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A One-Dimensional Family of $K3$ Surfaces with a $\mathbb{Z}_4$ Action

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Abstract. The minimal resolution of the degree four cyclic cover of the plane branched along a GIT stable quartic is a $K3$ surface with a non symplectic action of $\mathbb{Z}_4$. In this paper we study the geometry of the one-dimensional family of $K3$ surfaces associated to the locus of plane quartics with five nodes.

Introduction

Let $V \subset |\mathcal{O}_{\mathbb{P}^2}(4)|$ be the space of plane quartics with five nodes and $\mathcal{V}$ be the one-dimensional family given by the quotient of V for the action of $PGL(3, \mathbb{C})$. The minimal resolution X_C of the degree four cyclic cover of the plane branched along a quartic $C \in V$ is a $K3$ surface equipped with a non-symplectic automorphism group $G_C \cong \mathbb{Z}_4$. Since the isomorphism class of X_C only depends on the projective equivalence class of C, this construction gives a one-dimensional family $\mathcal{X}$ of $K3$ surfaces. Moreover, as proved in [1], it defines an injective period map $P \colon \mathcal{V} \longrightarrow \mathcal{M}$, where $\mathcal{M}$ is a moduli space for pairs (X_C, G_C) (as defined in [6]).

This paper describes the geometry of the family $\mathcal{X}$ by studying the structure of the moduli space $\mathcal{M}$, the behavior of the period map on the closure of $\mathcal{V}$, and the occurrence of singular $K3$ surfaces *i.e.,* with maximal Picard number.

In the first section we introduce the $K3$ surface associated to a GIT stable plane quartic according to the construction given by S. Kondō [9].

The period domain of polarized $K3$ surfaces associated to plane quartics with five nodes is isomorphic to the complex one-dimensional ball; the second section shows that their moduli space $\mathcal{M}$ is the Fricke modular curve of level two.

Any $K3$ surface X_C carries an elliptic fibration induced by the pencil of lines through one node of C. In Section 3 we prove that the fibration is isotrivial and the generic fiber is isomorphic to the elliptic curve $E = \mathbb{C}/\mathbb{Z}[i]$. In fact, after a base change and a normalization, the fibration is the product $B_C \times E$ where B_C is a genus two curve with splitting Jacobian $J(B_C) = E_C \times E_C$.

In Section 4 we describe the behavior of the period map on the closure $\overline{\mathcal{V}}$ of $\mathcal{V}$. We prove that the period map can be extended to $\overline{\mathcal{V}}$, giving an isomorphism with the projective line.

The last section shows that there is a correspondence between X_C and the Kummer surface $Km(E \times E_C)$. In particular, the occurrence of singular $K3$ surfaces in

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the family is due to the existence of isogenies between the elliptic curves E and E_C. This is also connected to the existence of certain rational "splitting curves" for C (see [2]). Finally, we give a partial characterization of transcendental lattices of singular $K3$ surfaces in the family. In particular we prove that the Fermat quartic, the Klein quartic, and Vinberg's $K3$ surface (see [24]) belong to the family.

1 Plane Quartics and $K3$ Surfaces

Let C be a *GIT stable* plane quartic, *i.e.,* having at most ordinary nodes and cusps (see [11]). The degree four cyclic cover of the plane branched along C has at most rational double points, hence its minimal resolution X_C is a $K3$ surface with an order four automorphism group G_C (see [1] and [9]). In this paper we consider the locus V of stable plane quartics with five nodes. By taking the quotient of V by the natural action of $PGL(3,\mathbb{C})$ we get a one-dimensional family $\mathcal{V}$. The isomorphism class of the cover only depends on the class of the quartic in $\mathcal{V}$, hence this construction defines a period map $P\colon \mathcal{V} \longrightarrow \mathcal{M}, [C] \longmapsto [(X_C, G_C)]$, where $\mathcal{M}$ is a moduli space parametrizing pairs (X_C, G_C) (the precise definition is given in Subsection 2.2).

We now choose a parametrization for $\mathcal{V}$. Consider the plane quartic C_α, $\alpha \in \mathbb{P}^1$ which is the union of the following conic and two lines:

$$Q : y^2 - xz = 0, L : y = 0, M_\alpha : \alpha x + 2y + z = 0.$$

This gives a one parameter non-constant family of plane quartics with five nodes, hence the general point in $\mathcal{V}$ is represented by a curve in this family.

We denote by π_α the degree four cyclic cover of the plane branched along C_α, $\pi_\alpha\colon Y_\alpha \longrightarrow \mathbb{P}^2$ and with X_α the minimal resolution of Y_α. By the previous remark, the general X_α is a $K3$ surface. Let $G_\alpha \cong \mathbb{Z}_4$ be the automorphism group of covering transformations on X_α induced by π_α.

2 The Period Domain

In this section we describe the moduli space parametrizing pairs (X_α, G_α), where X_α is a $K3$ surface associated to a plane quartic with five nodes and G_α is the corresponding order four covering transformation group.

Let σ_α be a generator of G_α and σ_α^* be the induced isometry on the cohomology lattice $H^2(X_\alpha, \mathbb{Z})$. In [1] it is proved that σ_α^* acts as a primitive 4-th root of unity on $H^{2,0}(X_\alpha)$. In fact we can assume $\sigma_\alpha^*(\omega_\alpha) = i\omega_\alpha$, where $\omega_\alpha \neq 0$ is a holomorphic two-form on X_α. In particular, the invariant lattice of the involution $\tau_\alpha = \sigma_\alpha^2$ is contained in the Picard lattice of X_α. In fact we will show that the invariant lattice is the Picard lattice of the generic $K3$ surface X_α.

2.1 The Generic Point

Let T and N be the transcendental lattice and the Picard lattice of the generic $K3$ surface X_α, respectively.

Lemma 2.1 *The isomorphism classes of T and N are given by*

$$T = A_1^{\oplus 2} \oplus A_1(-1)^{\oplus 2},\ N = U \oplus E_7^{\oplus 2} \oplus A_1^{\oplus 2}.$$

Moreover, in the natural basis of T, the action of the isometry σ_α^* is given by the matrix $J = A \oplus A$, where $A = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.

Proof Let $L_\pm(\alpha) \subset H^2(X_\alpha, \mathbb{Z})$ be the eigenspaces of the involution τ_α^*. It can be easily seen that the fixed locus of τ_α is the disjoint union of eight smooth rational curves. By [13, Theorem 4.2.2] this implies $r(L_+(\alpha)) = 18, \ell(L_+(\alpha)) = 4$, where $r(\cdot)$ denotes the rank of a lattice and $\ell(\cdot)$ the minimal number of generators of its discriminant group. Note that $L_+(\alpha)$ is a primitive sublattice of N.

