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DISCRETE LIAPUNOV FUNCTIONS WITH $\Delta^2 V > 0$

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Abstract

Consideration of functions whose second difference along the trajectories of a difference equation is positive gives a stability theorem for autonomous discrete-time systems. Such functions can be used to estimate domains of nonglobal stability.

1. Introduction

Recently Yorke [2] and Chow and Dunninger [1] have used conditions on the second derivative of Liapunov functions to establish stability theorems for autonomous differential equations. The main condition was that $\ddot{V} > 0$, along trajectories, except at the origin. This note proves a discrete analogue of these results for systems

$$x_{k+1} = f(x_k), \quad k = 0, 1, 2, \ldots, \tag{1}$$

where f: $R^n \to R^n$ and $f(0) = 0$. The result is used to estimate local stability regions and a simple example of the technique is given. A feature is that no assumption on the definiteness of the Liapunov function is required.

2. Stability theorem

Let $p(k, x)$ denote the orbit of (1) for which $x_0 = x$. By $\Delta^2 V(x_k)$ is meant the second difference

$$V(x_{k+2}) - 2V(x_{k+1}) + V(x_k)$$

along the trajectories of $x_{k+1} = f(x_k)$.

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THEOREM 1. *Suppose that $V: R^n \to R$ is a continuous function and that $\Delta^2 V(x_k) > 0$ for $x_k \neq 0$. Then for any $x \in R^n$:*

either $p(k,x)$ is unbounded, or $p(k,x) \to 0$, as $k \to \infty$.

Likewise if

$\Delta^2 V(x_k) < 0$ *for* $x_k \neq 0$.

PROOF. Since $\Delta^2 V(x) > 0$, it follows that $V(p(k,x))$ is a monotone function of k, provided that k is sufficiently large. To see this, observe that

$$V(x_{k+2}) - V(x_{k+1}) > V(x_{k+1}) - V(x_k).$$

If there exists an integer K so that $V(x_{K+1}) - V(x_K) \geqslant 0$, then, $V(x_{k+1}) > V(x_k)$ for $k > K$. On the other hand, if there is no such integer K, $V(x_{k+1}) < V(x_k)$ for all $k \geqslant 0$.

Suppose $V(x_k)$ is nonincreasing as $k \to \infty$. The details for the nondecreasing case are very much the same. Define

$$L(x) = \{u : p(k,x) \to u \text{ for some sequence } k_i \to \infty\}.$$

If $L(x)$ is empty, the Bolzano–Weierstrass theorem implies that $p(k,x)$ is unbounded as $k \to \infty$. If $L(x) \neq \phi$, for any $u \in L(x)$,

$$\lim_{k \to \infty} V(p(k,x)) = \lim_{i \to \infty} V(p(k_i,x)) = V(u).$$

since $V(p(k,x))$ is monotone in k and V continuous. It follows that $V(u)$ is a constant for all u in $L(x)$. But

$$\lim_{i \to \infty} p(j+k_i, x) = \lim_{i \to \infty} p(j, p(k_i,x)) = p(j,u),$$

and so $p(j,u)$ is contained in $L(x)$ for all j. Hence $V(p(k,u)) = V(u)$. Since $\Delta^2 V(p(k,u)) > 0$, for $p(k,u) = 0$, then $L(x) = \{0\}$. The proof for $\Delta^2 V < 0$ is very similar.

As an example, consider the second-order difference equation:

$$x_{k+2} = x_k + f(x_{k+1}).$$

If

$$\Delta(x_k f(x_k)) > 0,$$

then either x_k is unbounded as $k \to \infty$ or $x_k \to 0$ as $k \to \infty$.

To see this, put $y_k = x_{k+1}$, rewrite the system as

$$x_{k+1} = y_k, \quad y_{k+1} = x_k + f(y_k),$$

and define $V(x,y) = xy$. Then $\Delta^2 V(x_k, y_k) = \Delta(y_k f(y_k)) > 0$ and Theorem 1 applies. Observe that the Liapunov function V is indefinite.

Clearly the theorem is valid for both forward and backward differences.

