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INTEGRATED SOLUTIONS OF STOCHASTIC EVOLUTION EQUATIONS WITH ADDITIVE NOISE

A. FILINKOV AND I. MAIZURNA

We investigate the existence of a solution to the abstract stochastic evolution equation with additive noise:

$$dX(t) = AX(t)\,dt + BdW(t),\ X(0) = \xi,$$

in the case when A is the generator of an n-times integrated semigroup.

1. INTRODUCTION

Let H and U be real separable Hilbert spaces. We consider the stochastic differential equation

$$dX(t) = AX(t)\,dt + BdW(t),\ X(0) = \xi, \tag{1}$$

where $A : D(A) \subset H \to H$ is a closed linear operator and $B : U \to H$ is a bounded linear operator, $W(\cdot)$ is an U-valued cylindrical Wiener process in a probability space $(\Omega, \mathcal{F}, P)$ adapted to the filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ and ξ is an H-valued random variable.

Equation (1) was studied by many authors (see [**2**] and [**3**] and references therein) in the case when A is the generator of a C_0-semigroup. The novelty of this note is that we study this equation in the case when the operator A is the generator of an n-times integrated semigroup. We prove the existence of a weak n-integrated solution to (1) and discuss the existence a continuous version of this solution. Finally, we use the stochastic wave equation to illustrate our results.

2. PRELIMINARIES

By H-valued random variable, we understand an H-valued mapping $\xi : \Omega \to H$ which is measurable from $(\Omega, \mathcal{F})$ to $\big(H, \mathcal{B}(H)\big)$, where $\mathcal{B}(H)$ is the smallest σ-field containing all closed (or open) subsets of H. A stochastic process $X(\cdot)$ is said to be adapted to the filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ if, for any $t \geqslant 0$, $X(t)$ is $\{\mathcal{F}_t\}$-measurable. A stochastic process $X(\cdot)$ is called H-valued predictable if $X : [0,\infty) \times \Omega \to H$ (or $X : [0,T] \times \Omega \to H$) is

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$\mathcal{P}_\infty$-measurable (respectively $\mathcal{P}_T$-measurable), where $\mathcal{P}_\infty$ is a σ-field generated by sets of the form:

$$(s,t] \times F,\ 0 \leqslant s < t,\ F \in \mathcal{F}_s \text{ and } \{0\} \times F,\ F \in \mathcal{F}_0,$$

and $\mathcal{P}_T$ is the restriction of $\mathcal{P}_\infty$ to $[0,T]$.

We denote by $L^2(\Omega; H)$ the Banach space of all H-valued square integrable mappings endowed with the norm

$$\|X\|_2 := \left(E\left[\|X\|^2\right]\right)^{1/2},$$

and by $C_W\big([0,T]; H\big)$ the Banach space of all mappings $X : [0,T] \to L^2(\Omega; H)$, that are continuous and adapted to the filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$, endowed with the norm

$$\|X(\cdot)\|_{C_W([0,T];H)} := \sup_{t\in[0,T]} \left(E\left[\|X(t)\|^2\right]\right)^{1/2}.$$

Let furthermore $\{e_k\}_{k\in\mathbf{N}}$ be a complete orthonormal system in U and $\{\beta_k(\cdot)\}_{k\in\mathbf{N}}$ be a sequence of independent real Brownian motions on $(\Omega, \mathcal{F}, P)$ adapted to the filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$. For all $y \in U$ and $t \geqslant 0$ one can define the following random variables

$$\langle W(t), y\rangle = \sum_{k=1}^{\infty} \beta_k(t)\langle e_k, y\rangle,$$

which clearly belong to $L^2(\Omega)$. The formal sum

$$W(t) = \sum_{k=1}^{\infty} \beta_k(t) e_k, \qquad t \geqslant 0, \tag{2}$$

is called an U-valued cylindrical Wiener process. Note that the series in (2) is not convergent in $L^2(\Omega; U)$.

We now give a very brief summary of basic facts about integrated semigroups, which can be found, for example, in [**1**] and [**6**].