It can be easily seen that the period map P is not constant, *i.e.*, the family $\{X_\alpha\}_{\alpha\in\mathbb{P}^1}$ is one dimensional, hence the Picard number of the generic X_α is not maximal. Moreover, by [12, Theorem 3.1], the rank of the Picard lattice is even. This implies that the Picard number of the generic X_α is 18 and $N = L_+(\alpha)$, $T = L_-(\alpha)$. Theorem 3.1 in [12] also gives the existence of an isomorphism of $\mathbb{Z}[i]$-modules: $T \cong \mathbb{Z}[i] \oplus \mathbb{Z}[i]$.

Notice that an even symmetric lattice Λ which is a free $\mathbb{Z}[i]$-module of rank one with $i \in O(\Lambda)$, $i^2 = -id$ is of the form: $\Lambda \cong A_1(n) \oplus A_1(n), n \in \mathbb{Z}$, where the action of the isometry i is given by the matrix A. Hence, in a suitable integral basis, the transcendental lattice T has intersection matrix:

$$B = \begin{pmatrix} A_1(n)^{\oplus 2} & C \\ C & A_1(m)^{\oplus 2} \end{pmatrix},$$

where $C = \left(\begin{smallmatrix} b & c \\ -c & b \end{smallmatrix}\right)$. Since T is a 2-elementary lattice with $\ell(T) = 4$, we have $\det(B) = 2^4$. Moreover, its signature is $(2, 2)$. This implies $b = c = 0$ and $nm = -1$. Hence $T \cong A_1^{\oplus 2} \oplus A_1(-1)^{\oplus 2}$. In particular, the Picard lattice is an even hyperbolic 2-elementary lattice with the invariants $(s_+, s_-, \ell, \delta) = (1, 17, 4, 1)$. By [14, Theorem 3.6.2] it is isomorphic to the lattice $U \oplus E_7^{\oplus 2} \oplus A_1^{\oplus 2}$. ■

2.2 The Moduli Space

Let L be the abstract $K3$ lattice and σ^*, τ^* be the isometries of L induced by the generic $\sigma_\alpha^*, \tau_\alpha^*$. We still denote by N and T the positive and negative eigenlattices of τ^* in L respectively. By the remarks in the previous section it follows that the period domain for pairs (X_α, G_α) in the family is given by:

$$D = \{z \in \mathbb{P}(T \otimes C) : \sigma^*(z) = iz, (z, \bar{z}) > 0\}.$$

Since T has rank 4, it can be easily seen that D is a one-dimensional complex ball. By taking the quotient for the arithmetic group $\Gamma = \{\gamma \in O(T) : \gamma \circ \sigma^* = \sigma^* \circ \gamma\}$, we get the moduli space $\mathcal{M} = D/\Gamma$. Let $T_{-2} = \{\delta \in T : \delta^2 = -2\}$, $H_\delta = \delta^\perp \cap D$ and $\Delta = \bigcup_{\delta\in T_{-2}} H_\delta$.

Proposition 2.2 *The quotient $(D\backslash\Delta)/\Gamma$ parametrizes isomorphism classes of pairs (X_α, G_α), where $C_\alpha \in V$. Moreover, the period map $P\colon \mathcal{V} \longrightarrow (D\backslash\Delta)/\Gamma$ is an isomorphism.*

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Proof See [1, Theorem 3.5] and [6, Theorem 11.3]. ■

Lemma 2.1 allows us to describe in detail the structure of the moduli space $\mathcal{M}$. Consider the following subgroups of $SL(2,\mathbb{C})$:

$$G_0 = SU(1,1) \cap M(2,\mathbb{Z}[i]),$$
$$H_0 = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z}) : a+d \equiv b+c \equiv 0 \ (\mathrm{mod}\ 2) \right\},$$

where $M(2,\mathbb{Z}[i])$ is the group of 2×2 matrices with entries in $\mathbb{Z}[i]$.

Proposition 2.3 *We have the isomorphisms*

$$\mathcal{M} \cong B/G \cong S/H,$$

where $B = \{z \in \mathbb{C} : |z| < 1\}$ *is the complex 1-ball,* $S = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ *is the Siegel upper half space, and*

$$G = G_0 \cup LH_0, \quad H = H_0 \cup MH_0$$

where

$$L = \begin{pmatrix} e^{-i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}, \quad M = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Proof The period domain D is given by points $z = (z_1, \dots, z_4) \in \mathbb{P}(T \otimes \mathbb{C})$ such that:

(1) ${}^t z T \bar{z} > 0$
(2) $Jz = iz$.

Hence z is of the form:

$$z = (iz_2, z_2, iz_4, z_4), \quad |z_2|^2 - |z_4|^2 > 0.$$

Thus we get the isomorphism:

$$\psi_1 : D \longrightarrow B = \{w \in \mathbb{C} : |w| < 1\}, \quad z \mapsto z_4/z_2.$$

We are interested in the following subgroup of the isometries of T:

$$\Gamma = \{M \in O(T) : MJ = JM\}.$$

Under the identification $\mathbb{Z}[J] \cong \mathbb{Z}[i]$ we have the isomorphism $T \cong \mathbb{Z}[i]^2$ as $\mathbb{Z}[i]$-modules. It can be easily seen that in the natural basis for $\mathbb{Z}[i]^2$ the intersection form on T is given by $Q(z,w) = 2(z\bar{z} - w\bar{w})$. Then we get

$$\Gamma = U(Q) \cap M(2,\mathbb{Z}[i]) \cong U(1,1) \cap M(2,\mathbb{Z}[i]).$$

Let $M \in \Gamma$: $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, where $a,b,c,d \in \mathbb{Z}[i]$. The action of M on D induces an action of M on B which is given by the Möbius transformation

$$w \mapsto \psi_M(w) = \frac{c + dw}{a + bw}.$$

Since two matrices in $M(2,\mathbb{C})$ give the same Möbius transformation if and only if they are the same up to multiplication for a nonnegative scalar, the group of Möbius transformations of $\mathbb{C}$ is isomorphic to the quotient $\mathcal{T} \cong SL(2,\mathbb{C})/\pm I$. Consider the homomorphism:

$$\Phi\colon GL(2,\mathbb{C}) \longrightarrow \mathcal{T} \quad M \mapsto \frac{1}{\sqrt{\det(M)}}M.$$

Notice that the kernel of Φ is isomorphic to $\mathbb{C}^*$. Let $\Phi_{|\Gamma}$ be the restriction of Φ to Γ, then $\ker(\Phi_{|\Gamma})$ is isomorphic to the group of 4-th roots of unity. Notice that $G = \operatorname{Im}(\Phi_{|\Gamma}) \subset SU(1,1)/\pm I$ is given by:

$$G = \{M \in SU(1,1)/\pm I \mid \exists \epsilon \in \mathbb{C}^* : \epsilon M \in M(2,\mathbb{Z}[i])\}.$$

Let $\Gamma_0 = SU(1,1) \cap M(2,\mathbb{Z}[i]) \subset G$ and let G_0 be its image in $\mathcal{T}$. Let $L' = \left(\begin{smallmatrix} 1 & 0 \\ 0 & i \end{smallmatrix}\right) \in \Gamma$, then $\Phi(L') = [L]$, where

$$L = \begin{pmatrix} e^{-i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}.$$

Notice that $L^{-1}G_0L = G_0$ and $L^2 \in G_0$. In fact, if $M \in G$ then $LM \in G_0$. Hence, $G = G_0 \cup LG_0$.

