Can. J. Math. Vol. **50** (2), 1998 pp. 342–355.

SHAPE FIBRATIONS, MULTIVALUED MAPS AND SHAPE GROUPS

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ABSTRACT. The notion of shape fibration with the near lifting of near multivalued paths property is studied. The relation of these maps—which agree with shape fibrations having totally disconnected fibers—with Hurewicz fibrations with the unique path lifting property is completely settled. Some results concerning homotopy and shape groups are presented for shape fibrations with the near lifting of near multivalued paths property. It is shown that for this class of shape fibrations the existence of liftings of a fine multivalued map is equivalent to an algebraic problem relative to the homotopy, shape or strong shape groups associated.

1**. Introduction.** Inspired by works of R. C. Lacher [5, 6] on cellular maps, D. S. Coram and P. F. Duvall, Jr. [3] defined in 1977 the concept of approximative fibration, replacing the homotopy lifting property of Hurewicz fibrations by an approximate homotopy lifting property (AHLP), applying to a larger class of maps. This property is based on the notion of δ-closeness for maps: Given two maps $f, g\colon X \longrightarrow Y$ and given a covering δ of Y, f and g are δ-close (denoted $f \overset{\delta}{=} g$) if, for every $x \in X$, $f(x)$ and $g(x)$ are contained in a member of δ. A map $p\colon E \longrightarrow B$ has the AHLP($\mathcal{X}$) with respect to a class $\mathcal{X}$ of topological spaces if, for every open covering α of B, there are open coverings δ and β of E and B such that for every $X \in \mathcal{X}$, every map $g\colon X \longrightarrow E$ and every map $h\colon X \times I \longrightarrow B$ such that $pg \overset{\beta}{=} h_0$ there exists a homotopy $k\colon X \times I \longrightarrow E$ such that $k_0 \overset{\delta}{=} g$ and $pk \overset{\alpha}{=} h$. In this way they obtained a generalization of Hurewicz fibrations with similar properties.

In 1979, S. Mardešić and T. B. Rushing [8, 9] introduced shape fibrations between metric compacta, as those maps $p\colon E \longrightarrow B$ satisfying an approximate homotopy lifting property, but referred to sequences of maps and homotopies whose images are not contained in the spaces E and B but in systems of ANRs having E, B and p as limits. In its most simplified formulation this reduces to sequences of homotopies in neighborhoods of B in the Hilbert cube Q being lifted to homotopies in neighborhoods of E in Q. This notion was extended by Mardešić [7] to the general case of maps between arbitrary topological spaces using polyhedral resolutions.

Recently, Z. Čerin [2] has given a redefinition of approximate fibrations for arbitrary topological spaces, replacing the maps g, h and k in the definition of the AHLP($\mathcal{X}$) by relations (*i.e.* multivalued functions with nonempty images of points). Čerin's approach provides, in addition, a unifying theory for the main types of fibrations in the literature.

Received by the editors September 5, 1996; revised June 16, 1997 and October 30, 1997.
AMS subject classification: 54C56, 55P55, 55Q05, 55Q07, 55R05.
Key words and phrases: Shape fibration, multivalued map, homotopy groups, shape groups, strong shape groups.
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https://doi.org/10.4153/CJM-1998-018-7 Published online by Cambridge University Press

In fact, Čerin's approximate fibrations include, as particular cases, Coram and Duvall's approximate fibrations and Mardešić and Rushing's shape fibrations, as well as other types of fibrations, like weak shape fibrations and n-shape fibrations.

Given a topological space X, a class $\mathcal{G}$ of relations, and an open covering α of another space E, a relation f from X to E is an $\alpha\mathcal{G}$-relation if it is in the class $\mathcal{G}$ and there exists an open covering σ of X such that, for every member S of σ, $f(S)$ is contained in a member of α.

Let now $\tau = (\mathcal{G}, \mathcal{H}, \mathcal{K})$ be a triple of classes of relations and let $\mathcal{X}$ be a class of topological spaces. A map $p\colon E \longrightarrow B$ has the AHLP$(\mathcal{X}, \tau)$, if for every open covering α of B and every open covering δ of E there are open coverings β and ε of B and E, respectively, such that for every $X \in \mathcal{X}$, every $\varepsilon\mathcal{G}$-relation $g\colon X \longrightarrow E$ and every $\beta\mathcal{H}$-relation $h\colon X \times I \longrightarrow B$ such that $pg \overset{\beta}{=} h_0$ there exists a $\delta\mathcal{K}$-relation $k\colon X \times I \longrightarrow E$ such that $k_0 \overset{\delta}{=} g$ and $pk \overset{\alpha}{=} h$.

The main result in [2] establishes that this version of the original concept of approximate fibration is equivalent to Mardešić's shape fibrations.

THEOREM 1 (Z. ČERIN [2]). *A map $p\colon E \longrightarrow B$ is a shape fibration if and only if it has the* AHLP$(\mathcal{T}, \rho)$, *where $\mathcal{T}$ denotes the class of all topological spaces, and $\rho = (\mathcal{R}, \mathcal{R}, \mathcal{R})$, with $\mathcal{R}$ the class of all relations.*

In this paper, the spaces E and B will be metric compacta, while the AHLP will be applied only to the class $\mathcal{M}$ of metric spaces. In this case, Theorem 1 holds if we replace $\mathcal{R}$ by the class $\mathcal{U}$ of all upper semicontinuous multivalued functions, where an upper semicontinuous multivalued function from X to Y is a multivalued function satisfying that for every $x \in X$ and for every neighborhood V of $F(x)$ in Y there is a neighborhood U of x such that $F(U) = \bigcup_{y \in U} F(y)$ is contained in V.

Throughout the paper, we will suppose that all multivalued functions are upper semicontinuous, calling them multivalued maps for short.

We study in this paper an extension of the unique path lifting property of Hurewicz fibrations, adapted to shape fibrations, called the *near lifting of near multivalued paths property*. This notion was introduced in an earlier version of [2]. Our main results concern this property. We show that a shape fibration has this property if and only if its fibers are totally disconnected, and completely determine its relationship with Hurewicz fibrations with the unique path lifting property.

In the last part of the paper we study algebraic properties of shape fibrations with the near lifting of near multivalued paths property, related to homotopy groups, shape groups and strong shape groups. The main result of this part is a theorem giving necessary and sufficient algebraic conditions for the existence of liftings of fine multivalued maps defined in Peano continua and with values in the base space of a shape fibration with near lifting of near multivalued paths.

The author is grateful to the referee for useful suggestions.

https://doi.org/10.4153/CJM-1998-018-7 Published online by Cambridge University Press

2. **The near lifting of near multivalued paths property.** Let $F: X \longrightarrow Y$ be a multivalued map. F is said to be ε-small if $\operatorname{diam}\bigl(F(x)\bigr) < \varepsilon$ for every $x \in X$. Any δ-small multivalued map $\omega: I \longrightarrow E$, from the unit interval to a metric space E will be called a *δ-small path in E*, and any δ-small multivalued map $\omega: I^n \longrightarrow E$ will be called a *δ-small n-path in E*.

The following definition is the analogous property, for shape fibrations, of the unique path lifting property of Hurewicz's fibrations.