DEFINITION 1. Let $n \in \mathbb{N}$. A one-parameter family of bounded linear operators $\{V_n(t) \in \mathcal{L}(H) : t \in [0,\infty)\}$ is called an n-times integrated exponentially bounded semigroup if the following conditions hold

(a) $\big(1/(n-1)!\big)\int_0^s \big[(s-r)^{n-1}V_n(t+r) - (t+s-r)^{n-1}V_n(r)\big]dr = V_n(t)V_n(s)$, $s,t \geqslant 0$;

(b) $V_n(t)$ is strongly continuous with respect to $t \geqslant 0$;

(c) $\exists\, C > 0,\ a \in \mathbb{R} : \|V_n(t)\| \leqslant Ce^{at},\ t \geqslant 0$.

The semigroup $\{V_n(t) \in \mathcal{L}(H) : t \in [0,\infty)\}$ is called *non-degenerate* if

$$\forall\, t \geqslant 0,\ \ V_n(t)x = 0 \ \Rightarrow\ x = 0.$$

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If the semigroup is non-degenerate, then $V_n(0) = 0$ and the operator

$$R(\lambda) := \int_0^\infty \lambda^n e^{-\lambda t} V_n(t)\, dt, \qquad \operatorname{Re}\lambda > a$$

is invertible. The operator A defined by

$$(\lambda I - A)^{-1} x = \int_0^\infty \lambda^n e^{-\lambda t} V_n(t) x\, dt, \; x \in H$$

with the domain equal to the range of $(\lambda I - A)^{-1}$, is called the *generator* of $\{V_n(t) \in \mathcal{L}(H) : t \in [0, \infty)\}$.

Proposition 1. *Let A be a densely defined linear operator on H with nonempty resolvent set. Then the following statements are equivalent:*

1. *A is the generator of an n-times integrated semigroup $\{V_n(t) \in \mathcal{L}(H) : t \in [0, \infty)\}$;*
2. *for any $x \in \mathcal{D}(A^{n+1})$ the Cauchy problem*

(3) $$u'(t) = Au(t), \qquad t \geqslant 0, \; u(0) = x,$$

has a unique solution $u(\cdot) \in C\big([0,\infty], \mathcal{D}(A)\big) \bigcap C^1\big([0,\infty], H\big)$ satisfying

$$\exists K > 0, \; a \in \mathbb{R} : \big\|u(t)\big\| \leqslant K e^{at} \|x\|_{A^n},$$

where $\|x\|_{A^n} := \|x\| + \|Ax\| + \ldots + \|A^n x\|$.

In this case the solution of (3) has the form

$$u(t) = V_n^{(n)}(t) x, \; x \in \mathcal{D}(A^{n+1}).$$

Assume that the operator A in problem (1) generates an exponentially bounded n-times integrated semigroup $\{V_n(t) \in \mathcal{L}(H) : t \in [0, \infty)\}$. We consider the stochastic convolution

(4) $$W_n(t) := \int_0^t V_n(t-s) B\, dW(s) = \sum_{k=1}^\infty \int_0^t V_n(t-s) B e_k\, d\beta_k(s).$$

The series in (4) is convergent in $L^2(\Omega, H)$ due to the following lemma, which is an obvious generalisation of the corresponding result for the generators of C_0-semigroups, which can be found in [2].

Lemma 1. *Assume that $K(\cdot)x \in C\big([0,T]; H\big)$ for any $x \in H$, and that the linear operator*

(5) $$L_t x := \int_0^t K(s) B B^* K^*(s) x\, ds, \quad x \in H,$$

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is of trace class:

$$\operatorname{Tr} L_t = \sum_{k=1}^{\infty} \int_0^t \left\| K(s) B e_k \right\|_H^2 ds < \infty.$$

Then for all $t > 0$ the series

$$W_K(t) = \int_0^t K(t-s) B dW(s) = \sum_{k=1}^{\infty} \int_0^t K(t-s) B e_k d\beta_k(t)$$

is convergent on $L^2(\Omega; H)$ to a Gaussian random variable $W_K(t)$ with mean zero and covariance operator L_t. Moreover $W_K(\cdot)$ belongs to $C_W\big([0,T]; H\big)$ for any $T > 0$.

By Lemma 1, we also have that

$$\int_0^t \frac{(t-s)^n}{n!} B dW(s)$$

is a Gaussian random variable.

3. Main Results

DEFINITION 2. An H-valued predictable process $X(t)$ is said to be a weak n-integrated solution of (1) if the trajectories of $X(\cdot)$ are P-almost surely Bochner integrable and if for all $\nu \in D(A^*)$ and $t \in [0,T]$ the equality

$$\langle X(t), \nu \rangle = \left\langle \frac{t^n}{n!} \xi, \nu \right\rangle + \left\langle \int_0^t X(s)\, ds, A^* \nu \right\rangle + \left\langle \int_0^t \frac{(t-s)^n}{n!} B dW(s), \nu \right\rangle, \tag{6}$$

holds P-almost surely.