A biholomorphic map between B and $S = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$ is given by the Möbius transformation $\psi_2 = \psi_K$:

$$\psi_K\colon B \longrightarrow S \quad z \mapsto \frac{i+z}{1+iz}$$

associated with the matrix $K = \left(\begin{smallmatrix} 1 & i \\ i & 1 \end{smallmatrix}\right)$. The map ψ_K induces the isomorphism between the groups of automorphisms:

$$\Psi_K\colon \operatorname{Aut}(B) \longrightarrow \operatorname{Aut}(S) \quad \phi \mapsto \psi_K \phi \psi_K^{-1}.$$

Let ψ_M be the Möbius transformation corresponding to a matrix $M \in SU(1,1)$: $M = \left(\begin{smallmatrix} a & b \\ \bar{b} & \bar{a} \end{smallmatrix}\right)$, where $a, b \in \mathbb{C}$. Then the map $\Psi_K(\psi_M)$ is the Möbius transformation associated to the matrix $KMK^{-1} \in SL(2,\mathbb{R})$:

$$KMK^{-1} = \begin{pmatrix} \operatorname{Re}(a) + \operatorname{Im}(b) & \operatorname{Re}(b) + \operatorname{Im}(a) \\ \operatorname{Re}(b) - \operatorname{Im}(a) & \operatorname{Re}(a) - \operatorname{Im}(b) \end{pmatrix}.$$

Conversely, let $N \in SL(2,\mathbb{R})$: $N = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$. Then $N = KMK^{-1}$ where $M \in SU(1,1)$ is given by:

$$a = \frac{1}{2}[\alpha + \delta + i(\beta - \gamma)], \quad b = \frac{1}{2}[\beta + \gamma + i(\alpha - \delta)].$$

This gives an isomorphism between the groups of Möbius transformations associated to $SU(1,1)$ and that associated to $SL(2,\mathbb{R})$. The image of G_0 is the following subgroup of H:

$$H_0 = \left\{ \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in SL(2,\mathbb{Z}) : \alpha + \delta \equiv \beta + \gamma \equiv 0 \pmod 2 \right\}.$$

The image of L in $SL(2,\mathbb{R})$ is given by $\Psi_K(L) = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & -1 \\ 1 & 1 \end{smallmatrix}\right)$. Then we have

$$H = H_0 \cup \Psi_K(L)H_0. \qquad \blacksquare$$

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Consider the level 2 congruence subgroup

$$H[2] = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z}) \mid c \equiv 0 \pmod 2\}.$$

The order two element $F = \left(\begin{smallmatrix} 0 & -1/\sqrt{2} \\ \sqrt{2} & 0 \end{smallmatrix}\right) \in SL(2,\mathbb{R})$ lies in the normalizer of $H[2]$ in $SL(2,\mathbb{R})$ and it is called *Fricke involution*. The group

$$H[2]^+ = H[2] \cup FH[2] \subset SL(2,\mathbb{R})$$

is called *Fricke modular group of level 2* and the quotient $C(2)^+ = S/H[2]^+$ is the *Fricke modular curve of level 2*.

Corollary 2.4 *We have the isomorphisms:*

$$\mathcal{M} \cong C(2)^+ \cong \mathbb{A}^1.$$

Proof The group H_0 is conjugated to $H[2]$ in $SL(2,\mathbb{Z})$:

$$TH_0T^{-1} = H[2], \quad T = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Besides, it can be easily proved that $T(\Upsilon(L)H_0)T^{-1} = FH_0$. Hence the group H is isomorphic to the Fricke modular group of level two and $\mathcal{M}$ is isomorphic to $C(2)^+$. The last isomorphism follows by [5, Proposition 7.3 and Corollary 7.4]. ■

Remark 2.5 In [5] it is proved that the Fricke modular curve of level 2 is also the moduli space for the mirror family of degree 4 polarized $K3$ surfaces. It would be interesting to determine if there is any geometric correspondence between the two families.

3 An Elliptic Pencil

In this section we show that X_α carries a natural elliptic fibration induced by the pencil of lines through one of the nodes of C_α.

3.1 Definition

Note that the conic Q intersects the line L in $p_1 = (0:0:1)$ and $p_2 = (1:0:0)$.

Proposition 3.1 *The pencil of lines through the point p_1 induces an isotrivial elliptic fibration $\mathcal{E}_\alpha \colon X_\alpha \longrightarrow \mathbb{P}^1$. After a base change $B_\alpha \longrightarrow \mathbb{P}^1$ and a normalization, the fibration is the trivial fibration $E \times B_\alpha \longrightarrow B_\alpha$, where $E = \mathbb{C}/\mathbb{Z}[i]$ is the elliptic curve with $j = 1728$ and B_α is a genus two curve.*

Proof The pencil of lines through the point $p_1 \in Q \cap L$ is given by the equation $y = \lambda x$, where $\lambda \in \mathbb{P}^1$. We substitute $y = \lambda x$ in the equation of C_α and we restrict to the affine subset where $x = 1$: $\lambda(z - \lambda^2)(z + (2\lambda + \alpha)) = 0$. In general we have:

$$(z-a)(z-b) = z^2 - (a+b)z + ab = (z - \tfrac{1}{2}(a+b))^2 - (\tfrac{1}{2}(a-b))^2.$$

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Introducing a new variable z_1 by $z = \frac{1}{2}(a-b)z_1 + \frac{1}{2}(a+b)$, we get

$$(z-a)(z-b) = \tfrac{1}{4}(a-b)^2\,(z_1^2 - 1).$$

In our case $a = \lambda^2, b = -(2\lambda+\alpha)$, so

$$\lambda(z-\lambda^2)(z+(2\lambda+\alpha)) = \tfrac{1}{4}\lambda(\lambda^2+2\lambda+\alpha)^2(z_1^2-1).$$

Thus we are considering the the fibration on Y_α:

$$Y_\alpha \longrightarrow \mathbb{P}^1_\lambda, \qquad w^4 = \tfrac{1}{4}\lambda(\lambda^2+2\lambda+\alpha)^2(z_1^2-1).$$

This induces an elliptic fibration on X_α with fibers isomorphic to the elliptic curve $E : w^4 = z_1^2 - 1$. Notice that $E \cong \mathbb{C}/\mathbb{Z}[i]$ since E has an automorphism of order 4 which fixes a point $(z_1, w) \longmapsto (z_1, iw)$, (the point $(z_1, w) = (1, 0)$ is fixed).