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3. Nonglobal stability domains

Suppose $\Delta^2 V(x) > 0$ is known to hold only in some open region H containing the origin. Then, in general, stability is not global but a finite stability domain exists. Set

$$\Delta_{\max} = \max\{\Delta V(x)\colon x \in bdy\, H\}$$

and define

$$E_j = \{x \in H\colon \Delta V(p(j, x)) > \Delta_{\max}\}, \quad j = 0, 1, 2, \ldots.$$

THEOREM 2. *If the regions E_j are bounded and nonempty, then they are domains of asymptotic stability for the system $x_{k+1} = f(x_k)$.*

PROOF. If E_j is nonempty and $x \in E_j$, $p(j+k, x) \in H$ since $\Delta V(x)$ is nondecreasing along any trajectory of the system, in E_j. So

$$\Delta V(p(j+k+1), x) - \Delta V(j+k, x)) = \Delta^2 V(p(j+k, x)) > 0,$$

and $\Delta V(p(j+k, x)) > \Delta V(p(j, x)) > \Delta_{\max}$, $p(k, x) \in E_j$ and is thus bounded. By Theorem 1, $p(k, x) \to 0$ as $k \to \infty$.

Notes

1. A similar result holds for the regions

$$F_j = \{x \in H\colon \Delta V(p(j, k)) < \Delta_{\min}\},$$

$$\Delta_{\min} = \min\{\Delta V(x)\colon x \in bdy\, H\},$$

provided $\Delta^2 V < 0$ in H and F_j is nonempty and bounded.

2. If E_j is empty there may be no stability domain. Consider $x_{k+1} = y_k$, $y_{k+1} = x_k + f(y_k)$, where $f(y) = y + 1$ and $V(x, y) = xy$. Then

$$\Delta^2 V = (x+y+1)^2 + (x+y+1) - y(y+1)$$

$$= (x+y+1)^2 + x + 1 - y^2,$$

which is positive provided

$$y^2 > x + 1.$$

But the supremum of $\Delta V = y(y+1)$ on $y^2 = x+1$ is $\Delta_{\max} = \infty$, E_j is empty for all j, and $p(j, x) \to \infty$, as stated as one alternative in Theorem 1.

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4. Example

Consider the two-dimensional system

$$x_{k+1} = y_k,$$
$$y_{k+1} = ax_k - y_k^2, \quad |a| < 3^{-\frac{1}{2}},$$

and the Liapunov function

$$V(x, y) = 2a^2 x^2/(1+a^2) + y^2.$$

This system arises in a biological context. The equations are associated with the stability analysis of the population dynamics of a single species, two age-class model (see [3]).

We observe that

$$\Delta^2 V(x, y) = y^2(1-3a^2-4ax-2y^2)/(1+a^2) + (ay+(ax-y^2)^2)^2,$$

and $H = \{(x, y)\colon 1-3a^2-4ax-2y^2 > 0\}$ contains the origin. For definiteness set $a = \frac{1}{4}$, and it is easy to show that $\Delta_{\max} = -0.0364$, whence

$$E_0 = \{(x, y)\colon (y^2 - x/4)^2 - 2x^2/17 - 15y^2/17 > -0.0364\}.$$

The regions H, E_0 are shown in Fig. 1.

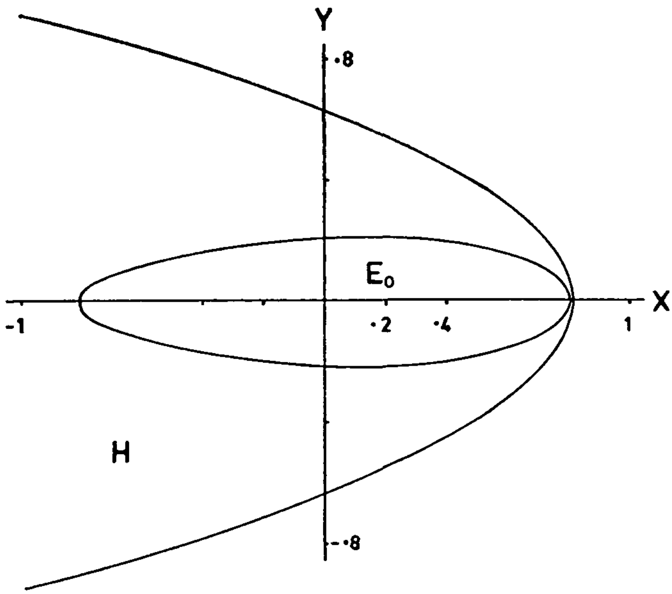

Fig. 1. Local stability region for the example. The region H is a domain where $\Delta^2 V > 0$ and E_0 is a finite stability domain as in Theorem 2.

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