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BEHAVIOR OF COEFFICIENTS OF INVERSES OF α-SPIRAL FUNCTIONS

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1. Preliminary remarks. If $f(z)$ is univalent (regular and one-to-one) in the open unit disk Δ, $\Delta = \{z \in \mathbf{C}:|z| < 1\}$, and has a Maclaurin series expansion of the form

$$(1.1) \qquad f(z) = z + a_2z^2 + a_3z^3 + \dots, \quad z \in \Delta,$$

then, as de Branges has shown, $|a_k| \leqq k$, for $k = 2, 3, \dots$ and the Koebe function.

$$(1.2) \qquad K(z) = \frac{z}{(1-z)^2} = \sum_{k=1}^{\infty} kz^k$$

serves to show that these bounds are the best ones possible (see [3]). The functions defined above are generally said to constitute the class $\mathcal{S}$.

If $f(z)$ is in $\mathcal{S}$, then its inverse $\check{f}(w)$ exists and has a series expansion

$$(1.3) \qquad \check{f}(w) = w + A_2w^2 + A_3w^3 + \dots$$

in some disk of positive radius centered at the origin. Using his parametric method, Loewner [10] showed that

$$(1.4) \qquad |A_n| \leqq \frac{1}{n}\binom{2n}{n+1}$$

for $n \geqq 2$ and that the sharp upper bound is achieved by the inverse of a suitable rotation of the Koebe function.

Recently there has been a good deal of interest in determining the behavior of the coefficients given in (1.3) when the corresponding function $f(z)$ is restricted to some proper subclass of $\mathcal{S}$. For example, it has been shown ([8], [1]) that $|A_k| \leqq 1$, $k = 2, 3, \dots, 8$, whenever $f(z)$ in $\mathcal{S}$ maps Δ onto a convex domain, but that $|A_{10}| > 1$ for some such function [5]. Other subclasses of $\mathcal{S}$ have been shown to have curious properties relating to the coefficients A_k, ([5], [6], [8], [9], [12]). Our purpose here is to report on the behavior of the coefficients A_k when $f(z)$ is spiral-like.

A function $f(z)$ as in (1.1) is spiral-like if for some real α, $|\alpha| < \pi/2$,

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$$\text{(1.5)} \qquad \operatorname{Re}\left\{e^{i\alpha}\frac{zf'(z)}{f(z)}\right\} > 0, \quad z \in \Delta.$$

The class of all such functions is often denoted by the symbol $\overset{\vee}{\mathscr{S}}_\alpha$ and Špaček, who introduced the class [13], showed that $\overset{\vee}{\mathscr{S}}_\alpha \subset \mathscr{S}$; it was later called the class of α-spiral functions [7]. For $\alpha = 0$ one obtains the class of starlike function $\mathscr{S}^*$, i.e., $\overset{\vee}{\mathscr{S}}_0 = \overset{*}{\mathscr{S}}$.

$\mathscr{P}$ will represent the family of all functions regular in Δ for which $P(0) = 1$ and $\operatorname{Re} P(z) > 0$, $z \in \Delta$. Then condition (1.5) can be restated in the equivalent form

$$\text{(1.6)} \qquad e^{i\alpha}\frac{zf'(z)}{f(z)} - i \sin \alpha = P(z) \cdot \cos \alpha$$

for $P(z)$ in $\mathscr{P}$ and z in Δ. (Note: In subsequent computations it will be convenient to replace $P(z)$ by its reciprocal; this is no restriction, since both $P(z)$ and its reciprocal are simultaneously in $\mathscr{P}$.)