To get the trivial fibration, we first make the base change

$$\mathbb{P}^1_\rho \longrightarrow \mathbb{P}^1_\lambda, \quad \rho \longmapsto \lambda = \rho^2,$$

which gives the equation

$$w^4 = \bigl(\tfrac{1}{2}\rho(\rho^4+2\rho^2+\alpha)\bigr)^2(z_1^2-1).$$

Next we consider the genus two curve $B_\alpha : \tau^2 = \rho(\rho^4+2\rho^2+\alpha))$. We make the base change $B_\alpha \longrightarrow \mathbb{P}^1_\rho, \quad (\rho,\tau) \longmapsto \rho$ and we define $w = \tau w_1/\sqrt{2}$, so we get $w_1^4 = z_1^2 - 1$. Hence the normalization of the pull-back of the family is the product $B_\alpha \times E$. ∎

Remark 3.2 The construction in the proof gives a rational map of degree four from $B_\alpha \times E$ to the quartic surface $Y_\alpha \subset \mathbb{P}^3$. In coordinates it is given by:

$$\Upsilon\colon B_\alpha \times E \longrightarrow Y_\alpha \subset \mathbb{P}^3,$$

$$((\rho,\tau),(z_1,w_1)) \longmapsto \begin{cases} x = 1, \\ y = \rho^2, \\ z = \frac{1}{2}(\rho^4+2\rho^2+\alpha)z_1 + \frac{1}{2}(\rho^4-2\rho^2-\alpha), \\ w = \tau w_1/\sqrt{2i}. \end{cases}$$

3.2 The Weierstrass Model

We now determine the Weierstrass model for the isotrivial elliptic fibration defined in Proposition 3.1.

Lemma 3.3 *The Weierstrass form for the elliptic fibration $\mathcal{E}_\alpha$ is given by*

$$Y_\alpha \longrightarrow \mathbb{P}^1_\lambda, \quad v^2 = u^3 - \lambda^3(\lambda^2+2\lambda+\alpha)^2 u.$$

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Proof Recall that $\mathcal{E}_\alpha$ is given by $w^4 = \frac{1}{4}\lambda(\lambda^2+2\lambda+\alpha)^2(z_1^2-1)$. We will apply the algorithm from [3] to find its Weierstrass form. Let $\beta = \big(\frac{1}{4}\lambda(\lambda^2+2\lambda+\alpha)\big)^{-2}$. Then the fibration can be rewritten as $z_1^2 = \beta w^4 + 1$. Introducing the new coordinates $w = 1/s, z_1 = t/s^2$ we get

$$t^2 = s^4 + \beta \Longrightarrow (t-s^2)(t+s^2) = \beta.$$

Next we put $x = t + s^2 \Longrightarrow t - s^2 = \beta/x,\ 2s^2 = x - \beta/x$. Multiply the last equation by x^2 and put $y = sx$:

$$2v^2x^2 = x^3 - \beta x \quad \Longrightarrow \quad 2y^2 = x^3 - \beta x.$$

We finally put $x = u/2,\ y = v/4$ and multiply the equation by 8, giving

$$v^2 = u^3 - 4\beta u, \quad \text{where} \quad u = 2\frac{z_1+1}{w^2}, \quad v = 4\frac{z_1+1}{w^3}.$$

Hence the family is

$$v^2 = u^3 - \frac{16}{\lambda(\lambda^2+2\lambda+\alpha)^2}u.$$

The transformation to the form $w^4 = \frac{1}{4}\lambda(\lambda^2+2\lambda+\alpha)^2(z_1^2-1)$ is given by

$$u = \Big(\frac{\lambda(\lambda^2+2\lambda+\alpha)}{w}\Big)^2\frac{z_1+1}{2}, \ v = \Big(\frac{\lambda(\lambda^2+2\lambda+\alpha)}{w}\Big)^3\frac{z_1+1}{2}.$$

The original variables x, y, z, and w can be obtained from

$$z_1 = 2z - \frac{\lambda^2-2\lambda-\alpha}{\lambda^2+2\lambda+\alpha}. \qquad \blacksquare$$

3.3 Singular Fibers

The singular fibers in a Weierstrass fibration with equation

$$v^2 = u^3 - f(\lambda)u, \quad f \in \mathbb{C}[\lambda],$$

correspond to the values λ where $f(\lambda) = 0$ and to $\lambda = \infty$ if $\deg(f)$ is not divisible by 4. Let $f = (\lambda - a)^k g$ with $g(a) \neq 0$, then we may always assume that $0 \leq k \leq 3$ and we have bad reduction in a only if $k \neq 0$. Thus we get three types of bad fibers for $k = 1, 2, 3$. We list their Kodaira types and corresponding Dynkin diagrams:

- $k = 1$: type III, A_1,
- $k = 2$: type I_0^*, D_4,
- $k = 3$: type III^*, E_7.

It is now easy to find the bad fibers in our case:

$$v^2 = u^3 - \lambda^3(\lambda^2+2\lambda+\alpha)^2u.$$

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Lemma 3.4 *The elliptic fibration $\mathcal{E}_\alpha$ has the following configuration of singular fibers:*

- *type III over $\lambda = \infty$,*
- *type I_0^* over the solutions λ_1, λ_2 of $\lambda^2 + 2\lambda + \alpha$,*
- *type III^* over $\lambda = 0$.*

Notice that, for $\lambda = 0$ we get the line L and for $\lambda = \infty$ we get the line $x = 0$, which is tangent to the conic Q in p_1. Finally, the values λ_i, $i = 1, 2$ correspond to lines through the intersection points of Q and M_α.

Remark 3.5 It follows from the Shioda–Tate formula [19, Corollary 1.5] that the Mordell–Weil group of the elliptic fibration $\mathcal{E}_\alpha$ has order 2. In fact, the two sections are given by the line M_α and the conic Q, which intersect each fiber in the two fixed points of the order 4 automorphism (defined by $(z_1, w) = (\pm 1, 0)$ or $(u, v) = (0, 0)$ and ∞).

4 Compactification

If the quartic C_α is not GIT stable, then X_α is not a $K3$ surface. However, we show that in some cases proper modifications of the family still give $K3$ surfaces in the limit. In other words, we study the behavior of the period map P on the closure $\overline{\mathcal{V}}$ of $\mathcal{V}$.

Note that M_α (see Section 1) is the pencil of lines through the point $(0, 1, -2)$ and the curve C_α is not stable if and only if M_α is tangent to Q or if it contains a point of $Q \cap L$. The line M_∞ is tangent to Q in the point $p_1 \in Q \cap L$. Furthermore, the line M_1 is tangent to Q in the point $(1, -1, 1)$, hence C_1 has a tacnode in this point. Finally, the line M_0 passes through the point $(1, 0, 0) \in Q \cap L$, hence C_0 has a triple point. Thus there are three quartics in the family which are not GIT stable: $\alpha = 0, 1, \infty$. Note that C_1 is semistable while C_0, C_∞ are not even semistable.

Lemma 4.1 *There exists a modification X'_α of the family X_α such that the fiber X'_∞ is a $K3$ surface with*

(i) *an elliptic fibration with the same configuration of singular fibers of Lemma* 3.4*;*
(ii) *an automorphism of order eight acting as the multiplication by a primitive* 8*-th root of unity on the holomorphic two-form;*
(iii) *Picard number* 18.