DEFINITION 1. Let E and B be compact metric spaces and let $p: E \longrightarrow B$ a map. We say that p has the near lifting of near multivalued paths property if for every $\varepsilon > 0$ there exists $\delta > 0$ such that given $\omega', \omega'': I \longrightarrow E$ δ-small paths such that

$$d\bigl(\omega'(0), \omega''(0)\bigr) < \delta \quad \text{and} \quad d\bigl(p\omega'(t), p\omega''(t)\bigr) < \delta$$

for every $t \in I$, then $d\bigl(\omega'(t), \omega''(t)\bigr) < \varepsilon$ for every $t \in I$.

The two following results are immediate.

PROPOSITION 1. *Let $p: E \longrightarrow B$ be a map between compact metric spaces, with the near lifting of near multivalued paths property. Then p has the unique path lifting property.*

PROPOSITION 2. *Let E and B be compact metric spaces and let $p: E \longrightarrow B$ be a map with the near lifting of near multivalued paths property. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that given a metric continuum X and given $F', F'': X \longrightarrow E$ δ-small multivalued maps such that $d\bigl(F'(x_0), F''(x_0)\bigr) < \delta$ for a point $x_0 \in X$ and such that $d\bigl(pF'(x), pF''(x)\bigr) < \delta$ for every $x \in X$, then $d\bigl(F'(x), F''(x)\bigr) < \varepsilon$ for every $x \in X$.*

In the next result we show that the near lifting of near multivalued paths property can be characterized by the stronger condition of the fibers being totally disconnected, in the case of shape fibrations. This result is the shape theoretical version of the well-known result stating that for a Hurewicz's fibrations the unique path lifting property is equivalent to fibers not having non constant paths.

As we already mentioned in the introduction, throughout this paper the spaces E and B will be metric compacta, while the AHLP will be applied only to the class $\mathcal{M}$ of metric spaces and to the class $\mathcal{U}$ of multivalued maps. In this particular case, the AHLP($\mathcal{M}, \upsilon$) property (where $\upsilon = (\mathcal{U}, \mathcal{U}, \mathcal{U})$) admits the following form, which we call MHLP property.

Let E and B be compact metric spaces and let $p: E \longrightarrow B$ be a map. We say that p satisfies the multivalued homotopy lifting property (MHLP) with respect to a topological space X if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every δ-small multivalued maps $F: X \longrightarrow E$ and $H: X \times I \longrightarrow B$ with $d\bigl(pF(x), H(x, 0)\bigr) < \delta$ for every $x \in X$, then there exists an ε-small multivalued map $H': X \times I \longrightarrow E$ such that $d\bigl(H'(x, 0), F(x)\bigr) < \varepsilon$, $d\bigl(pH'(x, t), H(x, t)\bigr) < \varepsilon$ for every $(x, t) \in X \times I$.

THEOREM 2. *Let E and B be compact metric spaces and let $p: E \longrightarrow B$ be a shape fibration. Then p has the near lifting of near multivalued paths property if and only if every non empty fiber is totally disconnected.*

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PROOF. Observe first that if $p\colon E \longrightarrow B$ is a shape fibration so is $p\colon E \longrightarrow p(E)$. Moreover, $p\colon E \longrightarrow B$ has the near lifting of near multivalued paths property if and only if $p\colon E \longrightarrow p(E)$ has the near lifting of near multivalued paths property. Hence we can suppose that p is surjective.

Suppose now that p has the near lifting of near multivalued paths property and that there exists $x \in X$ such that $p^{-1}(x)$ is not totally disconnected. Then there exists a connected subset A of $p^{-1}(x)$ with more than two points. Take $x', y' \in A$, $x' \neq y'$ and consider $\delta > 0$ associated with $\varepsilon = d(x', y')$ by the near lifting of near multivalued paths property.

Since A is connected, there exists a δ-small path $\omega'\colon I \longrightarrow A \subset p^{-1}(x)$ such that $\omega'(0) = x'$, $\omega'(1) = y'$, and we can consider, on the other hand, $\omega''\colon I \longrightarrow p^{-1}(x)$ given by $\omega''(t) = x'$ for every $t \in I$. Then ω' and ω'' are δ-small paths such that $\omega'(0) = \omega''(0)$ and $p\omega' = p\omega''$. Therefore $d\big(\omega'(t), \omega''(t)\big) < \varepsilon$ for every $t \in I$. In particular,

$$d\big(\omega'(1), \omega''(1)\big) = d(y', x') < \varepsilon.$$

This contradiction proves the first implication.

Let us prove the other implication. Consider $\varepsilon > 0$. We see first that there exists $0 < \eta < \varepsilon$ such that for every $A \subset B$ with $\operatorname{diam}(A) < \eta$ there exist $\{U_1, \ldots, U_n\}$ pairwise disjoint open subsets of E with diameter less than ε such that $p^{-1}(A) \subset U_1 \cup \cdots \cup U_n$ with $d(U_i, U_j) > \eta$ for every $i \neq j$.

For every $x \in B$, since $p^{-1}(x)$ is a totally disconnected compact set, there exist $\{U_1^x, \ldots, U_{n_x}^x\}$ pairwise disjoint open and closed subsets of $p^{-1}(x)$ with diameter less than $\frac{\varepsilon}{3}$, such that $p^{-1}(x) = U_1^x \cup \cdots \cup U_{n_x}^x$. There exists $0 < \varepsilon_x < \frac{\varepsilon}{3}$ such that $d(U_i^x, U_j^x) > 3\varepsilon_x$ for every $i \neq j$. Then $B_{\varepsilon_x}\big(p^{-1}(x)\big) = B_{\varepsilon_x}(U_1^x) \cup \cdots \cup B_{\varepsilon_x}(U_{n_x}^x)$, union of open balls in E with $d\big(B_{\varepsilon_x}(U_i^x), B_{\varepsilon_x}(U_j^x)\big) > \varepsilon_x$ for every $i \neq j$. Now, there exists $0 < \eta_x < \varepsilon_x$ such that $p^{-1}\big(B_{\eta_x}(x)\big) \subset B_{\varepsilon_x}\big(p^{-1}(x)\big)$.

If not, there would exist $(y_n) \subset E$ that, by the compactness of E, we can suppose converging to some $y \in E$, such that for every $n \in \mathbb{N}$ we have $y_n \in p^{-1}\big(B_{\frac{1}{n}}(x)\big)$ but $y_n \notin B_{\varepsilon_x}\big(p^{-1}(x)\big)$. But then by the first condition $\big(p(y_n)\big)$ converges to x and by the second $p(y) \neq x$ and this is in contradiction with the continuity of p. Hence there exists $0 < \eta_x < \varepsilon_x$ such that

$$p^{-1}\big(B_{\eta_x}(x)\big) \subset B_{\varepsilon_x}\big(p^{-1}(x)\big),$$

and, by the compactness of B, there exists $0 < \eta < \varepsilon$ such that for every $A \subset B$ with $\operatorname{diam}(A) < \eta$ there exists $x \in B$ such that

$$p^{-1}(A) \subset p^{-1}\big(B_{\eta_x}(x)\big) \subset B_{\varepsilon_x}\big(p^{-1}(x)\big) = B_{\varepsilon_x}(U_1^x) \cup \cdots \cup B_{\varepsilon_x}(U_{n_x}^x)$$

with $d\big(B_{\varepsilon_x}(U_i^x), B_{\varepsilon_x}(U_j^x)\big) > \eta$ for every $i \neq j$. Therefore for every $A \subset B$ with $\operatorname{diam}(A) < \eta$ there exist $\{U_1, \ldots, U_n\}$ pairwise disjoint open subsets of E with diameter less than ε such that $p^{-1}(A) \subset U_1 \cup \cdots \cup U_n$ with $d(U_i, U_j) > \eta$ for every $i \neq j$.