2. Our conclusions.

THEOREM 1. *If $f(z)$ is an α-spiral function, $\overset{\vee}{f}(w) = w + A_2w^2 + \dots$,*

$$\text{(2.1)} \qquad a = ie^{-i\alpha} \sin \alpha,\ A = |32a^2 - 52a + 21| \text{ and } B = |5 - 6a|;$$

then

$$\text{(2.2)} \qquad |A_2| \leqq |1 - a| \cdot 2;$$

and

$$\text{(2.3)} \qquad |A_3| \leqq B \cos \alpha,$$

which are both sharp, and

$$\text{(2.4)} \qquad |A_4| \leqq \begin{cases} \dfrac{4}{3}A \cos \alpha, \text{ if } 2B^2 + 5B + 2 \leqq A(B^2 + 3B + 3) \\[2ex] \dfrac{4}{3} \cos \alpha\left(\dfrac{A + B^2}{(1 + B)^3} + \dfrac{2B}{B + 1}\right), \text{ otherwise.} \end{cases}$$

THEOREM 2. *If $f(z)$ is an α-spiral function and $\overset{\vee}{f}(w) = w + A_2w^2 + \dots$, then*

$$\text{(2.5)} \qquad |A_n| \leqq \frac{e^{\pi n \cos\alpha \sin\alpha}}{n} \cdot \frac{\Gamma(1 + 2n \cos^2 \alpha)}{(\Gamma(1 + n \cos^2 \alpha))^2}$$

for $n = 2, 3, 4, \dots$.

THEOREM 3. *If $f(z)$ is a starlike function in $\mathscr{S}$, i.e., $f(z)$ is a zero-spiral function, and $\overset{\vee}{f}(w) = w + A_2w^2 + \dots$, then*

(2.6) $$|A_n| \leqq \frac{1}{n}\binom{2n}{n+1}, \quad \textit{for } n \geqq 2$$

and this bound is rendered sharp when $f(z)$ *is a properly chosen rotation of the Koebe function* (1.2).

This, of course, is not a new result, but a consequence of the work of Loewner cited above. However, the proof given here is a new one, relatively simple and applicable directly to the class $\mathcal{S}^*$.

3. Proof of theorem 1. If we let a member of $\mathcal{P}$ have the representation

(3.1) $$P(z) = 1 + C_1 z + C_2 z^2 + \dots, \ z \text{ in } \Delta,$$

let $w = f(z)$ and $z = \check{f}(w)$, recall that

$$(\check{f}(w))' = 1/f'(z),$$

and rewrite (1.6) accordingly; we obtain

(3.2) $$\check{f}(w)P(\check{f}(w)) = w(\check{f}(w))'(e^{-i\alpha}\cos\alpha + ie^{-i\alpha}\sin\alpha \cdot P(\check{f}(w))),$$

or

(3.3) $$\sum_{k=1}^{\infty} A_k w^k \cdot \left(1 + \sum_{k=1}^{\infty} C_k(\check{f}(w))^k\right) = \sum_{k=1}^{\infty} kA_k w^k \cdot \left(1 + \sum_{k=1}^{\infty} aC_k(\check{f}(w))^k\right);$$

and finally that

(3.4) $$\begin{cases} A_2 = (1 - a)C_1, \\ 2A_3 = (2 - 3a)C_1A_2 + (1 - a)C_2, \text{ and} \\ 3A_4 = (2 - 4a)C_1A_3 + (3 - 4a)C_2A_2 + (1 - 2a)C_1A_2^2 \\ \qquad + (1 - a)C_3. \end{cases}$$

The relations in (3.4) may be rewritten as

(3.5) $$\begin{cases} A_2 = (1 - a)C_1, \\ 2A_3 = (1 - a)((2 - 3a)C_1^2 + C_2), \text{ and} \\ 3A_4 = (1 - a)((1 - 2a)(3 - 4a)C_1^3 \\ \qquad + (4 - 6a)C_1C_2 + C_3). \end{cases}$$

(2.2) is now obtained from the first of these relations by an application of

Caratheodóry's well-known theorem which states that $|C_k| \leqq 2$ for all k, (see **[3]**, **[4]**, for example).

To justify (2.3) and (2.4) we call upon another result due to Caratheódory (stated here in a form due to Toeplitz); it appears in **[4]**.

