TOPOLOGICAL INVARIANTS OF WEIGHTED HOMOGENEOUS POLYNOMIALS

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Let $f:\mathbb{R}^n \to \mathbb{R}$ be a weighted homogeneous polynomial such that $df(0)=0$, let $L=\{x \in S^{n-1} \mid f(x)=0\}$, and let $\chi(L)$ be the Euler characteristic of L. The problem is how to calculate $\chi(L)$ in terms of f.

In the paper we shall show that there are maps $H_1, H_2:(\mathbb{R}^n,0)\to(\mathbb{R}^n,0)$ defined explicitly in terms of f such that $0 \in \mathbb{R}^n$ is isolated in $H_i^{-1}(0)$, $i=1,2$, and

$$\chi(L)=2-(\deg_0(H_1)+\deg_0(H_2)+\chi(S^{n-1})),$$

where $\deg_0(H_i)$ is the topological degree of the map

$$x \mapsto H_i(x)/\|H_i(x)\|$$

from a small sphere centered at the origin to S^{n-1}. If f is a homogeneous polynomial then the above formula is a consequence of results which have been proved in **[6]**, **[7]**, **[8]**.

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We start with a technical lemma.

LEMMA 1. *Let* $\omega_1, \omega_2, f:(\mathbb{R}^n,0)\to(\mathbb{R},0)$ *be polynomials such that* $\omega_1>0$, $\omega_2>0$ *everywhere, except at the origin, and* $\lim \omega_1(x)=\lim \omega_2(x)=+\infty$ *for* $\|x\|\to+\infty$. *Let* $N_r^i=\{x \mid \omega_i(x)=r,\ f(x)\le 0\}$. *If* $r>0$ *is small enough then* $H_*(N_r^1,\mathbb{Z}_2)\cong H_*(N_r^2,\mathbb{Z}_2)$. *In particular* $\chi(N_r^1)=\chi(N_r^2)$.

Proof. Let $M_r^i=\{x \mid 0<\omega_i(x)\le r,\ f(x)\le 0\}$, $i=1,2$. According to the local triviality of polynomial mappings (see **[3]**) there is $r_0>0$ such that $\omega_i: M_{r_0}^i \to (0,r_0]$ is a trivial fibration. Since $\omega_i\ge 0$, $\omega_i^{-1}(0)=\{0\}$, and $\omega_i(x)\to+\infty$ as $\|x\|\to+\infty$, there are constants r_1, r_2 such that $0<r_2<r_1<r_0$ and $M_{r_2}^1 \subset M_{r_1}^2 \subset M_{r_0}^1$. From the triviality of $\omega_1 \mid M_{r_0}^1$, the induced homomorphism $H_*(M_{r_2}^1,\mathbb{Z}_2)\to H_*(M_{r_0}^1,\mathbb{Z}_2)$ is an isomorphism. Hence the induced homomorphism $H_*(M_{r_2}^1,\mathbb{Z}_2)\to H_*(M_{r_1}^2,\mathbb{Z}_2)$ is injective. From the triviality of $\omega_i \mid M_{r_0}^i$ we may deduce that $H_*(M_{r_2}^1,\mathbb{Z}_2)\cong H_*(M_{r_0}^1,\mathbb{Z}_2)$, $H_*(M_{r_1}^2,\mathbb{Z}_2)\cong H_*(M_{r_0}^2,\mathbb{Z}_2)$ and so there is an injection $H_*(M_{r_0}^1,\mathbb{Z}_2)\to H_*(M_{r_0}^2,\mathbb{Z}_2)$. Using similar arguments we may prove that there exists the opposite injection, and then $H_*(M_{r_0}^1,\mathbb{Z}_2)\cong H_*(M_{r_0}^2,\mathbb{Z}_2)$. From the triviality of $\omega_i \mid M_{r_0}^i$ we also know that $N_{r_0}^i$ is a deformation retract of $M_{r_0}^i$, and then $H_*(N_{r_0}^1,\mathbb{Z}_2)\cong H_*(N_{r_0}^2,\mathbb{Z}_2)$. ■

Let $d_1,\ldots,d_n \ge 1$ be positive integers. For every $z=(z_1,\ldots,z_n)\in\mathbb{C}^n$ and $\lambda\in\mathbb{C}$ we shall denote $\lambda . z=(\lambda^{d_1}z_1,\ldots,\lambda^{d_n}z_n)$.