Take $0 < \eta' < \frac{\eta}{6}$ such that $\operatorname{diam}\big(p(K)\big) < \frac{\eta}{6}$ for every compact subset K of E with $\operatorname{diam}(K) < \eta'$. Take $0 < \delta' < \eta'$ associated with η' by the property MHLP with respect

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to I, and $0 < \delta < \frac{\delta'}{3}$ such that $\operatorname{diam}\bigl(p(K)\bigr) < \frac{\delta'}{3}$ for every compact subset K of E with $\operatorname{diam}(K) < \delta$. To complete the proof of the theorem it only rests to show that δ is associated with ε by the near lifting of near multivalued paths property.

Let $\omega', \omega'' \colon I \longrightarrow E$ be δ-small paths such that

$$d\bigl(\omega'(0), \omega''(0)\bigr) < \delta \quad \text{and} \quad d\bigl(p\omega'(t), p\omega''(t)\bigr) < \delta$$

for every $t \in I$. Fix $s \in I$ and consider $F \colon I \times \{0\} \cup \{0,1\} \times I \longrightarrow E$ given by

$$F(t,r) = \begin{cases} \omega'(s-2st) & \text{if } 0 \le t < \frac{1}{2}, r = 0 \\ \omega'(0) \cup \omega''(0) & \text{if } t = \frac{1}{2}, r = 0 \\ \omega''(2st - s) & \text{if } \frac{1}{2} < t \le 1, r = 0 \\ \omega'(s) & \text{if } t = 0 \\ \omega''(s) & \text{if } t = 1 \end{cases}$$

and $H \colon I \times I \longrightarrow B$ defined as

$$H(t,r) = \begin{cases} p\omega'(s - 2st + 2rst) & \text{if } 0 \le t \le \frac{1}{2} \\ p\omega'(2rs + 2st - s - 2rst) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

Then F and H are δ'-small multivalued maps such that $d\bigl(pF(t,r), H(t,r)\bigr) < \delta'$ for every $(t,r) \in I \times \{0\} \cup \{0,1\} \times I$. Then by the property MHLP, since there exists a homeomorphism of $I \times I$ into itself sending $I \times \{0\} \cup \{0,1\} \times I$ onto $I \times \{0\}$, there exists an η'-small multivalued map $H' \colon I \times I \longrightarrow E$ such that $d\bigl(H'(t,r), F(t,r)\bigr) < \eta'$ for every $(t,r) \in I \times \{0\} \cup \{0,1\} \times I$ and $d\bigl(pH'(t,r), H(t,r)\bigr) < \eta'$ for every $(t,r) \in I \times I$. In particular, $d\bigl(pH'(t,1), H(t,1)\bigr) = d\bigl(pH'(t,1), p\omega'(s)\bigr) < \eta' < \frac{\eta}{6}$, for every $t \in I$, and since H' is η'-small, then pH' is $\frac{\eta}{6}$-small and hence $H'(t,1) \in p^{-1}\Bigl(B_{\frac{\eta}{3}}\bigl(p\omega'(s)\bigr)\Bigr)$ for every $t \in I$. On the other hand ω'' is δ-small, hence $\operatorname{diam}\bigl(p\omega''(s)\bigr) < \frac{\delta'}{3} < \eta' < \frac{\eta}{6}$ and, since $d\bigl(p\omega'(s), p\omega''(s)\bigr) < \delta < \eta' < \frac{\eta}{6}$, then

$$\omega'(s), \omega''(s) \in p^{-1}\Bigl(B_{\frac{\eta}{3}}\bigl(p\omega'(s)\bigr)\Bigr).$$

Hence we can define $\bar{\omega} \colon I \longrightarrow p^{-1}\Bigl(B_{\frac{\eta}{3}}\bigl(\omega(s)\bigr)\Bigr)$ such that

$$\bar{\omega}(t) = \begin{cases} H'(0,1) \cup \omega'(s) & \text{if } t = 0 \\ H'(t,1) & \text{if } 0 < t < 1 \\ H'(1,1) \cup \omega''(s) & \text{if } t = 1. \end{cases}$$

It is easy to see that $\bar{\omega}$ is an η-small multivalued map.

On the other hand, since $\operatorname{diam}\Bigl(B_{\frac{\eta}{3}}\bigl(p\omega'(s)\bigr)\Bigr) < \eta$ there exist $\{U_1, \ldots, U_n\}$ pairwise disjoint open subsets of E with diameter less than ε such that

$$p^{-1}\Bigl(B_{\frac{\eta}{3}}\bigl(p\omega'(s)\bigr)\Bigr) \subset U_1 \cup \cdots \cup U_n$$

with $d(U_i, U_j) > \eta$ for every $i \neq j$. And since $\bar{\omega}$ is η-small, there exists U_i such that $\bar{\omega}(I) \subset U_i$. In particular, $\omega'(s), \omega''(s) \subset U_i$ and $d\bigl(\omega'(s), \omega''(s)\bigr) < \varepsilon$. This completes the proof of the theorem.

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REMARK 1. The property of p being a shape fibration may be replaced by the weaker condition of p having the property MHLP with respect to I. Moreover, this property is only necessary in the second part of the proof of the theorem. Therefore if a map $p: E \longrightarrow B$ has the near lifting of near multivalued paths property then every non empty fiber is totally disconnected.

The above theorem allows us to easily recognize shape fibrations with the near lifting of near multivalued maps property.

EXAMPLE 1. Let E be a dyadic solenoid and let B be the unit circumference. The canonical projection of E on B is a shape fibration whose fibers are Cantor sets and hence totally disconnected. Therefore p is a shape fibration with the near lifting of near multivalued paths property.

In the following theorem and the examples that follow it, we completely establish the relationship between the concept of shape fibration with the near lifting of near multivalued paths property and the classical concept of fibration with the unique path lifting property.

THEOREM 3. *Let E and B be compact metric spaces and let $p: E \longrightarrow B$ be a shape fibration with the near lifting of near multivalued paths property. Then p is a Hurewicz's fibration with the unique path lifting property.*

PROOF. We have to show that p has the homotopy lifting property with respect to all metric spaces. Let X be a metric space and let $f: X \longrightarrow E$ and $H: X \times I \longrightarrow B$ be maps such that $pf = H_0$.

Let $\{\varepsilon_n\}$ be a decreasing sequence satisfying that $\varepsilon_n < \frac{1}{2^{n+1}}$ and that $2\varepsilon_n$ is associated with $\frac{1}{2^{n+2}}$ by the near lifting of near multivalued paths property, for every $n \in \mathbb{N}$.

