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R-SEPARATING SETS

R. E. GOMORY, T. C. HU, AND J. M. YOHE

This study of r-separating sets was originally motivated by the maximum-flow-minimum-cut theorem of finite networks [**1**; **2**]. In working toward a continuous version of the maximum-flow-minimum-cut theorem from the usual discrete version, one is led directly to the notion of r-connectedness of a set as defined below. This notion of r-connectedness also has a very simple intuitive interpretation. Intuitively speaking, a set of points in the plane is r-connected if a person starting at any one point of the set is able to reach any other point of the set by jumping from point to point within the set, but never jumping a distance exceeding r in any one jump.

The notion of connectedness by jumps or steps is sometimes taken as a characterization of ordinary connectedness when the step size is allowed to become arbitrarily small; see Newman [**3**, p. 81].

More precisely, two points a and b in a set S are said to be r-connected if there is a finite sequence of points $a = p_0, p_1, \ldots, p_n = b$ with $p_i \in S$ and the distance $\rho(p_i, p_{i+1}) \leqq r$, $i = 0, \ldots, n - 1$. In this paper we will develop properties related to r-connectedness. We will deal mainly with the notion of r-separation (two points in a set S are r-separated if they are not r-connected) and with planar r-separating sets, which are, roughly, sets C whose removal from the plane R_2 r-separates two points in $R_2 - C$. The prototype of such sets might be an annulus which separates a from b. However, much more complicated r-separating sets are also possible (Figure 1). The study of r-separating sets also has interesting connections with the minimal surface problem; see, for example, [**2**, Chapter 12].

Of course, r-separating sets, as described, can have few interesting properties since almost any sufficiently large set will do. However, the set shown in Figure 1 has an additional property: it is irreducible; i.e., it contains no proper r-separating subset. It is the irreducible r-separating sets, which have a very detailed structure, that will be described below.

In Section 2 we will develop the general properties of irreducible r-separating sets. Among other things, we will prove that their boundaries are always well behaved. With this established we will be able, in Section 3, to exhibit a much more detailed structure. We will show, roughly, that all irreducible r-separating sets consist of simple tube-like sections of width r (such as the T_i in Figure 1) hooked together by polyhedra each having an even number of sides of length r (such as the P_i in Figure 1).

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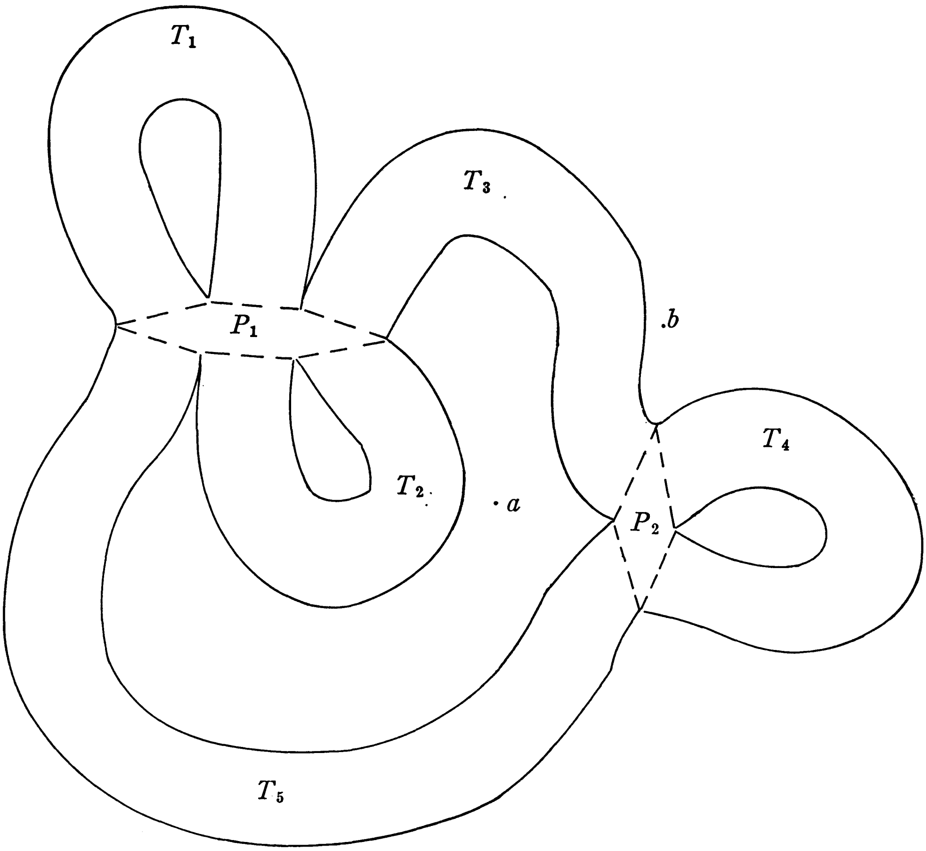

