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Isometries and Hermitian Operators on Zygmund Spaces

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Abstract. In this paper we characterize the isometries of subspaces of the little Zygmund space. We show that the isometries of these spaces are surjective and represented as integral operators. We also show that all hermitian operators on these settings are bounded.

1 Introduction

The Zygmund space $\mathcal{Z}$ is the set of all analytic functions f on the open unit disc $\triangle$ that satisfy the boundedness condition

$$\sup_{|z|<1}(1-|z|^2)|f''(z)| < \infty.$$

This space endowed with the norm $\|f\|_{\mathcal{Z}} = |f(0)| + |f'(0)| + \sup_{|z|<1}(1-|z|^2)|f''(z)|$ is a Banach space. We recall that the little Zygmund space is the closed subspace of $\mathcal{Z}$ defined by (see [17]):

$$\mathcal{Z}_0 = \Big\{ f \in \mathcal{Z} : \lim_{|z|\to 1^-}(1-|z|^2)|f''(z)| = 0 \Big\}.$$

Furthermore, we also consider the subspaces of the little Zygmund space

$$\mathcal{Z}_0^{(0,1)} = \{ f \in \mathcal{Z}_0 : f(0) = f'(0) = 0 \}$$

and

$$\mathcal{Z}_0^{i} = \{ f \in \mathcal{Z}_0 : f^{(i)}(0) = 0 \} \quad \text{with } i = 0,1, \quad f^{(0)} = f \quad \text{and} \quad f^{(1)} = f'.$$

Recently, there have been numerous papers on various aspects of classes of operators on Zygmund spaces; see [8] and the references therein. In this paper we characterize the surjective isometries supported by these spaces as well as classes of operators that are intrinsically related to the surjective isometries.

In Sections 2 and 3, we followed an approach by deLeeuw, Rudin and Werner for the characterization of the surjective isometries on function algebras (see [9]) to describe the surjective linear isometries supported by $\mathcal{Z}_0^{(0,1)}$. We start by defining an embedding of $\mathcal{Z}_0^{(0,1)}$ into a space of continuous functions $\mathcal{C}_0(\triangle)$; then, using the form

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of the extreme points of the unit ball of the dual space $\mathcal{C}_0(\triangle)^*$, given by [11, Theorem 2.3.16], we give a characterization for the extreme points of $(\mathcal{Z}_0^{(0,1)})_1^*$; see [5]. Given a Banach space X we use the subscript 1, X_1 to represent the unit ball of X.

The adjoint operator of a surjective linear isometry on a Banach space X determines a natural bijection on the set of extreme points of X_1^*. Hence the action of the adjoint operator on the set of extreme points often gives a representation for the isometries on X. We follow this path in our derivation of the form for the surjective isometries supported by $\mathcal{Z}_0^{(0,1)}$. We show in Section 3 that the isometries of $\mathcal{Z}_0^{(0,1)}$ are integral operators of translated weighted differential operators. The form of the isometries encountered in this new setting is quite different from the standard weighted composition operator type of isometries supported by several spaces of analytic functions, as pointed out in [16], see also [11,14] and [13]. Then we outline another scheme, as suggested by the referee, from which we conclude that all the isometries on these spaces are surjective.

In Section 4, we use our characterization of the isometries to describe the hermitian operators. In particular, we show that these spaces only support bounded hermitian operators, since these operators are the generators of uniformly continuous one-parameter group of isometries. We also conclude that bounded hermitian operators are trivial. We then employ a theorem in [12] to extend our representation for the hermitian operators on $\mathcal{Z}_0^{(0,1)}$ to the little Zygmund space $\mathcal{Z}_0$.

2 Extreme Points of $(\mathcal{Z}_0^{(0,1)})_1^*$

We embed $\mathcal{Z}_0^{(0,1)}$ into $\mathcal{C}_0(\triangle)$, the space of all continuous functions F defined on the unit disc and satisfying the boundary condition $\lim_{|z|\to 1} F(z) = 0$. This space is endowed with the norm $\|F\|_\infty = \max|F(z)|$. We define

$$\begin{aligned}\Phi\colon \mathcal{Z}_0^{(0,1)} &\to \mathcal{C}_0(\triangle)\\ f &\mapsto F = \Phi(f)\colon \triangle \to E,\end{aligned}$$

by $\Phi(f)(z) = (1-|z|^2)f''(z)$. The map Φ is a linear isometry with range space denoted by $\mathcal{Y}$.