By the property MHLP, for every $n \in \mathbb{N}$ there exists $H'_n: X \times I \longrightarrow E$ ε_n-small multivalued map such that $d\big(H'_n(x,0), f(x)\big) < \varepsilon_n$ and $d\big(pH'_n(x,t), H(x,t)\big) < \varepsilon_n$ for every $(x,t) \in X \times I$. Then for every $x \in X$ and every $n \in \mathbb{N}$ we have that $d\big(H'_n(x,0), H'_{n+1}(x,0)\big) < \varepsilon_n + \varepsilon_{n+1} \leq 2\varepsilon_n$ and $d\big(pH'_n(x,t), pH'_{n+1}(x,t)\big) < \varepsilon_n + \varepsilon_{n+1} \leq 2\varepsilon_n$, for every $t \in I$. By the near lifting of near multivalued paths property

$$d\big(H'_n(x,t), H'_{n+1}(x,t)\big) < \frac{1}{2^{n+2}}$$

for every $n \in \mathbb{N}$, for every $x \in X$ and every $t \in I$. Therefore, since H'_{n+1} is ε_{n+1}-small, $H'_{n+1}(x,t) \subset B_{\varepsilon_{n+1}+\frac{1}{2^{n+2}}} H'_n(x,t) \subset B_{\frac{1}{2^{n+1}}} H'_n(x,t)$ and hence

$$\overline{B}_{\frac{1}{2^{n+1}}} H'_{n+1}(x,t) \subset \overline{B}_{\frac{1}{2^n}} H'_n(x,t)$$

for every $n \in \mathbb{N}$, every $x \in X$ and every $t \in I$. Then, for every $(x,t) \in X \times I$, $\{\overline{B}_{\frac{1}{2^n}} H'_n(x,t)\}$ is a decreasing sequence of compact sets with diameter converging to zero. Therefore $H': X \times I \longrightarrow E$ defined by

$$H'(x,t) = \bigcap_{n=1}^{\infty} \overline{B}_{\frac{1}{2^n}} H'_n(x,t),$$

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for every $(x, t) \in X \times I$, is a well defined single-valued function. To see that H' is continuous, consider $(x_0, t_0) \in X \times I$ and $\varepsilon > 0$. Take $n \in \mathbb{N}$ such that $\frac{1}{2^n} < \frac{\varepsilon}{4}$ and $\varepsilon_n < \frac{\varepsilon}{4}$. Then, since $H'(x, t) \in \bar{B}_{\frac{1}{2^n}} H'_n(x, t) \subset B_{\frac{\varepsilon}{4}} H'_n(x, t)$ and $\operatorname{diam}\bigl(H'_n(x, t)\bigr) < \varepsilon_n < \frac{\varepsilon}{4}$, we have that $H'_n(x, t) \subset B_{\frac{\varepsilon}{2}} H'(x, t)$ and hence $B_{\frac{\varepsilon}{4}} H'_n(x, t) \subset B_{\frac{3\varepsilon}{4}} H'(x, t)$ for every $(x, t) \in X \times I$. Take $\delta > 0$ such that $H'_n(x, t) \subset B_{\frac{\varepsilon}{4}} H'_n(x_0, t_0)$ for every $(x, t) \in X \times I$ with $d(x, x_0) < \delta$ and $d(t, t_0) < \delta$. Then

$$H'(x, t) \in B_{\frac{\varepsilon}{4}} H'_n(x, t) \subset B_{\frac{\varepsilon}{4}} B_{\frac{\varepsilon}{4}} H'_n(x_0, t_0) \subset B_{\frac{\varepsilon}{4}} B_{\frac{3\varepsilon}{4}} H'(x_0, t_0) \subset B_{\varepsilon} H'(x_0, t_0)$$

for every $(x, t) \in X \times I$ with $d(x, x_0) < \delta$ and $d(t, t_0) < \delta$. Hence H' is continuous.

Moreover, since $d\bigl(H'_n(x, 0), f(x)\bigr) < \varepsilon_n$ for every $x \in X$, then $H'_n(x, 0) \subset B_{2\varepsilon_n}\bigl(f(x)\bigr)$ and hence

$$H'(x, 0) \in \bar{B}_{\frac{1}{2^n}} H'_n(x, 0) \subset B_{2\varepsilon_n + \frac{1}{2^n}}\bigl(f(x)\bigr)$$

for every $n \in \mathbb{N}$. Therefore $H'(x, 0) = f(x)$ for every $x \in X$.

Finally, for every $(x, t) \in X \times I$ and every $\gamma > 0$, by the continuity of H', there exists $n \in \mathbb{N}$ such that $\varepsilon_n < \frac{\gamma}{2}$ and $p\Bigl(B_{\varepsilon_n + \frac{1}{2^n}}\bigl(H'(x, t)\bigr)\Bigr) \subset B_{\frac{\gamma}{2}}\bigl(pH'(x, t)\bigr)$. On the other hand, since $H'(x, t) \in \bar{B}_{\frac{1}{2^n}} H'_n(x, t)$ then $H'_n(x, t) \subset B_{\varepsilon_n + \frac{1}{2^n}} H'(x, t)$. Therefore,

$$H(x, t) \in B_{\varepsilon_n}\bigl(pH'_n(x, t)\bigr) \subset B_{\varepsilon_n}\Bigl(pB_{\varepsilon_n + \frac{1}{2^n}}\bigl(H'(x, t)\bigr)\Bigr) \subset B_{\gamma}\bigl(pH'(x, t)\bigr).$$

Hence $pH'(x, t) = H(x, t)$ for every $(x, t) \in X \times I$. ∎

The converse of the above theorem is false as shown by the following examples.

EXAMPLE 2. Let K be the pseudoarc [1] and let B be any compact metric space. Consider the projection $p_1 : B \times K \longrightarrow B$ on the first component. Then p_1 is a shape fibration (it is also a Hurewicz's fibration) whose fibers are homeomorphic to K. But K is a continuum such that every path connected component is a point. Therefore p_1 is a shape fibration and a fibration with the unique path lifting property, but it has not the near lifting of near multivalued paths property.

EXAMPLE 3. Let K be the pseudoarc, $E = K \times \{0, 1\}$ and $B = K \vee K$ (considered as the space resultant from E after identifying $(x_0, 0)$ and $(x_0, 1)$ for some point $x_0 \in K$). Let $p : E \longrightarrow K$ be the projection. Then p is a fibration with the near lifting of near multivalued paths property but it is not a shape fibration.

EXAMPLE 4. Let K be the pseudoarc, $E = K \vee K$ and $p : E \longrightarrow K$. Then, since the path connected components of K are points, every homotopy from any topological space to K is fixed. Hence p is a fibration and, since its fibers are discrete, it has the unique path lifting property. However, p is not a shape fibration (see [8, Example 4]), and it has not the near lifting of near multivalued paths property.

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3. **Lifting of homotopic paths.** We show in this section that the near lifting of near multivalued paths property implies the homotopic lifting of homotopic multivalued paths. Moreover, since the liftings of homotopies by a shape fibration are not exact, but approximate, it will prove useful to consider, not only homotopies of paths with extreme points fixed, but also to allow small variations in the homotopies in these extreme points. This fact suggests the following definition.

DEFINITION 2. Let E be a metric space and let $\omega', \omega'' : I^n \longrightarrow E$ be δ-small n-paths, $n \in \mathbb{N}$. We say that ω' and ω'' are ε-homotopic (rel. ∂I^n) if there exists an ε-small multivalued map $H: I^n \times I \longrightarrow X$ such that $H(t,0) = \omega'(t)$ and $H(t,1) = \omega''(t)$ for every $t \in I^n$, and $H(t,s) = \omega'(t) = \omega''(t)$ for every $t \in \partial I^n$ and every $s \in I$. We say that ω' and ω'' are ε-homotopic (rel. $\{\partial I^n; \eta\}$) if there exists an ε-small multivalued map $H: I^n \times I \longrightarrow E$ such that $H_0 = \omega'$, $H_1 = \omega''$ and $\operatorname{diam}\big(H(\{t\} \times I)\big) < \eta$ for every $t \in \partial I^n$.