FIGURE 1

2. General properties of irreducible r-separating sets. We now turn to more exact definitions. Using $\rho(p, q)$ for the Euclidean distance between two points in the plane R_2, we will say that the sequence of points $p_0, p_1, \ldots, p_n$ forms an r-chain from p_0 to p_n if $\rho(p_i, p_{i+1}) \leqq r$. We will say that a and b are r-connected in a set S if there is an r-chain with the properties

(i) $p_0 = a$, $p_n = b$, and

(ii) $p_i \in S$ for all i.

If p, q are two points of R_2, then by $\overline{pq}$ we will mean the line segment from p to q. If $p_0, \ldots, p_n$ is an r-chain, we will also use r-chain to denote the path consisting of

$$\bigcup_{i=0}^{n-1} \overline{p_i p_{i+1}}.$$

The context should resolve any ambiguities. We will say that a set $C \subset R_2$ r-separates a and b if

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(i) C is closed and bounded,

(ii) a and b are not r-connected in $R_2 - C$.

We will be able to see in retrospect that condition (i) of this definition does not meaningfully affect the structure of the separating sets, but it does facilitate the analysis.

The r-component of a point a in a metric space S is the set of all points which can be r-chained to a in S. If C r-separates a and b in R_2, we will use A and B to designate the r-components of a and b, respectively, in $R_2 - C$.

An irreducible r-separating set C is defined as one that contains no r-separating set as a proper subset. It is the structure of these irreducible sets that will be analyzed.

THEOREM 1. *Every r-separating set contains an irreducible r-separating set.*

Theorem 1 follows by routine arguments from the following easily established lemma:

LEMMA 1. *If $\mathscr{C}$ is any collection of nested r-separating sets, i.e., for any C, $C' \in \mathscr{C}$ we have either $C' \subset C$ or $C \subset C'$, then*

$$C^* = \bigcap_{C \in \mathscr{C}} C$$

is an r-separating set.

A further useful property of C is given by Theorem 2, which we state without proof:

THEOREM 2. *If p belongs to the irreducible r-separating set C, then $\rho(p, A) \leqq r$ and $\rho(p, B) \leqq r$.*

Theorem 3 is somewhat analogous to the Jordan curve theorem in that it asserts that the removal of an irreducible r-separating set separates the plane into two r-connected open sets.

THEOREM 3. *If C is an irreducible r-separating set which r-separates a and b, then $R_2 - C = A \cup B$.*

Proof. Suppose U is a component of $R_2 - C$ which does not intersect $A \cup B$, and let D be a circular disk such that $\operatorname{Int} D \subset U$ and $F(D)$ contains a point p of $F(U)$. By Theorem 2, there is a point $a' \in \bar{A}$ such that $\rho(a', p) = r$; then $N_r(a')$ contains no point of $\operatorname{Int} D$, so $N_r(a')$ and D are externally tangent at p. Similarly, there is a point $b' \in \bar{B}$ such that $N_r(b')$ is externally tangent to D at p. This implies that $N_r(a') = N_r(b')$ and hence $a' = b'$, a contradiction.

We now know that an irreducible C splits R_2 into two r-connected sets, A and B, with nothing left over. We next approach the problems of obtaining more detailed properties of C. This is done by a detour. We will establish that $F(A)$ and $F(B)$ are well behaved; this will then give information on $F(C)$. Then with $F(C)$ established as well behaved it will become possible to establish more detailed properties of C.

LEMMA 2. *Let* $\overline{p_1, p_2}$ *and* $\overline{q_1, q_2}$ *be two intersecting closed line segments, both of length* $< r$. *Then for some one of the four end points, say* p_1, *either*

(1) p_1 *is an intersection point and it coincides with either* q_1 *or* q_2, *or*
(2) *the distances* $\rho(p_1, q_1)$ *and* $\rho(p_1, q_2)$ *are both* $< r$.
In either case, the distances $\rho(p_1, q_1)$ *and* $\rho(p_1, q_2)$ *are* $\leqq r$ *so that* p_1 *is within distance* r *of all the end points.*

Proof. Assume (by relabeling, if necessary) that p_1 is an end point nearest the point intersection. The conclusion follows easily from elementary geometrical considerations.