Let $f:\mathbb{R}^n\to\mathbb{R}$ be a weighted homogeneous polynomial such that $df(0)=0$ and $f(\lambda . x)=\lambda^d f(x)$, $d\ge 2$, for every $x\in\mathbb{R}^n$, $\lambda\in\mathbb{R}$. Let p be the smallest positive integer such that $2p>d$ and each d_i divides p.

Let $a_i = p/d_i$ and let

$$\omega = \omega(x_1, \ldots, x_n) = \frac{x_1^{2a_1}}{2a_1} + \ldots + \frac{x_n^{2a_n}}{2a_n}. \tag{1}$$

Clearly $\omega > 0$ except at the origin. Let $g_1 = f - \omega$ and let $I_1 \subset \mathbb{R}[[x_1, \ldots, x_n]]$ be the ideal generated by $\partial g_1/\partial x_1, \ldots, \partial g_1/\partial x_n$. Since $df(0) = 0$, I_1 is a proper ideal.

LEMMA 2. $\dim \mathbb{R}[[x_1, \ldots, x_n]]/I_1 < \infty$.

Proof. Because f is a weighted homogeneous polynomial,

$$\lambda^{d_i} \frac{\partial f}{\partial x_i}(\lambda . z) = \lambda^d \frac{\partial f}{\partial x_i}(z);$$

thus

$$\frac{\partial f}{\partial x_i}(\lambda . z) = \lambda^{d-d_i} \frac{\partial f}{\partial x_i}(z), \tag{2}$$

for every $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, $\lambda \in \mathbb{C}$. Hence

$$\frac{\partial g_1}{\partial x_i}(\lambda . z) = \lambda^{d-d_i}\left(\frac{\partial f}{\partial x_i}(z) - \lambda^{2a_id_i-d} z_i^{2a_i-1}\right).$$

According to (1) we have

$$2a_id_i - d = 2p - d \geq 1.$$

Now it is easy to see that the set

$$\bigcap_{i=1}^{n} \left\{(z, \lambda) \in \mathbb{C}^n \times \mathbb{C} \mid \|z\| = 1, \frac{\partial g_1}{\partial x_i}(\lambda . z) = 0\right\}$$

is compact in $\mathbb{C}^n \times \mathbb{C}$. The continuous map $S^{2n-1} \times \mathbb{C} \ni (z, \lambda) \mapsto \lambda . z \in \mathbb{C}^n$ is onto, so $\{z \in \mathbb{C}^n \mid \operatorname{grad} g_1(z) = 0\}$ is an algebraic, compact, and so finite subset of $\mathbb{C}^n$. This means that $0 \in \mathbb{C}^n$ is an isolated point in the set of critical points of g_1, and so

$$\dim_{\mathbb{R}} \mathbb{R}[[x_1, \ldots, x_n]]/I_1 = \dim_{\mathbb{C}} \mathbb{C}[[z_1, \ldots, z_n]] \Big/ \left(\frac{\partial g_1}{\partial x_1}, \ldots, \frac{\partial g_1}{\partial x_n}\right) < \infty. \quad \blacksquare$$

Let $H_1 = \operatorname{grad} g_1 : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$. According to Lemma 2, $0 \in \mathbb{R}^n$ is isolated in $H_1^{-1}(0)$, and so $\deg_0(H_1)$ is well-defined.

Let $A_1 = \{x \in S^{n-1} \mid f(x) \leq 0\}$. Then we have

LEMMA 3. $\chi(A_1) = 1 - \deg_0(H_1)$.

If f is a homogeneous polynomial of degree d then $d_1 = \ldots = d_n = 1$, $p = [d/2] + 1$ and $\omega = (x_1^{2p} + \ldots + x_n^{2p})/2p$. In that case the above formula has been proved in [**7**].