REMARK 2. It is easy to see that if $\omega', \omega'' : I^n \longrightarrow E$ are ε-small n-paths such that $\omega'|_{\partial I^n} = \omega''|_{\partial I^n}$ and ω' and ω'' are ε-homotopic (rel. $\{\partial I^n; \varepsilon\}$), then ω' and ω'' are ε-homotopic (rel. ∂I^n).

The next result, whose proof is left to the reader, is a technical lemma that we will need in the following sections.

LEMMA 1. *Let E and B be compact metric spaces and let $p: E \longrightarrow B$ be a map with the property* MHLP *with respect to I^n and with the near lifting of near multivalued paths property. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that given $\omega', \omega'' : I^n \longrightarrow E$ δ-small n-paths with $d\big(\omega'(0,\ldots,0), \omega''(0,\ldots,0)\big) < \delta$ and such that $p\omega'$ and $p\omega''$ are δ-homotopic (rel. $\{\partial I^n; \delta\}$), then ω' and ω'' are ε-homotopic (rel. $\{\partial I^n; \varepsilon\}$).*

Moreover, if $\omega'|_{\partial I^n} = \omega''|_{\partial I^n}$ and $p\omega'$ and $p\omega''$ are δ-homotopic (rel. $\{\partial I^n\}$), then ω' and ω'' are ε-homotopic (rel. $\{\partial I^n\}$).

4. **The lifting problem for fine multivalued maps.** The notion of fine multivalued map plays a leading role in shape theory as developed in [10, 11, 12, 4]. Given X and Y compact metric spaces, a fine multivalued map from X to Y is a multivalued map $F: X \times \mathbb{R}_+ \longrightarrow Y$ such that for every $\varepsilon > 0$ there is a $t_0 \in \mathbb{R}_+ = [0, \infty)$ such that $\operatorname{diam}\big(F(x,t)\big) < \varepsilon$ for every $x \in X$ and every $t \geq t_0$. Two fine multivalued maps F and G from X to Y are said to be homotopic if there exists a fine multivalued map $H: X \times [0,1] \times \mathbb{R}_+ \longrightarrow Y$ such that $H(x,0,t) = F(x,t)$ and $H(x,1,t) = G(x,t)$ for every $(x,t) \in X \times \mathbb{R}_+$. F and G are said to be *weakly homotopic* if for every $\varepsilon > 0$ there is a $t_0 \in \mathbb{R}_+$ such that $F|_{X \times [t_0,\infty)}$ and $G|_{X \times [t_0,\infty)}$ are ε-homotopic. In the above mentioned papers, the morphisms in the shape and strong shape categories are expressed as weak homotopy and homotopy classes of fine multivalued maps, respectively.

Given a compact metric space X and given $x \in X$ we can consider, for every $n \in \mathbb{N}$, the strong shape group $\Pi_n^s(X,x)$ and the shape group $\Pi_n(X,x)$, as formulated in [4], in terms of fine multivalued maps. Moreover, if $f: (X,x_0) \longrightarrow (Y,y_0)$ is a map between compact metric spaces then f induces homomorphisms

$$f_*^{hh}: \pi_n(X,x_0) \longrightarrow \pi_n(Y,y_0), \quad f_*^{hs}: \pi_n(X,x_0) \longrightarrow \Pi_n^s(Y,y_0),$$

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$$f_*^{hw}\colon \pi_n(X,x_0)\longrightarrow \Pi_n(Y,y_0),\quad f_*^{ss}\colon \Pi_n^s(X,x_0)\longrightarrow \Pi_n^s(Y,y_0),$$
$$f_*^{sw}\colon \Pi_n^s(X,x_0)\longrightarrow \Pi_n(Y,y_0),\quad f_*^{ww}\colon \Pi_n(X,x_0)\longrightarrow \Pi_n(Y,y_0).$$

It is easy to see using Lemma 1, although we will not make use of this fact, that if $p\colon E\longrightarrow B$ is a shape fibration with the near lifting of near multivalued paths property, then p_*^{hh}, p_*^{ss} and p_*^{ww} are monomorphisms.

On the other hand, a fine multivalued map $F\colon (X,x_0)\times \mathbb{R}_+\longrightarrow (Y,y_0)$ induces homomorphisms

$$F_*^{ww}\colon \Pi_n(X,x_0)\longrightarrow \Pi_n(Y,y_0),\quad F_*^{ss}\colon \Pi_n^s(X,x_0)\longrightarrow \Pi_n^s(Y,y_0),$$
$$F_*^{sw}\colon \Pi_n^s(X,x_0)\longrightarrow \Pi_n(Y,y_0),$$

given by $F_*^{ww}([\omega]_w)=[\sigma]_w$, $F_*^{ss}([\omega])=[\sigma]$ and $F_*^{sw}([\omega])=[\sigma]_w$, where

$$\sigma(t,r)=F\Big(\omega\big(t,\alpha(r)\big),r\Big)$$

with α stretching map (see [4]) associated with the pair (ω,F).

F also induces $F_*^{hs}\colon \pi_n(X,x_0)\longrightarrow \Pi_n^s(Y,y_0)$ and $F_*^{hw}\colon \pi_n(X,x_0)\longrightarrow \Pi_n(Y,y_0)$ given by $F_*^{hs}([\omega])=[\sigma]$ and $F_*^{hw}([\omega])=[\sigma]_w$, where $\sigma(t,r)=F(\omega(t),r)$.

In the following proposition we give necessary conditions, concerning the above homomorphisms, for the existence of a lifting of a fine multivalued map. This lifting problem is to be understood in terms of asymptoticity, where two fine multivalued maps $F,G\colon X\times\mathbb{R}_+\longrightarrow Y$ are asymptotic if for every $\varepsilon>0$ there is a $t_0\in\mathbb{R}_+$ such that $d\big(F(x,t),G(x,t)\big)<\varepsilon$ for every $x\in X$ and every $t\geq t_0$.

PROPOSITION 3. *Let $p\colon (E,e_0)\longrightarrow (B,b_0)$ be a map between compact metric spaces and let $F\colon (X,x_0)\times\mathbb{R}_+\longrightarrow (B,b_0)$ be a fine multivalued map defined in a compact metric space X. Suppose that there exists a fine multivalued map $\tilde{F}\colon (X,x_0)\times\mathbb{R}_+\longrightarrow (E,e_0)$ such that $p\tilde{F}$ and F are asymptotic. Then*

$$F_*^{hs}\big(\pi_n(X,x_0)\big)\subset F_*^{ss}\big(\Pi_n^s(X,x_0)\big)\subset p_*^{ss}\big(\Pi_n^s(E,e_0)\big),$$
$$F_*^{hw}\big(\pi_n(X,x_0)\big)\subset F_*^{sw}\big(\Pi_n^s(X,x_0)\big)\subset p_*^{sw}\big(\Pi_n^s(E,e_0)\big)\subset p_*^{ww}\big(\Pi_n(E,e_0)\big),$$
$$F_*^{sw}\big(\Pi_n^s(X,x_0)\big)\subset F_*^{ww}\big(\Pi_n(X,x_0)\big)\subset p_*^{ww}\big(\Pi_n(E,e_0)\big).$$

The above proposition admits the following converse result, in the case of Peano continua.