The purpose of the next lemma is to enable us to deal with r-chains that do not cross themselves. An r-chain $p_0, p_1, \ldots, p_n$ is said to cross itself if the path consisting of the union of segments $\overline{p_i p_{i+1}}$, $i = 0, \ldots, n-1$ is not an arc.

LEMMA 3. *If there is an r-chain* $p = p_0, p_1, \ldots, p_n = q$ *from a point* p *to a point* q, *then there is a second r-chain from* p *to* q, *whose vertices are a subset of the original* p_i, *which does not cross itself.*

The proof of this lemma is elementary, and we omit it.

An important property of the sets A and B which prevents them from becoming too unruly is given in Theorem 4.

THEOREM 4. *Let* p_1 *and* p_2 *be points of one component of* A *with* $\rho(p_1, p_2) \leqq r$. *With radius* r *draw two circular arcs* α_1 *and* α_2 *through both points. Each arc should be* $\leqq \pi r/3$ *in length. Then there is a path* P^* *from* p_1 *to* p_2 *that lies entirely in the closed sector bounded by* α_1 *and* α_2 *and consists only of points of* A (Figure 2).

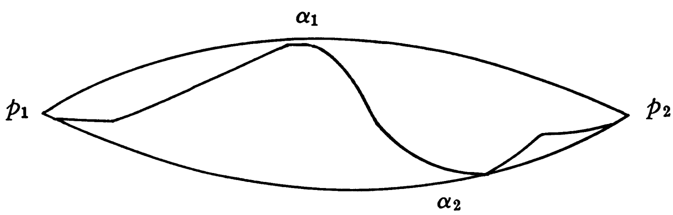

FIGURE 2

Proof. Since p_1 and p_2 belong to the same component of the open set A, there is some simple path P from p_1 to p_2 in A (Figure 3). This path can and will be taken to consist of straight line segments of length $\leqq r$.

We will proceed to analyze a special case from which the general theorem can be deduced.

We consider the case in which P does not touch the sector bounded by α_1 and α_2 except at p_1 and p_2. We can assume, without loss of generality, that b lies in the outer domain D_0 of the Jordan curve J formed from P and the segment $\overline{p_1 p_2}$.

Either α_1 or α_2 must lie inside J. Let us assume that the arc lying inside is α_2 as in Figure 3. Since all of α_2 lies within r of p_1, α_2 can consist only of points of

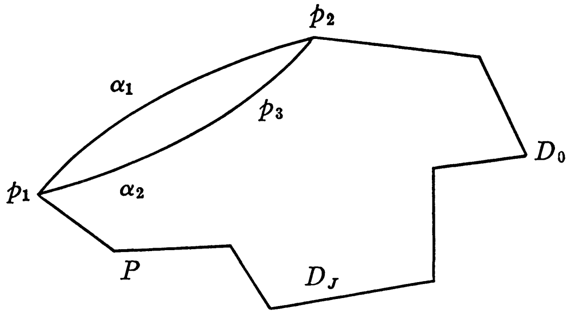

FIGURE 3

A or of C. If there are no C points on α_2, then the theorem holds; so let us suppose there is a point $p_3 \in C \cap \alpha_2$. Since $p_3 \in C$, there is a segment of length $< r$ connecting p_3 to a point q of B or of $F(B)$. Thus there is an r-chain $p_3 = q_0, q_1, \ldots, q_n = b$. This r-chain must cross J. If a segment of this chain crossed a segment of J other than $\overline{p_1p_2}$, we would have a point of A (namely, the intersection point) at a distance $< r$ from $\bar{B}$. Thus $\overline{p_3q}$ must cross $\overline{p_1p_2}$ as shown in Figure 4.