Proof We consider the elliptic fibration in Weierstrass form from Subsection 3.2

$$\mathcal{E}_\alpha \colon Y_\alpha \longrightarrow \mathbb{P}^1_\lambda, \qquad v^2 = u^3 - \lambda^3(\lambda^2 + 2\lambda + \alpha)^2 u.$$

We put

$$\alpha := \beta^{-8}, \qquad u := \beta^{-14} u, \quad v = \beta^{-21} v, \quad \lambda = \beta^{-4}\lambda.$$

Then, after multiplying the equation by β^{-42}, we get

$$Y'_\beta : v^2 = u^3 - \lambda^3(\lambda^2 + 2\beta^4\lambda + 1)^2 u.$$

This modified family has a good reduction for $\beta \to 0$. The fibration $Y'_\infty \longrightarrow \mathbb{P}^1_\lambda$ has 4 bad fibers with the same configuration of the general case. Moreover, the surface

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Y'_∞ has an extra automorphism φ given by

$$u := \zeta^2 u, \qquad v := \zeta^3 v, \qquad \lambda := -\lambda \qquad (\zeta^4 = -1).$$

Note that the holomorphic two form on X'_∞ is locally given by $\omega = (d\lambda \wedge du)/v$ and $\varphi^*\omega = (-\zeta^2/\zeta^3)\omega = -\zeta^{-1}\omega$. This implies that the transcendental lattice of X'_∞ allows the action of the ring $\mathbb{Z}[\zeta]$ (see [12, Theorem 3.1]), in particular its rank is a multiple of 4. ■

Lemma 4.2 *There exists a modification X''_α of the family X_α such that the fiber X''_0 is "Vinberg's K3 surface" and carries an elliptic fibration with two fibers of type III^* and one of type I_0^*.*

Proof We consider again the elliptic fibration in Weierstrass form from Subsection 3.2

$$\mathcal{E}_\alpha \longrightarrow \mathbb{P}^1_\lambda, \qquad v^2 = u^3 - \lambda^3(\lambda^2 + 2\lambda + \alpha)^2 u.$$

When $\alpha \to 0$, we get $\lambda^5(\lambda + 2)$, and changing coordinates allows us to reduce to the case $\lambda(\lambda + 2)$, which no longer gives a $K3$ surface. We consider the fibration near $\lambda = \infty$, so we put:

$$\mu = \lambda^{-1}, \quad u := u/\mu^4, \quad v := v/\mu^6$$

and multiply throughout by μ^{12}:

$$v^2 = u^3 - \mu(1 + 2\mu + \alpha\mu^2)^2 u.$$

We make a base change and a coordinate change:

$$\alpha = \beta^4, \qquad \mu := \mu/\beta^4, \quad u := u/\beta^6, \quad y := y/\beta^9,$$

and multiply throughout by β^{18}:

$$v^2 = u^3 - \mu(\beta^4 + 2\mu + \mu^2)^2 u.$$

It is now obvious that for $\beta \to 0$ we get an elliptic fibration on a $K3$ surface X''_0 associated to $v^2 = u^3 - \mu^3(2 + \mu)^2 u$. Notice that there are 2 fibers of type III^* over $\mu = 0, \infty$ and one of type I_0^* over $\mu = -2$. It follows from the Shioda–Tate formula [19, Corollary 1.5] that the rank of the Picard lattice of X''_0 is 20 and that the discriminant of the transcendental lattice is equal to 4. This implies that the surface X''_0 is "Vinberg's $K3$ surface" *i.e.,* the only $K3$ surface with transcendental lattice isomorphic to $A_1(-1)^{\oplus 2}$ (see [24]). ■

Remark 4.3 The degree four cyclic cover of the plane branched along the curve C_1 is a surface Y_1 with an elliptic singularity of type $\tilde{E}_7$. In fact, this surface is a degeneration of $K3$ surfaces of type II (in the sense of Kulikov) *i.e.,* the Picard–Lefschetz transformation T has infinite order and $N = log(T)$ satisfies $N^2 = 0, N \neq 0$ (see [10]). This implies that the corresponding point is mapped to the boundary by the period map. By GIT theory there exists a modification of the family such that Y_1 is replaced by the cover of a plane quartic in a minimal orbit. Such a surface has two elliptic singularities and it is birational to a ruled surface with elliptic base curve (see [18, Theorem 2.4]).

Let $\overline{\mathcal{V}}$ be the closure of $\mathcal{V}$ in the GIT quotient of the space of plane quartics.

Proposition 4.4 *The period map P can be extended to an isomorphism $P\colon \overline{\mathcal{V}} \longrightarrow \overline{\mathcal{M}}$, where $\overline{\mathcal{M}} = \mathcal{M} \cup \{\infty\} \cong \mathbb{P}^1$ is the Baily Borel compactification of $\mathcal{M}$. The stable reduction of C_0 is mapped to D/Γ, that of C_∞ to $(D\backslash\Delta)/\Gamma$ and the point C_1 to ∞.*

Proof It follows from [1, Theorem 3.5] that P gives an isomorphism between the closure of $\mathcal{V}$ in the GIT quotient of the space of plane quartics to the Baily Borel compactification of $\mathcal{M}$. Moreover, it is proved that points which are not GIT stable are mapped to the boundary. Lemmas 4.1 and 4.2 show that the family C_α has a stable reduction in $\alpha = \infty, 0$. In particular, it follows from Lemma 4.2 and [24] that the stable reduction of C_0 is the plane quartic with six nodes *i.e.,* the union of four lines. This implies that X_0 has a period point in Δ since the extra node gives a (-2) curve in $T \cap Pic(X_0)$. By Lemma 4.1, X_∞ is a $K3$ surface with Picard number 18, hence its period point is not in Δ (*i.e.,* the stable reduction of C_∞ has only 5 nodes). Finally, since C_1 is not GIT stable, it is mapped to the boundary (see Remark 4.3). ■

Remark 4.5 The quotient D/Γ parametrizes pairs (X_α, G_α), where X_α is a $K3$ surface associated to a plane quartic with at least 5 nodes (see [6]). The divisor Δ/Γ contains only one point, corresponding to the union of four lines.

5 Singular $K3$ Surfaces and Isogenies

We recall that a $K3$ surface is called *singular* if it has maximal Picard number (equal to 20). In this section we study the occurrence of singular $K3$ surfaces in the family, and we prove that this is connected to the existence of isogenies between certain elliptic curves.

5.1 The Curve B_α

We consider the genus two curve in Proposition 3.1:

$$B_\alpha : \tau^2 = \rho(\rho^4 + 2\rho^2 + \alpha).$$

It is convenient to take $\alpha = \beta^{-8}$.