THEOREM 1. *Let A be the generator of an n-times integrated semigroup $\{V_n(t) \in \mathcal{L}(H) : t \in [0,\infty)\}$ and let the operator L_t, defined by (5) with $K = V_n$, be of trace class. Then*

$$X(t) = V_n(t)\xi + \int_0^t V_n(t-s) B dW(s)$$

is a weak n-integrated solution of (1).

PROOF: Without loss of generality assume that $\xi = 0$. We show that equation (6) is satisfied by

$$W_n(t) = \int_0^t V_n(t-s) B dW(s).$$

Fix $t \in [0,T]$ and let $\nu \in D(A^*)$. Note that

$$\int_0^t \langle A^* \nu, W_n(s) \rangle\, ds = \int_0^t \left\langle A^* \nu, \int_0^t \mathcal{X}_{[0,s]}(r) V_n(s-r) B dW(r) \right\rangle ds.$$

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Hence by the stochastic Fubini theorem, we have

$$\begin{aligned}\int_0^t \langle A^*\nu, W_n(s)\rangle\, ds &= \int_0^t \Big\langle A^*\nu, \int_0^t \mathcal{X}_{[0,s]}(r)V_n(s-r)BdW(r)\Big\rangle\, ds \\ &= \int_0^t \Big\langle \int_0^t \mathcal{X}_{[0,s]}(r)B^*V_n^*(r-s)A^*\nu\, ds, dW(r)\Big\rangle \\ &= \int_0^t \Big\langle \int_r^t B^*V_n^*(s-r)A^*\nu\, ds, dW(r)\Big\rangle.\end{aligned}$$

Since A generates an n-times integrated semigroup $\{V_n(t)\in\mathcal{L}(H) : t\in[0,\infty)\}$, A^* generates the n-times integrated semigroup $\{V_n^*(t)\in\mathcal{L}(H^*) : t\in[0,\infty)\}$, where H^* is the dual space of H. Since A^* and $V_n^*(t)$ commute, using the properties of the n-times integrated semigroup $\{V_n^*(t)\in\mathcal{L}(H^*) : t\in[0,\infty)\}$ we obtain

$$\begin{aligned}\int_0^t \langle A^*\nu, W_n(s)\rangle\, ds &= \int_0^t \Big\langle \int_r^t \mathcal{X}_{[0,s]}(r)B^*A^*V_n^*(r-s)\nu\, ds, dW(r)\Big\rangle \\ &= \int_0^t \Big\langle \int_r^t \Big(B^*\frac{d}{ds}V^*(s-r)\nu - B^*\frac{(s-r)^{n-1}}{(n-1)!}\nu\Big)\, ds, dW(r)\Big\rangle \\ &= \int_0^t \Big\langle B^*V^*(t-r)\nu - B^*\frac{(t-r)^n}{n!}\nu, dW(r)\Big\rangle \\ &= \Big\langle \nu, \int_0^t V(t-r)BdW(r)\Big\rangle - \Big\langle \nu, \int_0^t \frac{(t-r)^n}{n!}BdW(r)\Big\rangle.\end{aligned}$$

Therefore $W_n(t) = \int_0^t V_n(t-s)BdW(s)$ is a weak n-integrated solution of (1). ∎

However, the solution X does not necessarily have a continuous version. The purpose of our next discussion is to find conditions under which the solutions have continuous versions.

Let A be the generator of an exponentially bounded n-times integrated semigroup $\{V_n(t)\in\mathcal{L}(H) : t\in[0,\infty)\}$. Hence A is also the generator of an exponentially bounded $(n+j)$-times integrated semigroups $\{V_{n+j}(t)\in\mathcal{L}(H) : t\in[0,\infty)\}$ for $j = 1, 2, \ldots$. In particular, A generates an exponentially bounded $2n$-times integrated semigroup $\{V_{2n}(t)\in\mathcal{L}(H) : t\in[0,\infty)\}$. It is shown in [5], that those semigroups satisfy the relation

$$V_{2n}(t+s) = V_n(t)V_n(s) + \sum_{j=0}^{n-1}\frac{1}{j!}\big(s^kV_{2n-j}(t) + t^kV_{2n-j}(s)\big). \tag{7}$$