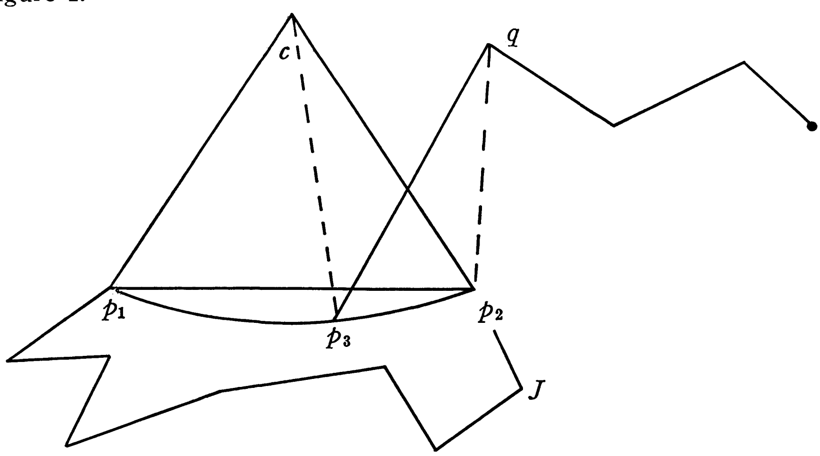

FIGURE 4

Let c be the center of the circle of which α_2 is an arc. q cannot lie in or on the isoscles triangle p_1cp_2, for then its distance from p_1 and p_2 would be r or less, but p_1 and p_2, as interior points of A, have distance $> r$ from $B \cup F(B)$. So q lies outside p_1cp_2 and the edge $\overline{p_3q}$ intersects either the edge $\overline{p_1c}$ or the edge $\overline{p_2c}$. Let us assume it is $\overline{pc_2}$. Then, applying Lemma 2 to the edges of $\overline{p_3q}$ and $\overline{p_2c}$, we find that neither q nor p_3 can coincide with c or p_2. Moreover, $\overline{p_3c}$ has length exactly r and $\overline{qp_2}$ must have length $\geqq r$, so none of the four vertices c, q, p_2 or p_3 can be within distance $< r$ of the two vertices of the opposite edge. Thus the existence of such a q contradicts Lemma 2 and we conclude that $p_3 \notin C$.

Since we have shown that $\alpha_2 \subset A$, it forms the P^* of our theorem in the special case we have been considering.

We next turn to the general case in which P intersects α_1 or α_2 (Figure 5).

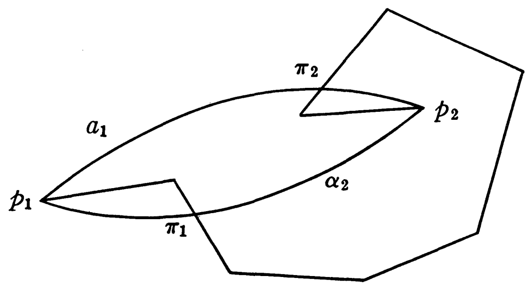

FIGURE 5

In this case P^* is formed by using the segmented path P up to the point π_1 where it first leaves the sector. All P is in A so in particular π_1 and the point of next return to the sector π_2 are in A. By connecting π_1 and π_2 by arcs of radius r we create the situation of the special case with π_1 and π_2 playing the role of p_1 and p_2. Consequently, one of these new arcs α' is in A and carries P^* up to π_2. If P does not go out of the sector again, then P^* is completed with the remainder of P; otherwise, the process is repeated. Since P consists of a finite number of straight line segments, there will only be a finite number of intersections with α_1 and α_2. Therefore, after a finite number of repetitions p_2 will be reached and the path P^* fully constructed. This establishes the theorem.

We are now in a position to say something about $F(A)$ and $F(B)$ and hence about $F(C)$.

THEOREM 5. *The boundary of each component of A or B is a simple closed curve.*

Proof. The proof is based on a converse to the Jordan curve theorem. A version by Newman [**3**, p. 166] asserts that if a domain in Z_2 (the two dimensional projective plane) is simply connected and uniformly locally connected, then its frontier is a simple closed curve, a point, or null. We will proceed to show that the components of A and B meet the conditions of this theorem. Theorem 4 shows that the components of A and B are uniformly locally connected, so it remains only to show simple connectedness. Consider a component of A_0 of A. If we draw any simple closed curve J in A_0, all of $B \cup C$ must be either in its outer domain R_0 or its inner domain R_I, since any attempt to split B between R_0 and R_I leads to the usual difficulties with some connecting chain crossing J which is in A, and if even one point of C were to lie in the other domain, it would require a point of B within distance r. Without loss of generality, suppose $(B \cup C)$ is in R_0; then J can be contracted in R_I; so A_0 is simply connected.

Thus, the theorem applies and each component of A (or B) has as its frontier a simple closed curve or a point or the null set. Possibilities other than the curve are easily ruled out; this establishes the theorem.

3. Structure of irreducible r-separating sets. With something now established about the regularity of A, B and C and their boundaries, we turn to a more complete analysis of the structure of C.

Let us define a *connector* to be a closed segment of length r connecting $F(A)$ and $F(B)$. Then it is easy to prove

THEOREM 6. *For every point p on $F(A)$ (or $F(B)$) there exists at least one connector with p as one of its ends and the other end a point of $F(B)$ (or $F(A)$). Furthermore, all points on the connector other than the endpoints belong to $C - F(C)$.*

Note that this theorem does not imply that every point of C must be on a connector.