Throughout this section we represent functions in $\mathcal{Z}_0^{(0,1)}$ with lower case letters and their images under Φ with upper case letters, *e.g.*, $F = \Phi(f)$.

Arens and Kelley's theorem (see [11, Corollary 2.3.6]) states that every extreme point of the unit ball of $\mathcal{Y}^*$ is of the form $e^{i\alpha}\delta_z$, with $z \in \triangle$ and $\delta_z\colon \mathcal{Y} \to \mathbb{C}$ given by $\delta_z(F) = F(z)$. This implies that the extreme points of $(\mathcal{Z}_0^{(0,1)})_1^*$ are of the form $\phi\colon \mathcal{Z}_0^{(0,1)} \to \mathbb{C}$ given by $\phi(f)(z) = e^{i\alpha}(1-|z|^2)f''(z)$.

We denote the set of extreme points of the unit ball of the dual space $\mathcal{Y}$ by $\mathrm{ext}(\mathcal{Y}_1^*)$, and in the next lemma we show that every functional of the form $e^{i\alpha}\delta_z$ is an extreme point of $\mathcal{Y}_1^*$.

Lemma 2.1 $\mathrm{ext}(\mathcal{Y}_1^*) = \{e^{i\theta}\delta_z : z \in \triangle \theta \in \mathbb{R}\}$.

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Proof Arens and Kelley's theorem states that $\operatorname{ext}(\mathcal{Y}_1^*) \subseteq \{e^{i\theta}\delta_z : z \in \triangle\}$. We now prove the reverse inclusion. Given a functional of the form $e^{i\alpha}\delta_z$, we assume that there exist ϕ_0 and ϕ_1 in $\mathcal{Y}_1^*$, such that

$$\delta_z = \frac{\phi_0 + \phi_1}{2}. \tag{2.1}$$

Since $\mathcal{Y}$ is a closed subspace of $\mathcal{C}_0(\triangle)$, the Hahn-Banach Theorem implies the existence of norm 1 extensions of ϕ_0 and ϕ_1 to $\mathcal{C}_0(\triangle)$, denoted by $\tilde{\phi_0}$ and $\tilde{\phi_1}$, respectively. These functionals are written as

$$\widetilde{\phi_0}(F) = \int_\triangle F d\nu \quad \text{and} \quad \widetilde{\phi_1}(F) = \int_\triangle F d\mu,$$

with ν and μ representing regular probability Borel measures on $\triangle$.

Given $z_0 \in \triangle \setminus \{0\}$, we consider the function

$$f_0(z) = (1 - |z_0|^2)\Big(-\frac{1}{\overline{z_0}}\Big)\Big[z + \frac{1}{\overline{z_0}}\operatorname{Log}(1 - \overline{z_0}z)\Big].$$

It is easy to check that $f_0 \in \mathcal{Z}_0^{(0,1)}$. Furthermore, $\|f_0\|_{\mathcal{Z}} = |F_0(z_0)| > |F_0(z)|$, where $F_0(z) = (1 - |z|^2)f_0''(z)$ for all $z \neq z_0$. We apply (2.1) to the function F_0 to conclude that $\tilde{\phi_0}(F_0) = \tilde{\phi_1}(F_0) = 1$. If $|\nu|(\triangle \setminus \{z_0\}) > 0$, then there exists a compact subset K of $\triangle \setminus \{z_0\}$ such that $|\nu|(K) > 0$. Clearly,

$$\sup_{z\in K}|F_0(z)| = \sup_{z\in K}(1 - |z|^2)|f_0''(z)| = \alpha < 1.$$

Hence,

$$\begin{aligned} 1 = \tilde{\phi_0}(F_0) = \Big|\int_\triangle F_0 d\nu\Big| &= \Big|\int_{\{z_0\}} F_0 d\nu + \int_K F_0 d\nu + \int_{(\triangle\setminus\{z_0\})\setminus K} F_0 d\nu\Big| \\ &\le |\nu|(\{z_0\}) + \alpha|\nu|(K) + |\nu|((\triangle \setminus \{z_0\}) \setminus K) < |\nu|(\triangle) = 1. \end{aligned}$$