THEOREM 4. *Let $p\colon (E,e_0)\longrightarrow (B,b_0)$ be a shape fibration between compact metric spaces, with the near lifting of near multivalued paths property. Let X be a Peano continuum and let $F\colon (X,x_0)\times\mathbb{R}_+\longrightarrow (B,b_0)$ be a fine multivalued map.*

Then there exists a fine multivalued map $\tilde{F}\colon (X,x_0)\times\mathbb{R}_+\longrightarrow (E,e_0)$ such that $p\tilde{F}$ and F are asymptotic if and only if $F_*^{hw}\big(\pi_1(X,x_0)\big)\subset p_*^{ww}\big(\Pi_1(E,e_0)\big)$.

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PROOF. The necessity is part of the above proposition. In order to prove the sufficiency consider a null sequence $\{\varepsilon_n\}$. Since $p^{-1}(x)$ is an empty or totally disconnected compact set, there exists, by the compactness of E and B, a null sequence $\{\eta_n\}$ with $0 < \eta_n < \varepsilon_n$, such that for every $A \subset B$ with $\operatorname{diam}(A) < \eta_n$ there exists a family $\{U_1, \dots, U_r\}$ of pairwise disjoint open subsets of E such that $p^{-1}(A) \subset U_1 \cup \cdots \cup U_r$, where $\operatorname{diam}(U_i) < \varepsilon_n$ for every $i \in \{1, \dots, r\}$ and $d(U_i, U_j) > \eta_n$ for every $i \neq j$.

On the other hand, there exists a null sequence $\{\mu_n\}$ with $0 < \mu_n < \eta_n$ such that given $\tau', \tau'' \colon I \longrightarrow E$ μ_n-small paths with $d\big(\tau'(0), \tau''(0)\big) < \mu_n$ and such that $p\tau'$ and $p\tau''$ are μ_n-homotopic (rel. $\big\{\{0,1\}; \mu_n\big\}$), then τ' and τ'' are η_n-homotopic (rel. $\big\{\{0,1\}; \eta_n\big\}$). Let $\{\psi_n\}$ be a null sequence with $0 < \psi_n < \frac{\mu_n}{5}$ such that $\operatorname{diam}\big(p(K)\big) < \frac{\mu_n}{5}$ for every $K \subset E$ with $\operatorname{diam}(K) < \psi_n$, and consider a null sequence $\{\delta_n\}$ with $\delta_n < \psi_n$ such that δ_n is associated with ψ_n by the property MHLP with respect to I.

Let $F \colon (X, x_0) \times \mathbb{R}_+ \longrightarrow (B, b_0)$ be a fine multivalued map defined in a Peano continuum X. Let $\{k_n\}$ be an unbounded increasing sequence such that $\operatorname{diam}\big(F(x,r)\big) < \delta_n$ for every $(x, r) \in X \times [k_n, \infty)$, and consider a point $(x, r) \in X \times (k_n, k_{n+1}]$, $n \in \mathbb{N}$. We are going to show how to define $\tilde{F}(x, r)$.

Since X is path connected there exists a map $\omega \colon I \longrightarrow X$ such that $\omega(0) = x_0$ and $\omega(1) = x$. Let $\sigma \colon I \longrightarrow B$ be the δ_n-small multivalued map given by $\sigma(t) = F\big(\omega(t), r\big)$. Then σ satisfies that $\sigma(0) = F(x_0, r) = b_0$ and $\sigma(1) = F(x, r)$. Moreover, the fact that $\sigma(0) = b_0 = p(e_0)$ implies the existence of a ψ_n-small multivalued map $\tau \colon I \longrightarrow E$ such that

$$d\big(\tau(0), e_0\big) < \psi_n, \; d\big(p\tau(t), \sigma(t)\big) < \psi_n,$$

for every $t \in I$. In particular $d\big(p\tau(1), F(x,r)\big) < \psi_n$.

Let $\omega' \colon I \longrightarrow X$ be another map such that $\omega'(0) = x_0$ and $\omega'(1) = x$. Let $\sigma' \colon I \longrightarrow B$ be given by $\sigma'(t) = F\big(\omega'(t), r\big)$. Let $\tau' \colon I \longrightarrow E$ be a ψ_n-small multivalued map such that

$$d\big(\tau'(0), e_0\big) < \psi_n, \quad d\big(p\tau'(t), \sigma'(t)\big) < \psi_n,$$

for every $t \in I$. We are going to see that $\operatorname{diam}\big(\tau(1) \cup \tau'(1)\big) < \eta_n$.

Consider the map $(\omega')^- \colon I \longrightarrow X$ given by $(\omega')^-(t) = \omega'(1-t)$. Then $\omega * (\omega')^- \colon I \longrightarrow X$ given by

$$\omega * (\omega')^-(t) = \begin{cases} \omega(2t) & \text{if } 0 \le t \le \frac{1}{2} \\ \omega'(2-2t) & \text{if } \frac{1}{2} \le t \le 1, \end{cases}$$

defines an element $[\omega * (\omega')^-]_h \in \pi_1(X, x_0)$ and hence

$$F_*^{hw}\big([\omega * (\omega')^-]_h\big) \in F_*^{hw}\big(\pi_1(X, x_0)\big) \subset p_*^{ww}\big(\Pi_1(E, e_0)\big) \subset \Pi_1(B, b_0).$$

This implies, since $\operatorname{diam}\Big(F\big(\omega * (\omega')^-(t), r'\big)\Big) < \delta_n$ for every $t \in I$ and every $r' \ge k_n$, that there exists a δ_n-small multivalued map $\kappa \colon (I, \{0,1\}) \longrightarrow (E, e_0)$ such that $\sigma * (\sigma')^-$ and $p\kappa$ are δ_n-homotopic (rel. $\{0,1\}$).

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Consider the multivalued maps $\overline{\tau}\colon I \longrightarrow E$ given by

$$\overline{\tau}(t) = \begin{cases} \tau(0) \cup \{e_0\} & \text{if } t = 0 \\ \tau(t) & \text{if } 0 < t < 1 \\ \tau(1) \cup \tau'(1) & \text{if } t = 1, \end{cases}$$

$\overline{\tau}'\colon I \longrightarrow E$ given by

$$\overline{\tau}'(t) = \begin{cases} \tau'(0) \cup \{e_0\} & \text{if } t = 0 \\ \tau'(t) & \text{if } 0 < t < 1 \\ \tau(1) \cup \tau'(1) & \text{if } t = 1, \end{cases}$$

and $\overline{\kappa}\colon I \longrightarrow E$ given by

$$\overline{\kappa}(t) = \begin{cases} \kappa(0) \cup \tau(0) = \tau(0) \cup \{e_0\} & \text{if } t = 0 \\ \kappa(t) & \text{if } 0 < t < 1 \\ \kappa(1) \cup \tau'(0) = \tau'(0) \cup \{e_0\} & \text{if } t = 1. \end{cases}$$

Then $p\overline{\tau} * (p\overline{\tau}')^{-}$ and $\sigma * (\sigma')^{-}$ are μ_n-homotopic (rel. $\left\{\{0,1\}; \frac{2\mu_n}{5}\right\}$). On the other hand, it is easy to see that $\sigma * (\sigma')^{-}$ and $p\overline{\kappa}$ are μ_n-homotopic (rel. $\left\{\{0,1\}; \frac{2\mu_n}{5}\right\}$). Hence $p\overline{\tau} * (p\overline{\tau}')^{-}$ and $p\overline{\kappa}$ are μ_n-homotopic (rel. $\left\{\{0,1\}; \mu_n\right\}$). Therefore, by Remark 2, $p\overline{\tau} * (p\overline{\tau}')^{-}$ and $p\overline{\kappa}$ are μ_n-homotopic (rel. $\{0,1\}$), and hence $p\overline{\tau}$ and $p\overline{\kappa} * p\overline{\tau}'$ are μ_n-homotopic (rel. $\{0,1\}$).