LEMMA 4. *Two connectors $\overline{p_2q_2}$ and $\overline{p_1q_1}$ with distinct end points can have no points in common.*

Proof. If two connectors have a point of intersection other than an end point, then it follows from Lemma 2 that there exists a point $p_1 \in F(A)$ with $\rho(p_1, p_2) < r$ and $\rho(p_1, q_2) < r$. This contradicts the assumption that A and B are r-separated.

Let $C_1 \subset C$ be the set of all points of C which lie on a connector, and let $C_2 = C - C_1$.

LEMMA 5. *C_1 is closed and C_2 is open.*

The proof of Lemma 5 is elementary, and we omit it.

LEMMA 6. *If $\overline{p_0q_0}$ is a connector, and $x_0 \in \operatorname{Int} \overline{p_0q_0}$ is an interior point of C_1, then all interior points of $\overline{p_0q_0}$ are interior points of C_1.*

Proof. (See Figure 6.) Since x_0 is interior to C_1, there exists an $\epsilon > 0$ such that $N_\epsilon(x_0) \subset C_1$. Suppose $y \in \operatorname{Int} \overline{p_0q_0}$ is not an interior point of C_1. Then there exists a sequence $\{y_i\} \to y$ with $y_i \notin C_1$; we may suppose that all of these points lie in one of the two half-planes determined by the line containing $\overline{p_0q_0}$, say H_0.

Pick $x_1 \in N_\epsilon(x_0) \cap H_0$. Then $\overline{x_0x_1} \subset N_\epsilon(x_0)$; we parameterize $\overline{x_0x_1}$ by α, $0 \leqq \alpha \leqq 1$. For each x_α we have a connector $\overline{p_\alpha q_\alpha}$ such that $\overline{p_\alpha q_\alpha} \cap \overline{x_0x_1} = x_\alpha$.
Now there exists a δ such that

$$N_\delta(y) \cap (\{p_0, q_0\} \cup N_\epsilon(x) \cup \overline{p_1q_1} = \emptyset.$$

Moreover, we want δ so small that every connector which intersects $N_\delta(y)$ must run through $N_\epsilon(x_0)$.

This choice of δ guarantees that no point of $N_\delta(y)$ may belong to A or B, since this would contradict r-separation. Hence all $y_i \in N_\delta(y)$ are C_2-points.