This leads to an absurdity and shows that $|\nu|(\triangle \setminus \{z_0\}) = 0$ and $\nu(\triangle \setminus \{z_0\}) = 0$. Therefore, $\nu(\{z_0\}) = 1$. A similar reasoning applies to μ. Given $F \in \mathcal{Y}$, we have

$$\begin{aligned} \delta_{z_0}(F) &= (1 - |z_0|^2)f''(z_0) = \frac{\phi_0(F) + \phi_1(F)}{2} \\ &= \frac{1}{2}\Big(\int_{\{z_0\}} F d\nu + \int_{\{z_0\}} F d\mu\Big) \\ &= \frac{1}{2}\Big[\nu(z_0)(1 - |z_0|^2)f''(z_0) + \mu(z_0)(1 - |z_0|^2)f''(z_0)\Big]. \end{aligned}$$

Therefore,

$$f''(z_0) = \frac{\nu(z_0)f''(z_0) + \mu(z_0)f''(z_0)}{2}.$$

Then $\nu = \mu$ and $\phi_0 = \phi_1$. This completes the proof. ∎

Lemma 2.1 implies that the extreme points of $(\mathcal{Z}_0^0)_1^*$ are precisely the functionals $\Upsilon_{z_0}(f) = e^{i\alpha}(1 - |z_0|^2)f''(z_0)$, with $\alpha \in \mathbb{R}$ and $z_0 \in \triangle$.

Remark 2.2 We observe that $(\mathcal{Z}_0^i)^* = (\mathbb{C} \oplus_1 \mathcal{Z}_0^{(0,1)})^* = \mathbb{C} \oplus_\infty (\mathcal{Z}_0^{(0,1)})^*$ $(i = 0, 1)$ and also

$$(\mathcal{Z}_0)^* = (\mathbb{C} \oplus_1 \mathbb{C} \oplus_1 \mathcal{Z}_0^{(0,1)})^* = \mathbb{C} \oplus_\infty \mathbb{C} \oplus_\infty (\mathcal{Z}_0^{(0,1)})^*.$$

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It follows that $\operatorname{ext}((\mathcal{Z}_0^i)_1^*)$ consists of functionals τ given by

$$\tau(f) = e^{i\theta_1} f^{(1-i)}(0) z_0^{(1-i)} + e^{i\theta_2}(1-|z_0|^2) f''(z_0), \text{ with } z_0 \in \triangle, \theta_{1,2} \in [0, 2\pi).$$

Moreover, $\operatorname{ext}((\mathcal{Z}_0)^*)_1$ is the set of all functionals

$$\tau(f) = e^{i\theta_0} f(0) + e^{i\theta_1} f'(0) z_0 + e^{i\theta_2}(1-|z_0|^2) f''(z_0),$$

with $\theta_k \in [0, 2\pi)$ $(k = 0, 1, 2)$.

3 Characterization of the Linear Isometries on $\mathcal{Z}_0^{(0,1)}$

In this section we show that linear surjective isometries on $\mathcal{Z}_0^{(0,1)}$ can be represented as integral operators. Then we follow a scheme suggested by the referee to show that this space only supports surjective isometries.

Given a surjective linear isometry $T: \mathcal{Z}_0^{(0,1)} \to \mathcal{Z}_0^{(0,1)}$, we denote by $S: \mathcal{Y} \to \mathcal{Y}$ the corresponding isometry on $\mathcal{Y}$ such that $S = \Phi \circ T \circ \Phi^{-1}$, where Φ represents the embedding considered in the previous section. The adjoint operator of S, $S^*: \mathcal{Y}^* \to \mathcal{Y}^*$ induces a permutation on the set of extreme points of $(\mathcal{Y}^*)_1$. This can be expressed as follows. For every $z \in \triangle$ and θ there exists a unique pair $(w, \alpha) \in \triangle \times [0, 2\pi)$ such that

$$S^*(e^{i\theta}\delta_z) = e^{i\alpha}\delta_w.$$

Equivalently,

$$(1-|z|^2) e^{i\theta} (Tf)''(z) = (1-|w|^2) e^{i\alpha} f''(w), \quad \text{for every } f \in \mathcal{Z}_0^{(0,1)}.$$