Consider now $\tilde{\tau}\colon I \longrightarrow E$ given by

$$\tilde{\tau}(t) = \begin{cases} \tau(0) \cup \{e_0\} & \text{if } t = 0 \\ \tau(t) & \text{if } 0 < t \leq 1, \end{cases}$$

and $\tilde{\tau}'\colon I \longrightarrow E$ given by

$$\tilde{\tau}'(t) = \begin{cases} \tau'(0) \cup \{e_0\} & \text{if } t = 0 \\ \tau'(t) & \text{if } 0 < t \leq 1. \end{cases}$$

Then $\tilde{\tau}$ and $\overline{\kappa} * \tilde{\tau}'$ are μ_n-small multivalued maps with $\tilde{\tau}(0) = \overline{\kappa}(0)$ such that $p\tilde{\tau}$ and $p\overline{\kappa} * p\tilde{\tau}'$ are μ_n-homotopic (rel. $\left\{\{0,1\}; \mu_n\right\}$). Therefore $\tilde{\tau}$ and $\overline{\kappa} * \tilde{\tau}'$ are η_n-homotopic (rel. $\left\{\{0,1\}; \eta_n\right\}$). In particular,

$$\operatorname{diam}\big(\tau(1) \cup \tau'(1)\big) < \eta_n.$$

Coming back to the definition of $\tilde{F}$ observe that $p\tau(1) \in B_{\frac{2\mu_n}{5}}\big(F(x,r)\big)$ and hence

$$p^{-1}\Big(B_{\frac{2\mu_n}{5}}\big(F(x,r)\big)\Big) \neq \emptyset.$$

On the other hand, since $\operatorname{diam}\Big(\overline{B_{\frac{2\mu_n}{5}}\big(F(x,r)\big)}\Big) < \mu_n$, we have that

$$p^{-1}\Big(\overline{B_{\frac{2\mu_n}{5}}\big(F(x,r)\big)}\Big) = K_1^{(x,r)} \cup \cdots \cup K_{n_{(x,r)}}^{(x,r)}$$

where $\operatorname{diam}(K_i^{(x,r)}) < \varepsilon_n$ for every $i \in \{1, \ldots, n_{(x,r)}\}$, and $d(K_i^{(x,r)}, K_j^{(x,r)}) > \eta_n$ for every $i \neq j$. Suppose that no $K_i^{(x,r)}$ can be decomposed in union of two subcompacta whose

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distance is bigger than η_n (this is always possible by the compactness of E). It is easy to see that there exists a unique decomposition of $p^{-1}\Big(\overline{B_{\frac{2\mu_n}{5}}\big(F(x,r)\big)}\Big)$ with these properties. On the other hand, since $\tau(1)\cup\tau'(1)\subset p^{-1}\Big(\overline{B_{\frac{2\mu_n}{5}}\big(F(x,r)\big)}\Big)$, there exists $K^{(x,r)}_{j_{(x,r)}}$ such that $\tau(1)\subset K^{(x,r)}_{j_{(x,r)}}$, for all the maps τ obtained according to the above construction.

Define $F'\colon X\times(k_1,\infty)\longrightarrow E$, a first approximation of $\tilde{F}$, by

$$F'(x,r)=K^{(x,r)}_{j_{(x,r)}}.$$

Then F' satisfies that $\operatorname{diam}\big(F'(x,r)\big)<\varepsilon_n$ for every $(x,r)\in X\times(k_n,\infty)$. We are going to see that F' is upper semicontinuous.

Take $(x,r)\in X\times(k_n,k_{n+1}]$ and let V be a neighborhood of $F'(x,r)$ in E. There exists an open neighborhood U of $p^{-1}\Big(\overline{B_{\frac{2\mu_n}{5}}\big(F(x,r)\big)}\Big)$ in E such that $U=U_1\cup\cdots\cup U_{n_{(x,r)}}$, where $\operatorname{diam}(U_i)<\varepsilon_n$ for every $i\in\{1,\ldots,n_{(x,r)}\}$, $d(U_i,U_j)>\eta_n$ for every $i\neq j$, and such that $U_{j_{(x,r)}}\subset V$. On the other hand, there exists $\alpha>0$ such that $\operatorname{diam}\Big(B_\alpha\big(F(x,r)\big)\Big)<\delta_n$ and such that

$$p^{-1}\Big(\overline{B_{\frac{2\mu_n}{5}+\alpha}\big(F(x,r)\big)}\Big)\subset U.$$

If not, there would exist $(y_k)\subset E$ that, by the compactness of E, we can take converging to some $y\in E$, such that for every $k\in\mathbb{N}$ we have $y_k\in p^{-1}\Big(\overline{B_{\frac{2\mu_n}{5}+\frac{1}{k}}\big(F(x,r)\big)}\Big)$ but $y_k\notin U$. But by the first condition, since $\big(p(y_k)\big)$ converges to $p(y)$, $y\in p^{-1}\Big(\overline{B_{\frac{2\mu_n}{5}}\big(F(x,r)\big)}\Big)$ and by the second $y\notin U$ and this is a contradiction.

Take $0<\beta<r-k_n$ such that $F(x',r')\subset B_\alpha\big(F(x,r)\big)$, for every $x'\in X$ with $d(x,x')<\beta$ and every $r'\in\mathbb{R}_+$ with $d(r,r')<\beta$. Let W be a path connected neighborhood of x contained in $B_\beta(x)$ (hence $\operatorname{diam}\Big(F\big(W\times B_\beta(r)\big)\Big)<\delta_n$). Take $x'\in W$ and $r'\in\mathbb{R}_+$ with $d(r,r')<\beta$. Then

$$F'(x',r')\subset p^{-1}\Big(\overline{B_{\frac{2\mu_n}{5}}\big(F(x',r')\big)}\Big)\subset p^{-1}\Big(\overline{B_{\frac{2\mu_n}{5}+\alpha}\big(F(x,r)\big)}\Big)\subset U,$$

and, by the construction of F', there exists $U_{j'}$ such that $F'(x',r')\subset U_{j'}$. We are going to see that $F'(x',r')\subset U_{j_{(x,r)}}\subset V$.