Proof. Set

$$V = \{(x, r, y) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \mid \omega(x) = r^{2p}, \operatorname{rank}(d\omega(x), df(x)) \leq 1, y = f(x)\}.$$

Thus V is an algebraic subset of $\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}$. Put $\Sigma_r = \{x \in \mathbb{R}^n \mid \omega(x) = r^{2p}\}$, where $r \neq 0$. Clearly Σ_r is a smooth manifold diffeomorphic to S^{n-1}. Let $\pi : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ be the natural projection. If $r \neq 0$ then

$$\pi(V) \cap \{r\} \times \mathbb{R} = \{r\} \times \{\text{critical values of } f \mid \Sigma_r\}.$$

The set of critical values of a polynomial mapping restricted to an algebraic manifold is finite (see [5, p.16]), so $\pi(V) \cap \{r\} \times \mathbb{R}$ is finite.

Let us take $(x, r, y) \in V$ and $\lambda \in \mathbb{R}$. Then

$$\begin{aligned}\omega(\lambda . x) &= \frac{(\lambda^{d_1}x_1)^{2a_1}}{2a_1} + \ldots + \frac{(\lambda^{d_n}x_n)^{2a_n}}{2a_n} \\ &= \frac{\lambda^{2d_1a_1}x_1^{2a_1}}{2a_1} + \ldots + \frac{\lambda^{2d_na_n}x_n^{2a_n}}{2a_n} \\ &= \lambda^{2p}\omega(x) = (\lambda r)^{2p},\end{aligned} \tag{i}$$

$$\begin{aligned}\det\begin{bmatrix} \frac{\partial\omega}{\partial x_i}(\lambda . x) & \frac{\partial\omega}{\partial x_j}(\lambda . x) \\ \frac{\partial f}{\partial x_i}(\lambda . x) & \frac{\partial f}{\partial x_j}(\lambda . x)\end{bmatrix} &= \det\begin{bmatrix} \lambda^{d_i(2a_i-1)}\frac{\partial\omega}{\partial x_i}(x) & \lambda^{d_j(2a_j-1)}\frac{\partial\omega}{\partial x_j}(x) \\ \lambda^{d-d_i}\frac{\partial f}{\partial x_i}(x) & \lambda^{d-d_j}\frac{\partial f}{\partial x_j}(x)\end{bmatrix} \\ &= \det\begin{bmatrix} \lambda^{2p-d_i}\frac{\partial\omega}{\partial x_i}(x) & \lambda^{2p-d_j}\frac{\partial\omega}{\partial x_j}(x) \\ \lambda^{d-d_i}\frac{\partial f}{\partial x_i}(x) & \lambda^{d-d_j}\frac{\partial f}{\partial x_j}(x)\end{bmatrix} \\ &= \lambda^{2p+d-d_i-d_j}\det\begin{bmatrix} \frac{\partial\omega}{\partial x_i}(x) & \frac{\partial\omega}{\partial x_j}(x) \\ \frac{\partial f}{\partial x_i}(x) & \frac{\partial f}{\partial x_j}(x)\end{bmatrix} = 0,\end{aligned} \tag{ii}$$

because $\operatorname{rank}(d\omega(x), df(x)) \leq 1$. Hence $\operatorname{rank}(d\omega(\lambda . x), df(\lambda . x)) \leq 1$ too. We have

$$f(\lambda . x) = \lambda^d f(x) = \lambda^d y. \tag{iii}$$

So, if $(x, r, y) \in V$ then $(\lambda . x, \lambda r, \lambda^d y) \in V$ too. Hence $\pi(V)$ is a finite union of curves and if $(r, y) \in \pi(V)$ and $\lambda \in \mathbb{R}$ then $(\lambda r, \lambda^d y) \in \pi(V)$. Because $2p > d$,

$$|y| > r^{2p} \tag{iv}$$

for every point $(r, y) \in \pi(V)$, $y \neq 0$, sufficiently close to the origin. Set

$$V' = \{(x, r, y) \mid \omega(x) = r^{2p}, \operatorname{rank}(d\omega(x), dg_1(x)) \leq 1, y = g_1(x)\}.$$