The values of α and w conceivably depend on the choice of θ and z. This determines the following two maps:

$$\sigma_0: \begin{array}{ccc} \mathbb{S}_1 \times \triangle & \to & \triangle \\ (e^{i\theta}, z) & \mapsto & w \end{array} \quad \text{and} \quad \Gamma_0: \begin{array}{ccc} \mathbb{S}_1 \times \triangle & \to & \mathbb{S}_1 \\ (e^{i\theta}, z) & \mapsto & e^{i\alpha}. \end{array}$$

We show in the following lemma that σ_0 is independent of the first coordinate, and then we consider $\sigma: \triangle \to \triangle$ given by $\sigma(z) = \sigma_0(1, z)$.

Lemma 3.1 *If $z \in \triangle$, then σ_0 restricted to the set $\{(e^{i\theta}, z) : \theta \in \mathbb{R}\}$ is constant and $\sigma: \triangle \to \triangle$, defined by $\sigma(z) = \sigma_0(1, z)$, is a disc automorphism.*

Proof We assume that there are points in $\mathbb{S}_1$, $e^{i\theta}$, and $e^{i\theta_1}$ such that $\sigma_0(e^{i\theta}, z) = w \neq w_1 = \sigma_0(e^{i\theta_1}, z)$, for some value of $z \in \triangle$. Hence

$$(1-|z|^2) e^{i\theta} (Tf)''(z) = (1-|w|^2) e^{i\alpha} f''(w), \tag{3.1}$$

$$(1-|z|^2) e^{i\theta_1} (Tf)''(z) = (1-|w_1|^2) e^{i\alpha_1} f''(w_1). \tag{3.2}$$

Substituting f with $f_0(z) = z^2/2$ into (3.1) and (3.2) we get

$$(1-|z|^2) e^{i\theta} (Tf_0)''(z) = (1-|w|^2) e^{i\alpha} \quad \text{and} \quad (1-|z|^2) e^{i\theta_1} (Tf_0)''(z) = (1-|w_1|^2) e^{i\alpha_1},$$

respectively.

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Therefore, $e^{i(\alpha-\theta)}(1-|w|^2) = e^{i(\alpha_1-\theta_1)}(1-|w_1|^2)$. This implies that $|w| = |w_1|$ and $e^{i(\alpha-\theta)} = e^{i(\alpha_1-\theta_1)}$. From (3.1) and (3.2) we conclude that $f''(w) = f''(w_1)$ for every $f \in \mathcal{Z}_0^{(0,1)}$. Hence, $w = w_1$ and σ_0 depends only on the value of z, as claimed.

Then, given σ as in the statement of the lemma, we write (3.1) as

$$(1-|z|^2)e^{i\theta}(Tf)''(z) = (1-|\sigma(z)|^2)e^{i\alpha}f''(\sigma(z)). \tag{3.3}$$

We apply the same reasoning to T^{-1} to determine ψ, a mapping from the open disc into itself, satisfying the equality

$$(1-|z|^2)e^{i\theta}(T^{-1}f)''(z) = (1-|\psi(z)|^2)e^{i\beta}f''(\psi(z)). \tag{3.4}$$

Equality (3.4) applied to Tf yields

$$\begin{aligned}(1-|z|^2)e^{i\theta}f''(z) &= \big(1-|\psi(z)|^2\big)\,e^{i\beta}(Tf)''(\psi(z))\\ &= e^{i(\beta-\theta+\alpha)}\big(1-|\sigma(\psi(z))|^2\big)\,f''\big(\sigma(\psi(z))\big);\end{aligned}$$

then for every $f \in \mathcal{Z}_0^{(0,1)}$ and $z \in \triangle$,

$$(1-|z|^2)e^{i\theta}f''(z) = e^{i(\beta-\theta+\alpha)}\big(1-|\sigma(\psi(z))|^2\big)\,f''\big(\sigma(\psi(z))\big).$$

Setting $f(z) = z^2/2$ in the equality displayed above, we obtain

$$(1-|z|^2)e^{i\theta} = e^{i(\beta-\theta+\alpha)}\big(1-|\sigma(\psi(z))|^2\big).$$