Now we define $\rho := \beta^{-2}\rho$, $\tau := \beta^{-5}\tau$ and the equation for B_α becomes

$$B_\beta : \tau^2 = \rho(\rho^4 + 2\beta^4\rho^2 + 1).$$

It is now easy to see that B_β carries the involution

$$\iota\colon B_\beta \longrightarrow B_\beta, (\rho, \tau) \longmapsto (\rho^{-1}, \tau\rho^{-3}).$$

The quotient by ι is the elliptic curve $E_\beta : v^2 = u(u^2 + 4u + 2(1 + \beta^4))$ with quotient map

$$f\colon B_\beta \longrightarrow E_\beta, \qquad (\rho, \tau) \longmapsto (u, v) = \left(\frac{2(1+\beta^4)\rho}{(\rho-1)^2}, \frac{2(1+\beta^4)\tau}{(\rho-1)^3}\right).$$

This formula shows that the hyperelliptic involution $(\rho, \tau) \longmapsto (\rho, -\tau)$ on B_β induces the involution $(u, v) \mapsto (u, -v)$ on E_β.

Lemma 5.1 *The Jacobian of B_β is isogenous to the product $E_\beta \times E_\beta$.*

Proof The reducibility of the Jacobian of B_β follows from [4, Theorem 14.1.1, Ch.14] since it is clear that B_β is equivalent to a curve of the form

$$y^2 = x(x-1)(x+1)(x-b)(x+b).$$

In fact, the curve B_β has the automorphism ι': $\iota'(\rho,\tau) = (-\rho, i\tau)$. This gives another map $f \circ \iota'\colon B_\beta \longrightarrow E_\beta$. Notice that $H^{1,0}(B_\beta) = \langle d\rho/\tau, \rho d\rho/\tau\rangle$. We have

$$\begin{aligned} H^{1,0}(B_\beta) &= \langle d\rho/\tau + \rho d\rho/\tau\rangle \oplus \langle d\rho/\tau - \rho d\rho/\tau\rangle, \\ &= f^* H^{1,0}(E_\beta) \oplus (f\circ \iota')^* H^{1,0}(E_\beta). \end{aligned}$$

Hence the Jacobian is isogenous to the product $E_\beta \times E_\beta$. ∎

5.2 The Elliptic Curve E

Notice that we have the isomorphism of genus one curves

$$\begin{gathered} E' = (y^2 = x^3 - x) \xrightarrow{\cong} E = (w_1^4 = z_1^2 - 1), \\ (x,y) \longmapsto (w_1, z_1) = (y/(\sqrt{2}x), (x + x^{-1})/2)). \end{gathered}$$

Moreover, the automorphism of order four on E $(z_1, w_1) \longmapsto (z_1, iw_1)$ is induced by the automorphism on E', $(x,y) \longmapsto (x^{-1}, iyx^{-2})$. We call *standard involution* the automorphism $(x,y) \longmapsto (x,-y)$ on $E' \cong E$ (sometimes we simply write $p \longmapsto -p$ for this map).

5.3 Isogenies

In Remark 3.2 we defined a map Υ from $B_\beta \times E$ to $Y_\beta \subset \mathbb{P}^3$. It can be proved that the image of $B_\beta \times E$ in $\mathbb{P}^3$ is the quotient by the order four automorphism

$$\phi\colon B_\beta \times E \longrightarrow B_\beta \times E, \qquad \big((\rho,\tau),(z_1,w_1)\big) \longmapsto \big((-\rho, i\tau),(z_1,-iw_1)\big).$$

Note that the square of the automorphism is the product of the hyperelliptic involution on B_β and the standard involution on E.

Remark 5.2 The rational map Υ has 9 base points, one of multiplicity 4 and 8 of multiplicity 2.

We now consider the Kummer surface associated to the abelian surface $E_\beta \times E$:

$$K_\beta = \mathrm{Km}(E_\beta \times E).$$

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From the previous remarks it follows that we have the diagram

$$\begin{array}{ccc} & (B_\beta \times E)/\phi^2 & \\ {}^{/\iota}\swarrow & & \searrow^{/\phi} \\ K_\beta & & X_\beta \end{array}$$

where the left arrow is the quotient by the involution ι and the right arrow is the quotient by ϕ (composed with birational maps). We now prove the following.

Theorem 5.3 *The K3 surface X_β is singular if and only if K_β is singular (*i.e., *E_β is isogenous to E).*

Proof Let $\omega = d\rho/\tau \in H^{1,0}(B_\beta)$, $\omega_i = dw_1/z_1 \in H^{1,0}(E)$. Notice that

$$(H^1(B_\beta) \otimes H^1(E))^\phi = \langle \omega \otimes \omega_i, \rho\omega \otimes \bar{\omega}_i, \bar{\omega} \otimes \bar{\omega}_i, \overline{\rho\omega} \otimes \omega_i \rangle.$$

Let $\widetilde{B_\beta \times E}$ be the blow up of $B_\beta \times E$ along the indeterminacy locus of Υ and $\widetilde{\Upsilon}$ be the map $\widetilde{B_\beta \times E} \to X_\beta$ induced by Υ. We have

$$\widetilde{\Upsilon}^*(T_\beta) \subset (H^1(B_\beta) \otimes H^1(E))^\phi \subset H^2(\widetilde{B_\beta \times E}),$$

and the first inclusion is an equality for general $\beta \in \mathbb{P}^1$. The transcendental lattice T_β has rank two if the space $(H^1(B_\beta, \mathbb{Q}) \otimes H^1(E, \mathbb{Q}))^\phi$ contains a cycle of type $(1,1)$. It can be proved by easy computations that $H^1(B_\beta, \mathbb{Q}) \otimes H^1(E, \mathbb{Q})$ is the direct sum of the eigenspaces (with respect to the eigenvalues ± 1) of the automorphism ϕ. Moreover, the involution ι interchanges the eigenspaces of ϕ. This implies that if $H^1(B_\beta, \mathbb{Q}) \otimes H^1(E, \mathbb{Q})$ contains a $(1,1)$ cycle, then the same is true for the positive eigenspace of ϕ. Notice that

$$H^1(B_\beta) \otimes H^1(E) \cong H^1(B_\beta)^* \otimes H^1(E) \cong Hom(H^1(B_\beta), H^1(E)).$$

Hence we can associate to each element $\omega \in H^1(B_\beta) \otimes H^1(E)$ a homomorphism $\psi_\omega \colon H^1(B_\beta) \to H^1(E)$. Moreover ω is of type $(1,1)$ if and only if ψ_ω preserves the Hodge decomposition *i.e.,* $\psi_\omega(H^{1,0}(B_\beta)) \subset H^{1,0}(E)$ (see [23]). By Lemma 5.1 $J(B_\beta) \cong E_\beta \times E_\beta$, hence this existence is equivalent to the existence of a homomorphism $\psi'_\omega : H^1(E_\beta, \mathbb{Q}) \longrightarrow H^1(E, \mathbb{Q})$ preserving the Hodge structure *i.e.,* of an isogeny between E_β and E. It is known that the Kummer surface associated to the product of two elliptic curves is singular if and only if the two curves are isogenous with complex multiplication [20, 21], and thus the result follows. ■

Assume now that β is such that there is an isogeny of elliptic curves $g \colon E_\beta \longrightarrow E$. Composing with the quotient map f (see Subsection 5.1) we have

$$h \colon B_\beta \longrightarrow E_\beta \longrightarrow E.$$

Let Γ_h be the graph of h. By the proof of Theorem 5.3, Γ_h is the $(1,1)$ cycle in $H^1(B_\beta) \otimes H^1(E)$ corresponding to g.