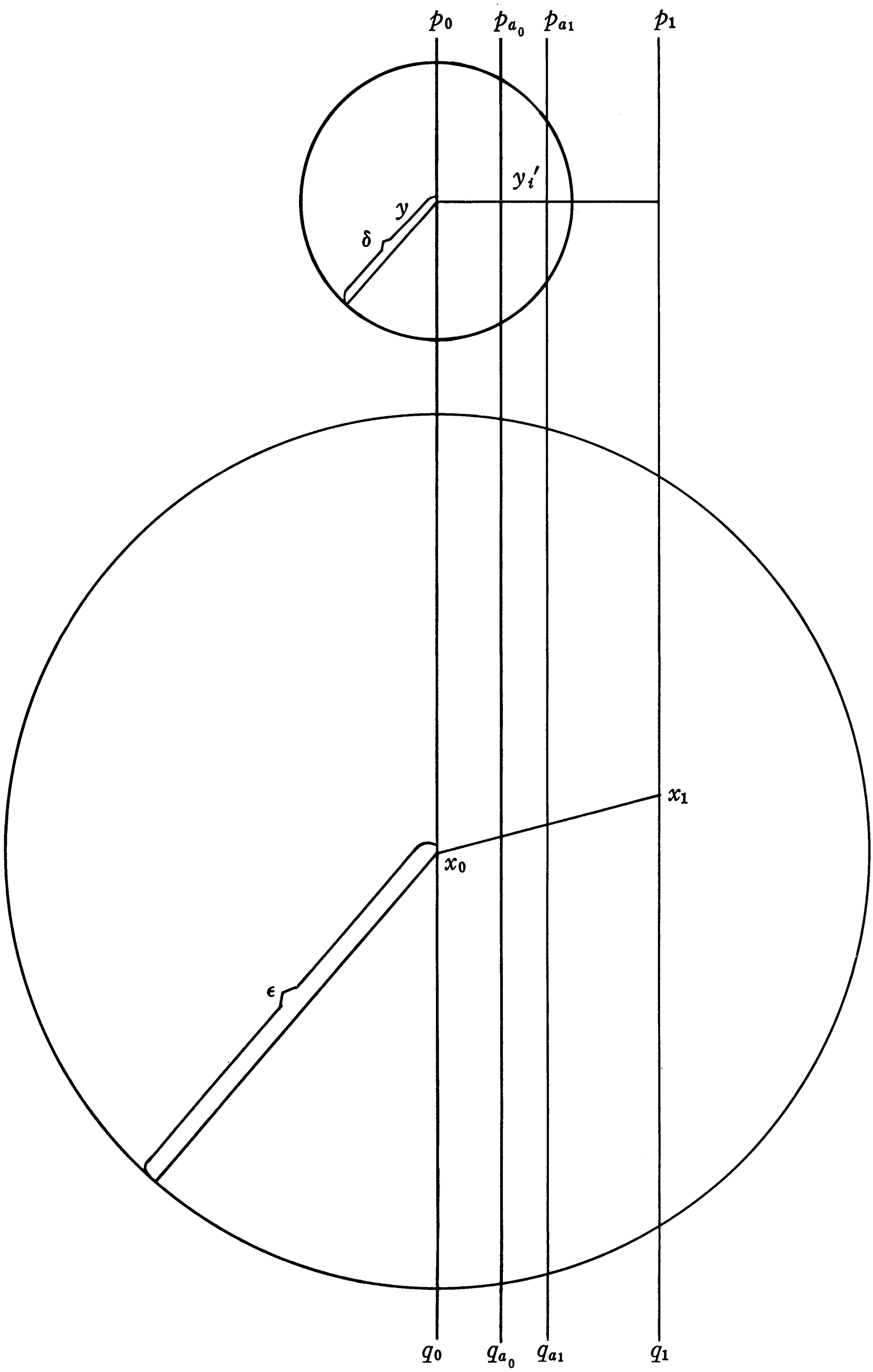

FIGURE 6

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Pick any such y_i, and from it draw a perpendicular line to $\overline{p_0q_0}$. Let the first connector this line hits be $\overline{p_{\alpha 0}, q_{\alpha 0}}$; similarly determine $\overline{p_{\alpha 1}q_{\alpha 1}}$ (these connectors must be $\overline{p_\alpha q_\alpha}$'s, since in the contrary case we will contradict r-separation).

Now $\overline{p_{\alpha 1}q_{\alpha 1}}$ and $\overline{p_{\alpha 0}q_{\alpha 0}}$ may not cross; moreover, no connectors are sandwiched between these two. But then the gap between these connectors near y_t cannot be closed by the time we reach $\overline{x_0x_1}$. The contradiction proves the lemma.

LEMMA 7. *Let $\tilde{C}$ be a component of C_2. Then each point of $F(\tilde{C})$ lies on a connector which lies on $F(\tilde{C})$.*

Proof. We first show that if a connector has an interior point on $F(\tilde{C})$ then the entire connector lies on $F(\tilde{C})$. By Lemma 6, we know that such a connector lies on $F(C_2)$. If a connector were to lie partially, but not completely, on $F(\tilde{C})$, then, since $F(\tilde{C}) \subset F(C_2) \subset C_1$, we would have two connectors which cross, contradicting Lemma 4.

Now suppose there is an $x \in F(\tilde{C})$ which does not belong to any connector which lies on $F(\tilde{C})$. Then x is the endpoint of a connector; moreover, there is a neighborhood N_x of x such that any connector which intersects N_x contains no boundary point of $\tilde{C}$ on its interior. For, supposing that no such neighborhoods exists, we can find a sequence of points converging to x such that each belongs to a connector containing a boundary point of $\tilde{C}$ on its interior, and therefore lying on $F(\tilde{C})$. These connectors converge to a connector containing x; this connector also lies on $F(\tilde{C})$.

We may suppose without loss of generality that $F(C) \cap N_x$ is an arc. Thus we can find a neighborhood N_x of x such that N_x is divided in two pieces by an arc on $F(C_2)$, where one of these pieces is in, say, A and the other is in C; moreover, each point p of $(C_1 - F(C)) \cap N_x$ is an interior point of $C - \tilde{C}$. Since components of C_2 are open, we can conclude that each point of $(C - \tilde{C} - F(C)) \cap N_x \subset F(C)$. Then we must have $F(\tilde{C}) \cap N_x = F(C) \cap N_x$. But this is impossible, since this implies x does not belong to a connector.

LEMMA 8. *If $p \in F(A)$ belongs to two connectors K_1 and K_2, and there exists an $\epsilon > 0$ such that $N_\epsilon(p) \cap C$ contains a sector S bounded by both $K_1 \cap C$ and $K_2 \cap C_2$, then the angle at the vertex of S is $\leqq \pi$.*

The proof of Lemma 8 is an elementary exercise in geometry.