This implies $|z| = |\sigma(\psi(z))|$ and $e^{i\theta} = e^{i(\beta-\theta+\alpha)}$. Therefore, $f''(z) = f''(\sigma(\psi(z)))$, which implies that $\sigma\circ\psi$ is the identity on $\triangle$, and then σ is surjective. Similar reasoning also shows that $\psi\circ\sigma$ is the identity on $\triangle$ and σ is injective. We now prove that σ is analytic. To this end, we replace f in (3.3) by the two functions $f_0(z) = \frac{z^2}{2}$ and $f_1(z) = z^3/6$. We obtain

$$\begin{aligned}(1-|z|^2)e^{i\theta}(Tf_0)''(z) &= (1-|\sigma(z)|^2)e^{i\alpha},\\ (1-|z|^2)e^{i\theta}(Tf_1)''(z) &= (1-|\sigma(z)|^2)e^{i\alpha}\sigma(z),\end{aligned}$$

respectively. For every $z \in \triangle$ we have $(Tf_0)''(z) \neq 0$. Therefore,

$$\sigma(z) = \frac{(Tf_1)''(z)}{(Tf_0)''(z)}.$$

This shows that σ is analytic and hence a disc automorphism. ∎

Theorem 3.2 *Let $T\colon \mathcal{Z}_0^{(0,1)} \to \mathcal{Z}_0^{(0,1)}$. Then T is surjective linear isometry if and only if there exist a disc automorphism σ and a real number α such that for every $f \in \mathcal{Z}_0^{(0,1)}$ and $z \in \triangle$,*

$$Tf(z) = e^{i\alpha}\int_0^z\big[(f'\circ\sigma)(\xi) - (f'\circ\sigma)(0)\big]\,d\xi.$$

Proof We first assume that T is a surjective linear isometry. It follows from Lemma 3.1 and its preamble that

$$(1-|z|^2)e^{i\theta}(Tf)''(z) = \big(1-|\sigma(z)|^2\big)\,\Gamma_0(\theta,z)\big(f''(\sigma(z))\big), \tag{3.5}$$

for every $f \in \mathcal{Z}_0^{(0,1)}$ and $z \in \triangle$.

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In particular, for $f_0(z) = z^2/2$, we have

$$\Big|\frac{(Tf_0)''(z)}{\sigma'(z)}\Big| = 1.$$

Since the mapping $z \to \frac{(Tf_0)''(z)}{\sigma'(z)}$ is analytic on the open disc, the Maximum Modulus Principle for analytic functions implies that it must be constant; then there exists $\eta \in [0, 2\pi)$ such that

$$\frac{(Tf_0)''(z)}{\sigma'(z)} = e^{i\eta}.$$

The equality displayed in (3.5) applied to f_0 yields

$$e^{i\theta} e^{i\eta} = \frac{|\sigma'(z)|}{\sigma'(z)} \Gamma_0(\theta, z).$$

Substituting this relation in (3.5), we have

$$(Tf)''(z) = e^{i\eta} \sigma'(z) f''(\sigma(z)).$$

Integrating this last equality twice and since $(Tf)'(0) = Tf(0) = 0$, we obtain

$$(Tf)(z) = e^{i\eta} \int_0^z [f'(\sigma(\xi)) - f'(\sigma(0))]\, d\xi.$$

It is easy to check that T of the form displayed in the statement of the theorem is an isometry. To this end and since $(1 - |z|^2)|\sigma'(z)| = |\sigma(z)|$, we have

$$\begin{aligned}\sup_{|z|<1}(1 - |z|^2)|(Tf)''(z)| &= \sup_{|z|<1}(1 - |z|^2)|f''(\sigma(z))\sigma'(z)| \\ &= \sup_{|z|<1}(1 - |z|^2)|f''(z)|.\end{aligned}$$

It is straightforward to check that the inverse of T is given as

$$(T^{-1}f)(z) = e^{-i\eta} \int_0^z \big[f'(\sigma^{-1}(\xi)) - f'(\sigma^{-1}(0)) \big]\, d\xi.$$

This completes the proof. ∎

The proof of the previous theorem requires surjectivity of the given isometry. In the remainder of this section we outline an approach suggested by the referee that does not rely on this surjectivity assumption. Consequently, we derive a stronger result.