Define

$$W_{2n}(t) := \int_0^t V_{2n}(t-s)BdW(s) = \sum_{k=1}^{\infty}\int_0^t V_{2n}(t-s)Be_k d\beta_k(s). \tag{8}$$

By Lemma 1, $W_{2n}(t)$ is a Gaussian random variable with the law $\mathcal{N}(0, L_t^{2n})$, where

$$L_t^{2n}x := \int_0^t V_{2n}(s)BB^*V_{2n}^*(s)x\, ds, \quad x\in H,$$

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given that L_t^{2n} is of trace class. We now show that W_{2n} has a continuous version. The following theorem is a generalisation of the corresponding result of Da Prato and Zabczyk [2] for generators of C_0-semigroups.

Theorem 2. *Assume that there is $\alpha \in (0,1/2)$ and $T \in (0,\infty)$ such that*

$$\int_0^T s^{-2\alpha}\mathrm{Tr}\big[V_n(s)BB^*V_n^*(s)\big]\,ds = C_{\alpha,T}^n < \infty, \tag{9}$$

and for $j = 0,1,2,\dots,n-1$,

$$\int_0^T s^{-2\alpha}\mathrm{Tr}\big[V_{2n-j}(s)BB^*V_{2n-j}^*(s)\big]\,ds = C_{\alpha,T}^{2n-j} < \infty. \tag{10}$$

Then $W_{2n}(t)$ defined by (8) has a continuous version.

Proof: Using (7) we can write

$$\begin{aligned} W_{2n}(t) &= \int_0^t V_{2n}(t-\sigma+\sigma-s)BdW(s) \\ &= \int_0^t V_n(t-\sigma)V_n(\sigma-s)BdW(s) \\ &\quad + \int_0^t \Big(\sum_{j=0}^{n-1}\frac{1}{j!}(\sigma-s)^jV_{2n-j}(t-\sigma)+(t-\sigma)^jV_{2n-j}(\sigma-s)\Big)BdW(s). \end{aligned}$$

Using the factorisation formula

$$\frac{\pi}{\sin(\pi\alpha)} = \int_s^t (t-\sigma)^{\alpha-1}(\sigma-s)^{-\alpha}\,d\sigma, \quad \alpha\in(0,1),\ 0\leqslant s\leqslant t,$$

we obtain

$$\begin{aligned} W_{2n}(t) &= \frac{\sin(\pi\alpha)}{\pi}\int_s^t (t-\sigma)^{\alpha-1}(\sigma-s)^{-\alpha}\,d\sigma \int_0^t V_n(t-\sigma)V_n(\sigma-s)BdW(s) \\ &\quad + \frac{\sin(\pi\alpha)}{\pi}\int_s^t (t-\sigma)^{\alpha-1}(\sigma-s)^{-\alpha}\,d\sigma \sum_{j=0}^{n-1}\int_0^t \frac{1}{j!}\big((\sigma-s)^jV_{2n-j}(t-\sigma)\big)BdW(s) \\ &\quad + \frac{\sin(\pi\alpha)}{\pi}\int_s^t (t-\sigma)^{\alpha-1}(\sigma-s)^{-\alpha}\,d\sigma \int_0^t \sum_{j=0}^{n-1}\frac{1}{j!}(t-\sigma)^jV_{2n-j}(\sigma-s)BdW(s). \end{aligned}$$

By the stochastic Fubini theorem, we have

$$\begin{aligned} W_{2n}(t) &= \frac{\sin(\pi\alpha)}{\pi}\int_0^t V_n(t-\sigma)(t-\sigma)^{\alpha-1}\int_0^\sigma V_n(\sigma-s)(\sigma-s)^{-\alpha}BdW(s)\,d\sigma \\ &\quad + \frac{\sin(\pi\alpha)}{\pi}\sum_{j=0}^{n-1}\frac{1}{j!}\int_0^t V_{2n-j}(t-\sigma)(t-\sigma)^{\alpha-1}\int_0^\sigma (\sigma-s)^{j-\alpha}BdW(s)\,d\sigma \\ &\quad + \frac{\sin(\pi\alpha)}{\pi}\sum_{j=0}^{n-1}\frac{1}{j!}\int_0^t (t-\sigma)^{j+\alpha-1}\int_0^\sigma V_{2n-j}(\sigma-s)(\sigma-s)^{-\alpha}BdW(s)\,d\sigma. \end{aligned} \tag{11}$$