Since $g_1 = f - \omega$, we have $\operatorname{rank}(d\omega(x), dg_1(x)) = \operatorname{rank}(d\omega(x), df(x))$, and then

$$V' = \{(x, r, y) \mid \omega(x) = r^{2p}, \operatorname{rank}(d\omega(x), df(x)) \leq 1, y = f(x) - r^{2p}\}.$$

Define $\Theta(r, y) = (r, y - r^{2p})$. Then

$$\pi(V') = \Theta(\pi(V)). \tag{v}$$

Let $N = \{x \in \Sigma_r \mid f(x) \leq 0\}$, $N_1 = \{x \in \Sigma_r \mid g_1(x) \leq 0\} = \{x \in \Sigma_r \mid f(x) \leq \omega(x)\}$. Clearly $N \subset \text{int}(N_1)$.

From (iv) and (v), if $r > 0$ is small enough then the function $g_1 \mid \Sigma_r$ has no critical points in $N_1 - N$. In particular, the function g_1 has an isolated critical point at the origin. The set N is closed, semialgebraic and hence, according to [4], can be triangulated. So N is a deformation retract of N_1, and then $\chi(N) = \chi(N_1)$. Let $M_1 = \{x \mid \|x\| = r, g_1(x) \leq 0\}$. According to [1], [9], $\chi(M_1) = 1 - \deg_0(H_1)$. Of course $M_1 = \{x \mid \|x\|^{2p} = r^{2p}, g_1(x) \leq 0\}$, $N_1 = \{x \mid \omega(x) = r^{2p}, g_1(x) \leq 0\}$ and then from Lemma 1 we have $\chi(M_1) = \chi(N_1)$ (where $\omega_1 = \|x\|^{2p}$, $\omega_2 = \omega$). The polynomial f is weighted homogeneous and so $\chi(A_1) = \chi(N)$. Hence $\chi(A_1) = 1 - \deg_0(H_1)$. ∎

Let $g_2 = -(f + \omega) = -f - \omega$, let $A_2 = \{x \in S^{n-1} \mid f(x) \geq 0\}$, and let $I_2 \subset \mathbb{R}[[x_1, \ldots, x_n]]$ be the ideal generated by $\partial g_2/\partial x_1, \ldots, \partial g_2/\partial x_n$. Using the same arguments as above we can prove that $\dim \mathbb{R}[[x]]/I_2 < \infty$, so $0 \in \mathbb{R}^n$ is isolated in $H_2^{-1}(0)$, where $H_2 = \text{grad}\, g_2 : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$. Clearly $A_2 = \{x \in S^{n-1} \mid -f(x) \leq 0\}$, so from Lemma 3 we have

LEMMA 4. $\chi(A_2) = 1 - \deg_0(H_2)$. ∎

Let $L = \{x \in S^{n-1} \mid f(x) = 0\}$. Since $L = A_1 \cap A_2$ and $S^{n-1} = A_1 \cup A_2$, we have $\chi(L) = \chi(A_1) + \chi(A_2) - \chi(S^{n-1})$; thus

THEOREM 5. $\chi(L) = 2 - (\deg_0(H_1) + \deg_0(H_2) + \chi(S^{n-1}))$. ∎

If d is odd then $\tau : \tau(x) = (-1) . x$ is an involution on S^{n-1} such that $f(\tau(x)) = (-1)^d f(x) = -f(x)$. It follows that $\tau(A_1) = A_2$ and then A_1 is homeomorphic to A_2. From Lemmas 3, 4 and Theorem 5 we have

COROLLARY 6. *If d is odd then*

$$\chi(L) = 2(1 - \deg_0(H_1)) - \chi(S^{n-1}). \quad ∎$$

According to Lemma 2, dimensions of local algebras associated to H_1, H_2 are finite, so we may compute $\deg_0(H_i)$ using the Eisenbud-Levine algorithm (see [2]). In the following examples we shall apply a computer program by Andrzej Łęcki from the Institute of Mathematics in Gdańsk which is able to calculate $\deg_0(H_i)$ using that algorithm.