We recall the definition of Bloch space,

$$\mathcal{B} = \Big\{ f \colon \triangle \to \mathbb{C} : f \text{ is analytic and } \sup_{|z|<1}(1 - |z|^2)|f'(z)| < \infty \Big\}.$$

The little Bloch space is the subspace of all functions in $\mathcal{B}$ such that

$$\lim_{|z|\to 1}(1 - |z|^2)|f'(z)| = 0,$$

endowed with the norm $\|f\|_{\mathcal{B}_0} = |f(0)| + \sup_{|z|<1}(1 - |z|^2)|f'(z)|$. Then $\mathcal{B}_{00}$ is the closed subspace of $\mathcal{B}_0$ with 0 fixed.

Lemma 3.3 $\mathcal{Z}_0^{(0,1)} \simeq \mathcal{B}_{00}$.

Proof The mapping $\Phi\colon \mathcal{Z}_0^{(0,1)} \to \mathcal{B}_{00}$ such that $\Phi(f) = f'$ defines a surjective isometry between $\mathcal{Z}_0^{(0,1)}$ and $\mathcal{B}_{00}$. To prove that Φ is surjective, let $f \in \mathcal{B}_{00}$ and define $g(z) = \int_0^z f(\xi)d\xi$. Then $\Phi(g) = f$. ∎

Corollary 3.4 *Every isometry T on $\mathcal{Z}_0^{(0,1)}$ is surjective, and there exist $\alpha \in \mathbb{R}$ and σ a disc automorphism such that for every $f \in \mathcal{Z}_0^{(0,1)}$ and $z \in \triangle$,*

$$(Tf)(z) = e^{i\alpha} \int_0^z \big[f'(\sigma(\xi)) - f'(\sigma(0))\big] d\xi.$$

Proof Given an isometry T on $\mathcal{Z}_0^{(0,1)}$, $\Phi T \Phi^{-1}$ is an isometry on $\mathcal{B}_{00}$. Hence,

$$\Phi T \Phi^{-1}(f)(z) = \lambda\big(f \circ \sigma(z) - f(\sigma(0))\big)$$

where λ is a modulus 1 complex number and σ a disc automorphism. If we denote $\Phi^{-1} f = g$, then $g' = f$ and

$$T(g)(z) = \lambda \int_0^z [g' \circ \sigma(\xi) - g'(\sigma(0))] d\xi.$$

Since $\mathcal{B}_{00}$ supports only surjective isometries (see [7, Corollary 1]), it follows from the Lemma 3.3 that the same is true for $\mathcal{Z}_0^{(0,1)}$. ∎

It is easy to see that the space $\mathcal{Z}_0$ endowed with the norm $\|f\| = |f(0)| + |f'(0)| + \sup_{|z|<1}(1-|z|^2)|f''(z)|$ is isometric to the ℓ_1-sum $\mathbb{C} \oplus_1 \mathbb{C} \oplus_1 \mathcal{Z}_0^{0,1}$. An isometry T on $\mathcal{Z}_0$ induces $T_0\colon \mathcal{Z}_0^{(0,1)} \to \mathcal{Z}_0^{(0,1)}$ defined by $(T_0 f)(z) = (Tf)(z) - (Tf)(0) - (Tf)'(0)z$. It is easy to check that T_0 is an isometry and thus surjective. Corollary 3.4 implies the existence of $\alpha \in \mathbb{R}$ and a disc automorphism σ such that for every $f \in \mathcal{Z}_0^{(0,1)}$ and $z \in \triangle$,

$$(T_0 f)(z) = e^{i\alpha} \int_0^z \big[f'(\sigma(\xi)) - f'(\sigma(0))\big] d\xi.$$

Therefore,

$$(Tf)(z) = (Tf)(0) + (Tf)'(0)z + e^{i\alpha} \int_0^z [f'(\sigma(\xi)) - f'(\sigma(0))] d\xi,$$

which implies that T is surjective. These considerations lead to the following result.