Let $\omega'\colon I\longrightarrow X$ be a map such that $\omega'(0)=x_0$ and $\omega'(1)=x'$. Let $\sigma'\colon I\longrightarrow B$ be given by $\sigma'(t)=F\big(\omega'(t),r'\big)$. Let $\tau'\colon I\longrightarrow E$ be a ψ_n-small (ψ_{n+1}-small if $r'>k_{n+1}$) multivalued map such that $d\big(\tau'(0),e_0\big)<\psi_n,\ d\big(p\tau'(t),\sigma'(t)\big)<\psi_n$, for every $t\in I$.

Since W is path connected, there exists a map $\omega''\colon I\longrightarrow W\subset B_\beta(x)$ such that $\omega''(0)=x'$ and $\omega''(1)=x$. Let $\sigma\colon I\longrightarrow B$ be given by

$$\sigma(t)=\begin{cases}F\big(\omega'(2t),r\big) & \text{if } 0\le t\le\frac{1}{2}\\ F\big(\omega''(2t-1),r\big) & \text{if } \frac{1}{2}\le t\le 1.\end{cases}$$

Then σ and σ' are δ_n-homotopic by a homotopy H such that $H(0,s)=e_0$ for every $s\in I$ and such that $H(\{1\}\times I)\subset F\big(W\times B_\beta(r)\big)$ and hence, since $\operatorname{diam}\Big(F\big(W\times B_\beta(r)\big)\Big)<\delta_n$, σ and σ' are $\frac{\mu_n}{5}$-homotopic (rel. $\big\{\{0,1\};\frac{\mu_n}{5}\big\}$).

Let $\tau: I \longrightarrow E$ be a ψ_n-small multivalued map such that

$$d\big(\tau(0), e_0\big) < \psi_n,\ d\big(p\tau(t), \sigma(t)\big) < \psi_n$$

for every $t \in I$. Then τ and τ' are μ_n-small paths such that $d\big(\tau(0), \tau'(0)\big) < \mu_n$ and such that $p\tau$ and $p\tau'$ are μ_n-homotopic (rel. $\big\{\{0,1\};\mu_n\big\}$). Then τ and τ' are η_n-homotopic (rel. $\big\{\{0,1\};\eta_n\big\}$).

In particular, $\operatorname{diam}\big(\tau(1) \cup \tau'(1)\big) < \eta_n$. And since $\tau(1) \subset U_{j_{(x,r)}}$ and $\tau'(1) \subset U$, then $\tau'(1) \subset U_{j_{(x,r)}}$, and hence $F'(x', r') \subset U_{j_{(x,r)}} \subset V$. Therefore F' is upper semicontinuous.

On the other hand, since for x_0 we can consider $\omega_0: I \longrightarrow X$ such that $\omega_0(t) = x_0$ for every $t \in I$, and $\tau_0: I \longrightarrow E$ given by $\tau_0(t) = e_0$, then $e_0 \in F'(x_0, r)$ for every $r \in (k_1, \infty)$.

Finally F' can be extended to a fine multivalued map $F': X \times \mathbb{R}_+ \longrightarrow E$ such that $e_0 \in F'(x_0, r)$ for every $r \in \mathbb{R}_+$. Moreover, since $p\big(F'(x,r)\big) \subset \overline{B_{\frac{2\mu_n}{5}}\big(F(x,r)\big)}$, for every $(x,r) \in X \times (k_n, \infty)$, then pF' and F are asymptotic.

Let now $\{\phi_n\}$ and $\{\gamma_n\}$ be null sequences such that $F'(\overline{B}_{\gamma_n}(x_0) \times [n, n+1]) \subset \overline{B}_{\phi_n}(e_0)$ for every $n \in \mathbb{N} \cup \{0\}$, and define $F'': X \times \mathbb{R}_+ \longrightarrow E$ such that

$$F''(x,r) = \begin{cases} F'(x,r) & \text{if } (x,r) \notin \bigcup_{n \in \mathbb{N} \cup \{0\}} \overline{B}_{\gamma_n}(x_0) \times [n, n+1] \\ \overline{B}_{\phi_0}(e_0) & \text{if } (x,r) \in \overline{B}_{\gamma_0}(x_0) \times [0,1] \\ \overline{B}_{\phi_n}(e_0) & \text{if } (x,r) \in \overline{B}_{\gamma_n}(x_0) \times (n, n+1]. \end{cases}$$

Then F'' is a fine multivalued map (since $\bigcup_{n \in \mathbb{N} \cup \{0\}} \overline{B}_{\gamma_n}(x_0) \times [n, n+1]$ is closed in $X \times \mathbb{R}_+$) such that $F' \subset F''$ and hence pF'' and F are asymptotic and $e_0 \in F''(x_0, r)$ for every $r \in \mathbb{R}_+$.

Finally define $\tilde{F}: (X, x_0) \times \mathbb{R}_+ \longrightarrow (E, e_0)$ by

$$\tilde{F}(x,r) = \begin{cases} F''(x,r) & \text{if } (x,r) \notin \bigcup_{n \in \mathbb{N} \cup \{0\}} B_{\gamma_n}(x_0) \times [n, n+1) \\ e_0 & \text{if } (x,r) \in B_{\gamma_n}(x_0) \times [n, n+1). \end{cases}$$

Then $\tilde{F}$ is a fine multivalued (since $\bigcup_{n \in \mathbb{N} \cup \{0\}} B_{\gamma_n}(x_0) \times [n, n+1)$ is open in $X \times \mathbb{R}_+$) and satisfies that $\tilde{F} \subset F''$, and hence $p\tilde{F}$ and F are asymptotic. This completes the proof of the theorem. ∎

COROLLARY 1. *Let $p: (E, e_0) \longrightarrow (B, b_0)$ be a surjective shape fibration between compact metric spaces, with the near lifting of near multivalued paths property. Let X be a Peano continuum and let $F: (X, x_0) \times \mathbb{R}_+ \longrightarrow (B, b_0)$ be a fine multivalued map such that*

$$F_*^{hs}\big(\pi_1(X, x_0)\big) \subset p_*^{ss}\big(\Pi_1^s(E, e_0)\big).$$

Then there exists a fine multivalued map $F': (X, x_0) \times \mathbb{R}_+ \longrightarrow (E, e_0)$ such that pF' and F are asymptotic.

PROOF. It suffices to observe that if $F_*^{hs}\big(\pi_1(X, x_0)\big) \subset p_*^{ss}\big(\Pi_1^s(E, e_0)\big)$, then

$$F_*^{hw}\big(\pi_1(X, x_0)\big) \subset p_*^{ww}\big(\Pi_1(E, e_0)\big)$$

holds too. ∎

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REMARK 3. Theorem 4 still holds if p has the property MHLP with respect to I, and if we replace the condition

$$F_*^{hw}\big(\pi_1(X,x_0)\big) \subset p_*^{ww}\big(\Pi_1(E,e_0)\big)$$

by any of the conditions

$$F_*^{ww}\big(\Pi_1(X,x_0)\big) \subset p_*^{ww}\big(\Pi_1(E,e_0)\big) \quad \text{or} \quad F_*^{ss}\big(\Pi_1^s(X,x_0)\big) \subset p_*^{ss}\big(\Pi_1^s(E,e_0)\big).$$

We close this section with the following open question.

PROBLEM 1. Does Theorem 4 hold for arbitrary continua X, if we replace the homotopy group $\pi_1(X,X_0)$ by the shape group $\Pi_1(X,X_0)$?

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https://doi.org/10.4153/CJM-1998-018-7 Published online by Cambridge University Press