LEMMA 9. *Each component of C_2 is convex.*

Proof. Suppose x, y belong to the same component $\tilde{C}$ of C_2, but $\overline{xy} \not\subset \tilde{C}$. Let $x', y' \in \overline{xy}$ be such that $\overline{x'y'} \cap \tilde{C} = \emptyset$. There is a path α on $F(\tilde{C})$ which connects x' and y' such that the interior of the simple closed curve $\overline{x'y'} \cup \alpha$ lies outside of $\tilde{C}$. The path α consists of connectors (and portions of connectors) by Lemma 7; it follows that there must be a pair of adjacent connectors K_i, K_{i+1}, an $\epsilon > 0$, and a sector S of $N_\epsilon(p) \cap C$ such that the angle at the vertex of S is greater than π. This contradicts Lemma 8. (See Figure 7.)

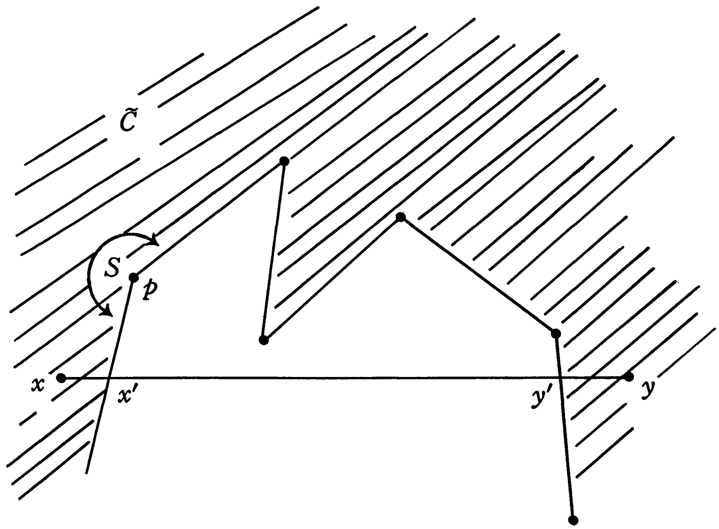

FIGURE 7

LEMMA 10. *The closure of each component of C_2 is a convex polyhedron with an even number of sides each of length r.*

Proof. Since C is compact, the closure of a component $\tilde{C}$ of C_2 is closed and bounded. By Lemma 7, the boundary of $\tilde{C}$ consists of connectors, each of which is of length r. By Lemma 9, $\tilde{C}$ is convex and hence so is its closure. Hence $\tilde{C}$ can have only finitely many sides. Since vertices of $F(\tilde{C})$ are alternately in A and B, it follows that there must be an even number of sides.

Definition. A *nondegenerate tube of width r* is the closure of a differentiable embedding h of $[0, 1] \times (-r/2, r/2)$ or $S^1 \times (-r/2, r/2)$ into R_2 which is

(a) an isometry on the second factor,

(b) such that for each $\alpha \in [0, 1]$, $\{\alpha\} \times (-r/2, r/2)$ is normal to $h([0, 1] \times \{0\})$, and

(c) each point of

$$\overline{h([0, 1] \times (-r/2, r/2))} - h([0, 1] \times (-r/2, r/2))$$

is a boundary point of $R_2 - h([0, 1] \times (-r/2, r/2))$.

A *tube of width r* is either a nondegenerate tube of width r or a connector.

THEOREM 7. *If C is an irreducible r-separating set, then $C = C_1 \cup C_2$, where*

(a) *the closure of each component of $C_1 - F(C)$ is a tube of width r which has one boundary component on A and the other on B,*

(b) *each component of C_2 is the interior of a convex polyhedron with an even number of sides, each of length r, which intersects $F(C)$ only in its vertices.*

Proof. Each point of C_1 lies on a connector by the definition of C_1. If $\tilde{C}_1$ is a component of $C_1 - F(C)$, then $\tilde{C}_1$ consists either of the interior of a single connector, in which case there is nothing to prove, or, since no two connectors

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may intersect except in a point of $F(C)$, an interval (or circle) of connectors. Certainly since each connector has length r, we may parameterize the interior of each connector via an isometry of $(-r/2, r/2) \rightarrow R_2$. By deciding in advance to map the positive side of this interval toward A, say, we may assure the possibility of a cohesive array of intervals. That these mappings may actually be extended to a differentiable embedding of $[0, 1]$ (or S) $\times$ $(-r, r)$ follows from the smoothness of the center line (the radius of curvature of the center line must be $\geqq r/2$ at each point) and the fact that the isometries match up there.

The normality condition follows from the fact that $\rho(a, B) = r$ for all $a \in \text{Bd } A$.

The properties of components of C_2 then follow from Lemma 10, Lemma 7 and the fact that interior points of connectors are also interior points of C. The theorem is proved.