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Writing

$$U_n(\sigma) = \int_0^\sigma V_n(\sigma - s)(\sigma - s)^{-\alpha} B dW(s),$$

and for $j = 0, 1, 2, \ldots, n-1$

$$U_j(\sigma) = \int_0^\sigma (\sigma - s)^{j-\alpha} B dW(s),$$
$$U_{2n-j}(\sigma) = \int_0^\sigma V_{2n-j}(\sigma - s)(\sigma - s)^{-\alpha} B dW(s),$$

and

$$P_n(t) = \frac{\sin(\pi\alpha)}{\pi} \int_0^t V_n(t-\sigma)(t-\sigma)^{\alpha-1} U_n(\sigma)\, d\sigma,$$
$$P_{2n-j}(t) = \frac{\sin(\pi\alpha)}{\pi} \int_0^t \frac{1}{j!} V_{2n-j}(t-\sigma)(t-\sigma)^{\alpha-1} U_j(\sigma)\, d\sigma,$$
$$P_j(t) = \frac{\sin(\pi\alpha)}{\pi} \int_0^t \frac{1}{j!} (t-\sigma)^{j+\alpha-1} U_{2n-j}(\sigma)\, d\sigma,$$

we can write $W_{2n}(t)$ as

$$W_{2n}(t) = P_n(t) + \sum_{j=0}^{n-1} P_{2n-j}(t) + \sum_{j=0}^{n-1} P_j(t).$$

As in Lemma 1 $U_n(\sigma)$ is a Gaussian random variable $\mathcal{N}(0, S_\sigma^n)$ for all $\sigma \in [0, T]$, where

$$S_\sigma^n x := \int_0^\sigma s^{-2\alpha} V_n(s) B B^* V^*(s) x\, ds.$$

Accordingly, for all $j = 0, 1, 2, \ldots, n-1$, $U_{2n-j}(\sigma)$ and $U_j(\sigma)$ are Gaussian random variables $\mathcal{N}(0, S_\sigma^{2n-j})$ and $\mathcal{N}(0, S_\sigma^j)$ respectively, where

$$S_\sigma^{2n-j} x := \int_0^\sigma s^{-2\alpha} V_{2n-j} B B^* V_{2n-j}^*(s) x\, ds$$
$$S_\sigma^j x := \int_0^t s^{2j-2\alpha} B B^* x\, ds.$$

By (9), for any $m > 0$, there exists a constant $D_{m,\alpha}^n$ such that for all $\sigma \in [0, T]$ we have

$$E\Big[\|U_n(\sigma)\|^{2m}\Big] \leqslant D_{m,\alpha}^n \sigma^m.$$

By (10), for $j = 0, 1, 2, \ldots, n-1$, there exist constants $D_{m,\alpha}^{2n-j}$ and $D_{m,\alpha}^j$ such that for all $\sigma \in [0, T]$ we have

$$E\Big[\|U_{2n-j}(\sigma)\|^{2m}\Big] \leqslant D_{m,\alpha}^{2n-j} \sigma^m,$$
$$E\Big[\|U_j(\sigma)\|^{2m}\Big] \leqslant D_{m,\alpha}^j \sigma^m.$$

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This implies

$$\int_0^T E\Big[\big\|U_n(\sigma)\big\|^{2m}\Big]\,d\sigma \leqslant \frac{D^n_{m,\alpha}}{m+1}T^{m+1},$$

and for $j = 0, 1, 2, \dots, n-1$

$$\int_0^T E\Big[\big\|U_{2n-j}(\sigma)\big\|^{2m}\Big]\,d\sigma \leqslant \frac{D^{2n-j}_{m,\alpha}}{m+1}T^{m+1},$$

$$\int_0^T E\Big[\big\|U_j(\sigma)\big\|^{2m}\Big]\,d\sigma \leqslant \frac{D^{j}_{m,\alpha}}{m+1}T^{m+1}.$$