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Lemma 5.4 *The image $\Upsilon(\Gamma_h)$ is a rational curve in $\mathbb{P}^3$.*

Proof As observed in Subsection 5.1, the hyperelliptic involution i on B_β induces the standard involution on E_β. Since g is an isogeny (so a homomorphism of groups) the hyperelliptic involution on E_β composed with g is the standard involution on E. Thus if $(p, h(p)) \in \Gamma_h \cong B_\beta$, then also $(i(p), h(i(p)) = (i(p), -h(p))$ lies in Γ_h. This means that the graph Γ_h is invariant under ϕ^2, therefore the composition

$$B_\beta \cong \Gamma_h \hookrightarrow B_\beta \times E \longrightarrow Y_\beta \subset \mathbb{P}^3$$

factors over $B_\beta/i \cong \mathbb{P}^1$. In particular, the image of the graph is a rational curve. ■

5.4 A Special Case

We now consider a special example where the isogeny $g \colon E_\beta \longrightarrow E$ is an isomorphism:

Example. We consider the curve B_β from Subsection 5.1 with $\bar\beta^4 = 7/9$. Then we have

$$B_{\bar\beta} : \tau^2 = \rho(\rho^4 + \tfrac{14}{9}\rho^2 + 1),$$
$$E_{\bar\beta} : v^2 = u(u + \tfrac{4}{3})(u + \tfrac{8}{3}).$$

Notice that, by putting $u = \frac{4}{3}x - \frac{4}{3}$, we get an isomorphism with the curve $E' : y^2 = x(x^2 - 1)$.

We fix the isomorphisms $E_{\bar\beta} \cong E' \cong E$ (the last one as in Subsection 5.2). We denote by $D_{\bar\beta}$ the projection to $\mathbb{P}^2$ of the image of Γ_h in $\mathbb{P}^3$. Then we have the following.

Lemma 5.5 *The image of $\Gamma_h \subset B_{\bar\beta} \times E$ in $\mathbb{P}^3$ is a rational curve of degree six. Moreover, the inverse image of the curve $D_{\bar\beta}$ splits in four components on the quartic surface $Y_{\bar\beta} \subset \mathbb{P}^3$.*

Proof An explicit computation gives that the curve $D_{\bar\beta} \subset \mathbb{P}^2$ is the image of the following map:

$$\psi \colon \mathbb{P}^1_r \longrightarrow \mathbb{P}^2, \quad r \longmapsto \begin{cases} x = 49(r-1)^2, \\ y = 63r^2(r-1)^2, \\ z = 3r^2(48 - 32r + 75r^2 - 54r^3 + 27r^4). \end{cases}$$

Recall that $Y_{\bar\beta}$ totally ramifies over the plane quartic:

$$Q : (y^2 - xz) \cdot y \cdot (\bar\alpha x + 2y + z) = 0, \quad \bar\alpha = 81/49.$$

Substituting for x, y, and z, we get:

$$\left(-2352(r-1)^2 r^2 (3 - 2r + 3r^2)\right) \cdot \left(63r^2(r-1)^2\right) \cdot \left(3(3 - 2r + 3r^2)^3\right).$$

Thus this product is a fourth power in $\mathbb{C}[r]$, hence the 4:1 cover of the curve splits into 4 components. ■

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We remark that the curve $D_{\bar{\beta}}$ defined in the previous section defines a 2-section for the elliptic fibration $\mathcal{E}_{\bar{\beta}}$ *i.e,* it meets every fiber in two points.

Consider the 2:1 base change $D_{\bar{\beta}} \longrightarrow \mathbb{P}^1_\lambda$ given by the projection of the 2-section to the base. The pull-back $\mathcal{E}_r$ of the Weierstrass fibration $\mathcal{E}_{\bar{\beta}} \longrightarrow \mathbb{P}^1_\lambda$ along this base change has two 'new' sections which are the irreducible components of the pull-back of the 2-section $D_{\bar{\beta}}$. The sum of these sections of $\mathcal{E}_r \longrightarrow \mathbb{P}^1_r$ actually defines a section of $\mathcal{E}_{\bar{\beta}} \longrightarrow \mathbb{P}^1_\lambda$.

Lemma 5.6 *The elliptic fibration $\mathcal{E}_\beta$ with $\beta = \bar{\beta}$ has a new section. The inverse image by π_β of the image of this curve in $\mathbb{P}^2$ splits in four components.*

Proof The parameter λ was defined as y/x, (see Section 3), hence the base change $D_{\bar{\beta}} \longrightarrow \mathbb{P}^1$ is defined by $\lambda = y/x = (9/7)r^2$. On the other hand, the coordinates of the 2-section are polynomials in r, so we need to make a base change with $\sqrt{\lambda}$ or, equivalently, with r. Then the pull-back surface $\mathcal{E}_r$ has the sections $r \longmapsto x_i(r)$ and $r \longmapsto x_i(-r)$ (here $x_i = x, y, z, w$ or u, v in the Weierstrass model).

Let $u_1 = u(r)$, $u_2 = u(-r)$, $v_1 = v(r)$, $v_2 = v(-r)$. The coordinates (u_3, v_3) of the sum of the two sections in the Weierstrass model can be found by using the formula

$$u_3 = (v_2 - v_1)^2/(u_2 - u_1)^2 - u_1 - u_2$$

from [22] (the v_3-coordinate is easy to find from the Weierstrass equation of $\mathcal{E}_\beta$).

The coordinate u_3 is a function of r^2 (since u_1, v_1 and u_2, v_2 are permuted under $r \mapsto -r$), hence we get a section of $\mathcal{E}_\beta$. Explicitly, the section of the fibration (note $\bar{\alpha} = 81/49$)

$$v^2 = u^3 - 7^{-4}\lambda^3(81 + 98\lambda + 49\lambda^2)^2 u,$$

is given by

$$u = \frac{(27 + 7\lambda)^2(81 + 98\lambda + 49\lambda^2)}{2^4 7^{7/2}},$$

$$v = \frac{(81 - 7\lambda)(27 + 7\lambda)(81 + 98\lambda + 49\lambda^2)^2}{2^6 7^{21/4}}.$$

The equation of the corresponding curve in Y_α can be easily found by using the inverse transformations. The projection of this curve to $\mathbb{P}^2$ is given by (after having replaced r^2 by r throughout):

$$\zeta \colon \mathbb{P}^1_r \longrightarrow \mathbb{P}^2, \quad r \longmapsto \begin{cases} x = 49(-9 + r)^2, \\ y = 63r(-9 + r)^2, \\ z = 9r^2(729 + 94r + 9r^2). \end{cases}$$

Recall that Y_α totally ramifies over

$$(y^2 - xz) \cdot y \cdot (\alpha x + 2y + z), \qquad \bar{\alpha} = 81/49.$$

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Substituting for x, y, and z, we get:

$$-2^8 3^6 7^2 r^3(-9+r)^2 \cdot 63r(-9+r)^2 \cdot 81(r+3)^4.$$

Thus this product is a fourth power in $\mathbb{C}[r]$, hence the 4:1 cover of the curve splits into four components in Y_α.