Corollary 3.5 *Every isometry T on $\mathcal{Z}_0$ is surjective and there exist $\alpha_j \in \mathbb{R}$, $j = 1, 2, 3$, and σ a disc automorphism such that for every $f \in \mathcal{Z}_0$ and $z \in \triangle$,*

$$(Tf)(z) = e^{i\alpha_1} f(0) + e^{i\alpha_2} f'(0)z + e^{i\alpha_3} \int_0^z \big[f'(\sigma(\xi)) - f'(\sigma(0))\big] d\xi,$$

or

$$(Tf)(z) = e^{i\alpha_1} f'(0) + e^{i\alpha_2} f(0)z + e^{i\alpha_3} \int_0^z \big[f'(\sigma(\xi)) - f'(\sigma(0))\big] d\xi.$$

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4 Existence of Hermitian Operators

It is known that hermitian operators on a Banach space X are generators of strongly continuous one parameter groups of bounded operators on X; see [1]. A remarkable result by Lotz (see [15]) asserts that every strongly continuous one parameter group of operators on a Banach with the Grothendieck property and the Dunford-Petit property is uniformly continuous. Such a Banach space will be referred to as a GDP space. See [15] for the relevant definitions and properties. We also refer the reader to [2]. The space $\mathcal{Z}^{(0,1)}$ is isometric to $\mathcal{B}_0$, thus every strongly continuous one parameter groups of operators on $\mathcal{Z}^{(0,1)}$ is uniformly continuous. This also implies that $\mathcal{Z}^{(0,1)}$ only supports bounded hermitian operators. Furthermore the space $\mathcal{Z}_0^{(0,1)}$ is a closed and complemented subspace of $\mathcal{Z}^{(0,1)}$, hence is also a GDP Banach space. We conclude that every hermitian operators on $\mathcal{Z}_0^{(0,1)}$ is bounded. We summarize these considerations in the next result.

Proposition 4.1 *Every hermitian operator A on $\mathcal{Z}_0^{(0,1)}$ is bounded, and there exists $\alpha \in \mathbb{R}$ such that for every $f \in \mathcal{Z}_0^{(0,1)}$ and $z \in \triangle$,*

$$(Af)(z) = \alpha f(z).$$

Proof Given a hermitian operator A, there exists a one-parameter group of surjective isometries $\{T_t\}_t$ such that

$$A(f)(z) = \Big(-i\frac{d}{dt}T_t\Big)|_{t=0} f(z).$$

It follows from Corollary 3.4 that $(T_t f)(z) = e^{i\alpha_t} \int_0^z [f'(\sigma_t(\xi)) - f'(\sigma(0))]d\xi$. Since $\{T_t\}_t$ is a one-parameter group then σ_t is a one-parameter group of disc automorphisms and $\alpha_t = \alpha \cdot t$, for some real number α. Therefore,

$$\begin{aligned} Af(z) &= -i\Big[\, i\alpha \int_0^z f'(\xi)\, d\xi + \partial_t \sigma_t(z)|_{t=0} f'(z)\Big] \\ &= \alpha f(z) - i\partial_t\sigma_t(z)|_{t=0} f'(z). \end{aligned}$$

Since $\mathcal{Z}_0^{(0,1)}$ is a GDP space, it supports only bounded hermitian operators. This implies that σ_t must be the trivial group. Hence, $Af = \alpha f$. ∎

We observe that similar results also hold for $\mathcal{Z}_0$ and the subspaces $\mathcal{Z}_0^{(i)}$ (with $i = 0,1$). As mentioned before these spaces are isometric to ℓ_1 sums of $\mathbb{C}$ and $\mathcal{Z}_0^{(0,1)}$. Then an application of a theorem by Hornor and Jamison [14] yields that all these spaces only support bounded hermitian operators and they are of the form $(Af)(z) = af(0) + bf'(0) + cf(z)$ with $a, b, c \in \mathbb{R}$ and $f \in \mathcal{Z}_0$.

Corollary 4.2 *If A is a hermitian operator on $\mathcal{Z}_0$, then A is bounded and there exist $a, b, c \in \mathbb{R}$ such that for every $f \in \mathcal{Z}_0$ and $z \in \triangle$,*

$$(Af)(z) = af(0) + bf'(0) + cf(z).$$

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