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EXAMPLE 1. Let $f(x, y, z) = x^2y - y^4 - yz^3$. The polynomial f is weighted homogeneous, where $d_1 = 3$, $d_2 = d_3 = 2$, $d = 8$. Clearly $p = 6$ has the property (1), and so $a_1 = 2$, $a_2 = a_3 = 3$. Hence $\omega = x^4/4 + y^6/6 + z^6/6$, and

$$H_1 = \text{grad}(f - \omega) = (2xy - x^3, x^2 - 4y^3 - z^3 - y^5, -3yz^2 - z^5),$$
$$H_2 = \text{grad}(-f - \omega) = (-2xy - x^3, -x^2 + 4y^3 + z^3 - y^5, 3yz^2 - z^5).$$

Thanks to the computer program by Andrzej Łęcki we have been able to calculate that $\deg_0(H_1) = \deg_0(H_2) = 1$. From Theorem 5, $\chi(L) = -2$. Since

$$L = \{(x, y, z) \in S^2 \mid y(x^2 - y^3 - z^3) = 0\}$$

is homeomorphic to a union of two circles with two common points, the solution is correct.

EXAMPLE 2. Let $f(x, y, z) = x^3 + x^2z - y^2$. The polynomial f is weighted homogeneous, where $d_1 = d_3 = 2$, $d_2 = 3$, $d = 6$. Then $p = 6$, $a_1 = a_3 = 3$, $a_2 = 2$ and $\omega = x^6/6 + y^4/4 + z^6/6$. Hence

$$H_1 = \text{grad}(f - \omega) = (3x^2 + 2xz - x^5, -2y - y^3, x^2 - z^5),$$
$$H_2 = \text{grad}(-f - \omega) = (-3x^2 - 2xz - x^5, 2y - y^3, -x^2 - z^5).$$

A computer has calculated that $\deg_0(H_1) = 1$, $\deg_0(H_2) = -1$, and so $\chi(L) = 0$. The reader may easily check that the solution is correct.

EXAMPLE 3. Let $f(x, y, z) = x^3 - xy^2 + xyz + 2x^2y - 2y^3 - y^2z - xz^2 + yz^2$. The polynomial f is homogeneous of degree $d = 3$, so $d_1 = d_2 = d_3 = 1$, $p = 2$, $a_1 = a_2 = a_3 = 2$ and $\omega = (x^4 + y^4 + z^4)/4$. Hence $H_1 = (3x^2 - y^2 + yz + 4xy - z^2 - x^3,\ -2xy + xz + 2x^2 - 6y^2 - 2yz + z^2 - y^3,\ xy - y^2 - 2xz + 2yz - z^3)$. A computer has calculated that $\deg_0(H_1) = 3$. According to Corollary 6, $\chi(L) = -6$. The reader may check that $f = (x - y)(x + y + z)(x + 2y - z)$, so the solution is correct.

REFERENCES

1. V. I. Arnold, Index of a singular point of a vector field, the Petrovski-Oleinik inequality, and mixed hodge structures, *Functional Anal. Appl.* **2** (1978), 1–11.

2. D. Eisenbud and H. I. Levine, An algebraic formula for the degree of a C^∞ map germ, *Ann. of Math. (2)* **7** (1977), 19–44.

3. R. Hardt, Semi-algebraic local triviality in semi-algebraic mappings, *Amer. J. Math.* **102** (1980), 291–302.

4. S. Łojasiewicz, Triangulation of semi-analytic sets, *Ann. Scuola Norm. Sup. Pisa (3)* **18** (1964), 449–474.

5. J. Milnor, *Singular points of complex hypersurfaces* (Princeton University Press, 1968).

6. Z. Szafraniec, On the Euler characteristic of analytic and algebraic sets, *Topology* **25** (1986), 411–414.

7. Z. Szafraniec, On the Euler characteristic of complex algebraic varieties, *Math. Ann.* **280** (1988), 177–183.

8. Z. Szafraniec, On the Euler characteristic mod 2 of real projective varieties, *Math. Proc. Cambridge Philos. Soc.* **104** (1988), 479–481.

9. C. T. C. Wall, Topological invariance of the Milnor number mod 2, *Topology* **22** (1983), 345–350.

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