Therefore $U_n(\cdot)\omega$, $U_{2n-j}(\cdot)\omega$ and $U_j(\cdot)\omega$ are in $L^{2m}\big([0,T];H\big)$ for almost all $\omega \in \Omega$ and $j = 0, 1, 2, \dots, n-1$. Furthermore, by Hölder's inequality and taking into account the exponential boundedness of $V_i(t)$ we have

$$\begin{aligned}\big\|P_n(t)\big\| &\leqslant \frac{M_T}{\pi}\left(\int_0^t \big[(t-\sigma)^{\alpha-1}\big]^{2m/(2m-1)}\,d\sigma\right)^{(2m-1)/2m}\|U_n\|_{L^{2m}([0,T];H)} \\ &= \frac{M_T}{\pi}\left(\frac{2m-1}{2m\alpha-1}\right)^{(2m-1)/2m} t^{\alpha-(1/2m)}\|U_n\|_{L^{2m}([0,T];H)},\end{aligned}$$

where $M_T = \sup_{t\in[0,T]}\big\|V_n(t)\big\|$. Accordingly, for all $j = 0, 1, 2, \dots, n-1$ we have

$$\begin{aligned}\big\|P_{2n-j}(t)\big\| &\leqslant \frac{M_T^j}{\pi j!}\left(\int_0^t \big[(t-\sigma)^{\alpha-1}\big]^{2m/(2m-1)}\,d\sigma\right)^{(2m-1)/2m}\|U_j\|_{L^{2m}([0,T];H)} \\ &= \frac{M_T^j}{\pi j!}\left(\frac{2m-1}{2m\alpha-1}\right)^{(2m-1)/2m} t^{\alpha-(1/2m)}\|U_j\|_{L^{2m}([0,T];H)},\end{aligned}$$

where $M_T^j = \sup_{t\in[0,T]}\big\|V_{2n-j}(t)\big\|$, and furthermore

$$\begin{aligned}\big\|P_j(t)\big\| &\leqslant \frac{1}{\pi j!}\left(\int_0^t \big[(t-\sigma)^{j+\alpha-1}\big]^{2m/(2m-1)}\,d\sigma\right)^{(2m-1)/2m}\|U_{2n-j}\|_{L^{2m}([0,T];H)} \\ &= \frac{1}{\pi j!}\left(\frac{2m-1}{2mj+2m\alpha-1}\right)^{(2m-1)/2m} t^{j+\alpha-(1/2m)}\|U_{2n-j}\|_{L^{2m}([0,T];H)}.\end{aligned}$$

Hence $P_n(\cdot)\omega \in C\big([0,T];H\big)$ for almost all $\omega \in \Omega$ and for all $j = 0, 1, 2, \dots, n-1$, $P_{2n-j}(\cdot)\omega, P_j(\cdot)\omega \in C\big([0,T];X\big)$ for almost all $\omega \in \Omega$. Thus

$$W_{2n}(\cdot)\omega = \left(P_n + \sum_{j=0}^{n-1} P_{2n-j} + \sum_{j=0}^{n-1} P_j\right)(\cdot)\omega \in C\big([0,T];H\big)$$

for almost all $\omega \in \Omega$ and (11) defines a continuous version of W_{2n}. ∎

COROLLARY 1. *Let A be the generator of an n-times integrated semigroup $\big\{V_n(t) \in \mathcal{L}(H); t \in [0,\infty)\big\}$ and all the assumptions in Theorem 2 hold. Then*

$$X(t) = V_{2n}(t)\xi + \int_0^t V_{2n}(t-s)BdW(s)$$

is a weak $2n$-integrated solution of (1) which has a continuous version.

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4. Example

Consider the stochastic wave equation

$$dY_t'(t,x) = \frac{d^2}{dx^2}Y(t,x)\,dt + dW(t,x)\,, \quad t \in [0,T],\ x \in \Omega = (0,1),$$
$$Y(t,0) = Y(t,1) = 0, \quad t \in [0,T],$$
$$Y(0,x) = Y_0(x)\,,\ Y_t'(0,x) = Y_1(x)\,,\ x \in \Omega,$$

where $dW(t,x)$ is white noise. Define the operator $\mathcal{A} = \dfrac{d^2}{dx^2}$ in $L^2(\Omega)$ with the domain

$$D(\mathcal{A}) = H^2(\Omega) \cap H_0^1(\Omega),$$

where $H^2(\Omega)$ and $H_0^1(\Omega)$ are the classical Sobolev spaces. The operator $-\mathcal{A}$ has a self-adjoint compact inverse and therefore its spectrum consists of discrete eigenvalues. Eigenfunctions and eigenvalues of $-\mathcal{A}$ can be obtained by solving

$$\frac{d^2 e_k}{dx^2} = -\mu_k e_k, \ \ e_k(0) = e_k(1) = 0,\ k \in \mathbb{N},$$