Remark 5.7 (i) By [2, Theorem 4.5] one expects that, for any isogeny g, the curve $\Upsilon(\Gamma_h)$ is a "splitting curve" for C_β *i.e.*, its inverse image by the cover π_β is the union of four distinct curves.

(ii) The elliptic fibration $\mathcal{E}_{\tilde{\alpha}}$ on the $K3$ surface $X_{\tilde{\alpha}}$ has a fiber of type III, one of type III^*, two of type I_0^* and a section. The Shioda–Tate formula [19, Corollary 1.5] implies that the Picard number of $X_{\tilde{\alpha}}$ is at least 19. Since the transcendental lattice is a $\mathbb{Z}[i]$ module, the Picard number is an even integer, hence the $K3$ surface is indeed singular.

5.5 The Transcendental Lattice

In this section we prove the following.

Proposition 5.8 *The intersection matrix of the transcendental lattice of a singular $K3$ surface X_α is of the form $T_n = \left(\begin{smallmatrix} 2n & 0 \\ 0 & 2n \end{smallmatrix}\right)$, $n \in \mathbb{Z}$, $n > 0$. Conversely, if n is a positive integer with $n \not\equiv 2 \pmod 4$, then the rank two lattice T_n is the transcendental lattice of a $K3$ surface X_α.*

Proof Let T_α be the transcendental lattice of a singular $K3$ surface X_α in the family. The surface X_α carries an order four automorphism σ such that the induced isometry σ^* on $H^2(X_\alpha, \mathbb{Z})$ satisfies $(\sigma^*)^2 = -id$ on T_α. Moreover, the transcendental lattice is isomorphic to $\mathbb{Z}[i]$ if we identify i with σ^*. It follows (as in the proof of Lemma 2.1) that $T_\alpha \cong A_1(-n)^{\oplus 2}$ for some n positive integer.

Conversely, we now construct explicitly a period point in D such that the corresponding $K3$ surface is singular.

We start by proving that for any positive integer n there exists $a = (a_1, \dots, a_4) \in \mathbb{Z}^4$ such that $n = a_1^2 + a_2^2 - a_3^2 - a_4^2$, with ${a_1}^2 + {a_2}^2 > {a_3}^2 + {a_4}^2$ and such that the rank two lattice $\Lambda(a) = \langle a, \sigma^*(a) \rangle$ is primitive in $\mathbb{Z}^4$. This is equivalent to the request that the rank two minors of the matrix $\left(\begin{smallmatrix} a_1 & a_2 & a_3 & a_4 \\ a_2 & -a_1 & -a_4 & a_3 \end{smallmatrix}\right)$ have no common factors. Let $n = 2k+1$ be an odd integer, then we can choose $a_1 = (k+1)^2$, $a_3 = k^2$, and $a_2 = a_4 = 0$. For $n = 2(k+1)$ with odd k we can choose $a_2 = 1$, $a_4 = 0$, and a_1, a_3 as before. Assume that $k = 2\ell$ is even, then $n = 2(2\ell+1) = 4\ell + 2$. Notice that $(a_1^2 + a_2^2) - (a_3^2 + a_4^2) \equiv 2 \pmod 4$. Since $x^2 \in \{0, 1\}$ for $x \in \mathbb{Z}_4$ we have only two possibilities:

$$(a_1^2, a_2^2, a_3^2, a_4^2) \equiv (1, 1, 0, 0) \pmod 4, \tag{1}$$

$$(a_1^2, a_2^2, a_3^2, a_4^2) \equiv (0, 0, 1, 1) \pmod 4. \tag{2}$$

In case (1) we have that a_1, a_2 are odd and a_3, a_4 are even. Hence we immediately get that $\Lambda(a)$ is not primitive (all minors are even integers). The second case is analogous.

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We now assume that $n \not\equiv 2 \pmod 4$ and we choose $a_1, \dots, a_4$ as before. Define $z(a) \in T \otimes \mathbb{C}$ by (with respect to the usual basis):

$$z(a) = (a_1 + ia_2, a_2 - ia_1, a_3 + ia_4, a_4 - ia_3).$$

We consider the following sublattices of L: $N(a) = z(a)^\perp \cap L_{K3}$, $T(a) = N(a)^\perp$. Notice that $T(a)$ is the lattice generated by $a, \sigma^*(a)$ with intersection matrix given by:

$$T(a) \cong \begin{pmatrix} 2({a_1}^2 + {a_2}^2 - {a_3}^2 - {a_4}^2) & 0 \\ 0 & 2({a_1}^2 + {a_2}^2 - {a_3}^2 - {a_4}^2) \end{pmatrix}.$$

By the surjectivity of the period map the point $z(a)$ is the period point of a $K3$ surface $X_{\alpha(a)}$ in the family (see Proposition 2.2). By construction the transcendental lattice of $X_{\alpha(a)}$ is $T(a)$. ■

We now give some examples of $K3$ surfaces with transcendental lattice isomorphic to T_n:

(a) $n = 1$ for *Vinberg's K3 surface* (see [24]),
(b) $n = 3$ for the $K3$ surface described in [7],
(c) $n = 4$ for the *Fermat quartic* (see [16]),
(d) $n = 7$ for the *Klein quartic* (see [17]).

Lemma 5.9 *Let $E \cong \mathbb{C}/\mathbb{Z} + i\mathbb{Z}$ and $E' \cong \mathbb{C}/\mathbb{Z} + mi\mathbb{Z}$, then the Kummer surface* $\mathrm{Km}(E \times E')$ *has transcendental lattice of the form T_{2m}, $m \in \mathbb{Z}$, $m > 0$.*

Proof This follows easily from the proof of [20, Theorem 4]. ■

Lemma 5.10 *The family $\{X_\alpha\}$ contains the $K3$ surfaces* (a), (b), (c), d)*, and all Kummer surfaces in Lemma* 5.9 *with even m. The surface* $\mathrm{Km}(E \times E)$ *is not in the family.*

Proof The first assertion is a corollary of Proposition 5.8. Note that the transcendental lattice of $X = \mathrm{Km}(E \times E)$ is isomorphic to T_2 (see also [8]), so Proposition 3.4 cannot be applied. Assume that $X = X_\alpha$, $\alpha \in \mathbb{P}^1$. Notice that X cannot correspond to the fibers $\alpha = 0, \infty$, since it is singular and is not isomorphic to Vinberg's $K3$ surface. Hence the elliptic fibration $\mathcal{E}_\alpha$ on X has the same configuration of singular fibers as the general case *i.e.,* Lemma 3.4.

All Jacobian fibrations on X are classified in [15, Table 4.1]. In particular this table shows that there exists no jacobian fibration on X with the configuration of singular fibers given in Lemma 3.4. This gives a contradiction, hence X is not in the family. ■

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