We note that we may actually have an infinite number of C_2-components (Figure 8) and that these components may get arbitrarily near one another; note also that a connector which is on $F(C_2)$ need not be on the boundary of

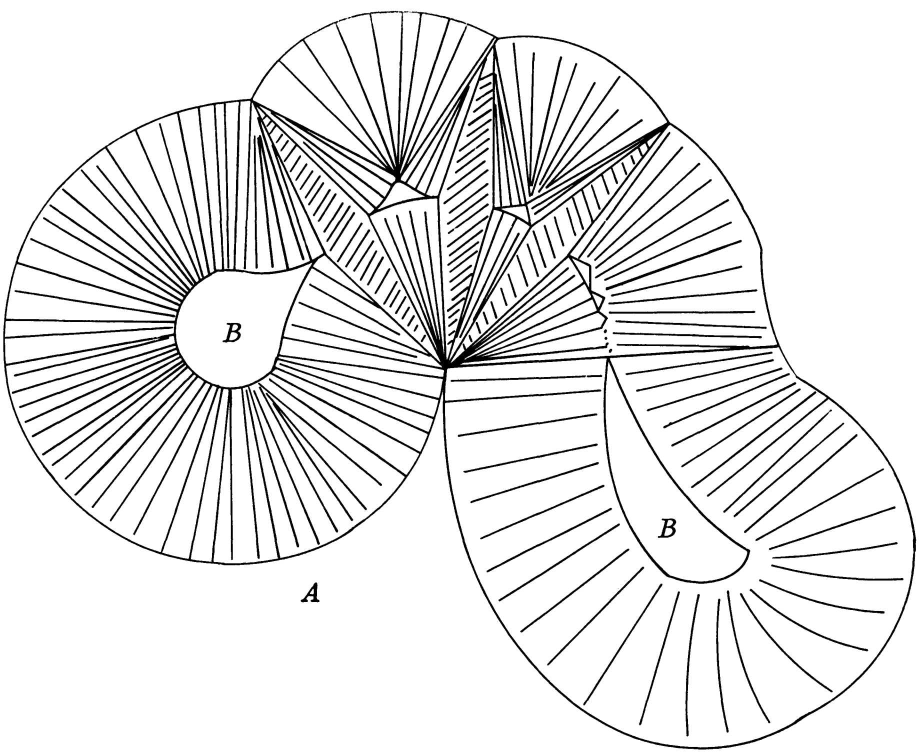

FIGURE 8

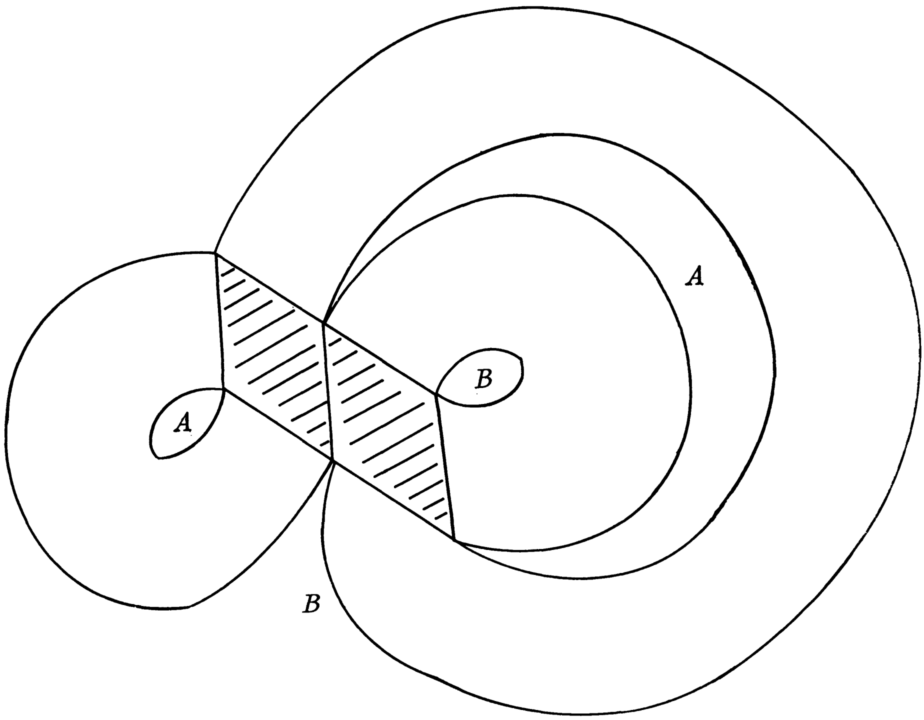

FIGURE 9

any component of C_2. We also observe that isolated connectors are indeed possible (Figure 9).

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