which gives

$$\mu_k = k^2\pi^2 > 0,\ e_k = \sqrt{2}\sin k\pi x,\ k \in \mathbb{N}.$$

Note that $\{e_k\}_{k=1}^{\infty}$ forms an orthonormal basis in $L^2(\Omega)$. Denote $L^2(\Omega) =: U,\ L^2(\Omega) \times L^2(\Omega) =: H$, and let $W(\cdot)$ be a U-valued cylindrical Wiener process. Setting

$$X(t) = \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} = \begin{pmatrix} Y(t,x) \\ Y_t'(t,x) \end{pmatrix},\ X_0 = \begin{pmatrix} Y_0(x) \\ Y_1(x) \end{pmatrix},$$

where $X(t),\ X_0 \in H$, we can rewrite the wave equation in the form (1):

$$dX(t) = AX(t)\,dt + BdW(t)\,, \quad X(0) = X_0.$$

The operator A is defined by

$$AX = \begin{pmatrix} X_2 \\ \mathcal{A}X_1 \end{pmatrix} = \begin{pmatrix} 0 & I \\ \mathcal{A} & 0 \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix},$$

$$\mathcal{D}(A) = \mathcal{D}(\mathcal{A}) \times L^2(\Omega) \subset H,$$

and $B \in \mathcal{L}(U,H)$ is defined by

$$Bu = \begin{pmatrix} 0 \\ u \end{pmatrix}.$$

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Then (see, for example, [4]) A generates an exponentially bounded non-degenerate 1-time integrated semigroup

$$V(t) = \begin{pmatrix} \mathcal{S}(t) & \int_0^t \mathcal{S}(s)\,ds \\ \mathcal{C}(t) - I & \mathcal{S}(t) \end{pmatrix}, \quad t \geqslant 0,$$

on H. Here $\mathcal{C}$ and $\mathcal{S}$ are cosine and sine operator-functions defined by

$$\mathcal{C}(t)v := \sum_{k=1}^{\infty} \cos\left(\sqrt{\mu_k} t\right) v_k e_k, \quad \mathcal{S}(t)v := \sum_{k=1}^{\infty} \frac{\sin\left(\sqrt{\mu_k} t\right)}{\sqrt{\mu_k}} v_k e_k,$$

where $v \in L^2(\Omega)$ and $v_k = \langle v, e_k \rangle_{L^2(\Omega)}$.

Note that if we consider a smaller space $H_1 := H_0^1(\Omega) \times L^2(\Omega)$, then the operator A with the domain $\mathcal{D}(A) = \mathcal{D}(\mathcal{A}) \times H_0^1(\Omega) \subset H_1$, is the generator of a C_0-semigroup on H_1.

Now, since we have that

$$\sum_{k=1}^{\infty} \int_0^t \left\| V(s) B e_k \right\|^2 ds \leqslant C(T) \sum_{k=1}^{\infty} \frac{1}{\mu_k} < \infty,$$

then the process

$$X(t) = V(t)\xi + \int_0^t V(t-s) B\,dW(s)$$

is a weak 1-integrated solution of (1), that is,

$$\langle X(t), \nu \rangle = \langle t X_0, \nu \rangle + \left\langle \int_0^t X(s)\,ds, A^* \nu \right\rangle + \left\langle \int_0^t (t-s) B\,dW(s), \nu \right\rangle, \quad \nu \in D(A^*).$$

References

[1] W. Arendt, 'Vector valued Laplace transforms and Cauchy problems', *Israel J. Math.* **59** (1987), 321–349.

[2] G. Da Prato and J. Zabczyk, *Stochastic equations in infinite dimensions* (Cambridge University Press, Cambridge, 1992).

[3] G. Da Prato and J. Zabczyk, *Ergodicity for infinite-dimensional systems* (Cambridge University Press, Cambridge, 1996).

[4] H. Kellerman and M. Hieber, 'Integrated semigroups', *J. Funct. Anal.* **84** (1989), 160–180.

[5] G. Lumer, 'Generalized evolution operators and (generalized) C-semigroups', in *Semigroup Theory and Evolution Equations (Delft, 1989)* (Dekker, New York, 1991), pp. 337–345.

[6] F. Neubrander, 'Integrated semigroups and their applications to the abstract Cauchy problem', *Pacific J. Math.* **135** (1988), 111–155.

Department of Pure Mathematics
The University of Adelaide
Adelaide, S.A. 5005
Australia

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