb{S}$. ■

As an immediate consequence, we get a new proof of the next result [2, Theorem 2.8].

Corollary 6 *A self-reciprocal polynomial whose coefficients are ± 1 has at least one zero on $\mathbb{S}$.*

Proof of Corollary 6 Let α be a self-reciprocal polynomial whose coefficients are ± 1. If the degree of α is odd, it is straightforward to show that -1 is a root of α. If the degree of α is even, then the condition $|a_m| \leq 2\,|a_0|$ in Corollary 5 is satisfied, so α has a root on $\mathbb{S}$. ■

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References

[1] P. B. Borwein, *Computational Excursions in Analysis and Number Theory.* CMS Books in Mathematics 10, Springer-Verlag, New York (2002).
[2] T. Erdélyi, *On the zeros of polynomials with Littlewood-type coefficient constraints.* Michigan Math. J. **49**(2001), no. 1, 97–111.
[3] P. Erdős, *Some unsolved problems.* Michigan Math. J. **4**(1957), 291–300.
[4] A. D. Gilbert and C. J. Smyth, *Zero-mean cosine polynomials which are non-negative for as long as possible.* J. London Math. Soc. (2) **62**(2000), no. 2, 489–504.
[5] J. Konvalina and V. Matache, *Palindrome-polynomials with roots on the unit circle.* C. R. Math. Rep. Acad. Sci. Canada **26**(2004), no. 2, 39–44.
[6] J. E. Littlewood, *On polynomials* $\sum^n \pm z^m$, $\sum^n e^{\alpha_m i} z^m$, $z = e^{\theta_i}$. J. London Math. Soc. **41**(1966), 367–376.
[7] ———, *Some Problems in Real and Complex Analysis.* D.C. Heath, Lexington, MA, 1968.
[8] G. Pólya and G. Szegö, *Problems and Theorems in Analysis.* Volume II, Springer-Verlag, New York, 1976.
[9] L. Robinson, *Polynomials with plus or minus one coefficients: growth properties on the unit circle.* M.Sc. thesis, Simon Fraser University 1997.
[10] J. Spencer, *Six standard deviations suffice.* Trans. Amer. Math. Soc. **289**(1985), no. 2, 679–706.

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