LEMMA. *The power series for* $P(z)$ *given in* (3.1) *converges in* Δ *to a function in* $\mathcal{P}$ *if and only if the Toeplitz determinants*

$$(3.6) \quad D_n = \begin{vmatrix} 2 & C_1 & C_2 & \dots & C_n \\ C_{-1} & 2 & C_1 & \dots & C_{n-1} \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ C_{-n} & C_{-n+1} & C_{-n+2} & \dots & 2 \end{vmatrix}, n = 1, 2, 3, \dots,$$

with $C_{-k} = \bar{C}_k$, *are all non-negative. They are strictly positive except for*

$$P(z) = \sum_{k=1}^{m} \rho_k P_0(e^{it}z),$$

$$(3.7) \quad P_0(z) = \frac{1+z}{1-z} = 1 + 2z + 2z^2 + \dots,$$

$\rho_k > 0$, t_k *real and* $t_k \neq t_j$ *for* $k \neq j$; *in this exceptional case* $D_n > 0$ *for* $n < m - 1$ *and* $D_n = 0$ *for* $n \geqq m$.

With no restriction we may assume that $C_1 > 0$ and write

$$(3.8) \quad D_2 = \begin{vmatrix} 2 & C_1 & C_2 \\ C_1 & 2 & C_1 \\ \bar{C}_2 & C_1 & 2 \end{vmatrix} = 8 + 2\,\mathrm{Re}\{C_1^2 C_2\} - 2|C_2|^2 - 4C_1^2 \geqq 0$$

from which we conclude that

$$(3.9) \quad 2C_2 = C_1^2 + x(4 - C_1^2), \quad \text{for some } x, |x| \leqq 1.$$

This representation for C_2 and (3.6) gives

$$(3.10) \quad 4A_3 = (1 - a)((4 - 6a + 1)C_1^2 + x(4 - C_1^2))$$

and the bound $|x| \leqq 1$, along with an application of the triangle inequality, gives

$$(3.11) \quad 4|A_3| \leqq |1 - a| \cdot |(|5 - 6a| - 1)C_1^2 + 4| \leqq 4|1 - a| \cdot |5 - 6a|,$$

because $|5 - 6a| \geqq 1$. Equality holds true in (2.2) and (2.3) when $f(z)$, and consequently $\check{f}(w)$, is the solution of (1.6) with $P(z)$ replaced by $P_0(z)$.

To arrive at (2.4) we appeal once again to the lemma. $D_3 \geqq 0$, in (3.6), is equivalent to

$$(3.12)\quad |(4C_3 - 4C_1C_2 + C_1^3)(4 - C_1^2) + C_1(2C_2 - C_1^2)^2| \leqq 2(4 - C_1^2)^2 - 2|2C_2 - C_1^2|^2;$$

and using (3.9) we rewrite (3.12) as

$$(3.13)\quad 4C_3 = C_1^3 + 2(4 - C_1^2)C_1x - C_1(4 - C_1^2)x^2 + 2(4 - C_1^2)(1 - |x|^2)z,$$

for some z, $|z| \leqq 1$. Combining (3.13) with (3.9) and (3.13) yields the equation

$$(3.14)\quad \frac{12A_4}{1 - a} = (21 - 52a + 32a^2)C_1^3 + 2C_1(4 - C_1^2)(5 - 6a)x - C_1(4 - C_1^2)x^2 + 2(4 - C_1^2)(1 - |x|^2)z.$$

Letting $|x| = \rho$, recalling the definitions of A and B given in (2.1), applying the triangle inequality and replacing $|z|$ by its maximum value 1, we may find an upper bound for the right side of (3.14) by maximizing the function

$$(3.15)\quad \phi(\rho) = AC_1^3 + 2C_1(4 - C_1^2)B\rho + C_1(4 - C_1^2)\rho^2 + 2(4 - C_1^2)(1 - \rho^2).$$

If $C_1 = 2$, then $|\phi(\rho)| \leqq 8A$, and if $C_1 = 0$, then $|\phi(\rho)| \leqq 8$; consequently, we assume $0 < C_1 < 2$.

$$\phi'(\rho) = 2(4 - C_1^2)(C_1B + \rho(C_1 - 2))$$

and $\phi(\rho)$ achieves its maximum when

$$\rho_0 = \frac{C_1B}{2 - C_1}.$$

If $\rho_0 \leqq 1$, then

$$C_1 \leqq \frac{2}{B + 1}$$

and in this case we have

$$(3.16)\quad \frac{12|A_4|}{|1 - a|} \leqq \phi\left(\frac{C_1B}{2 - C_1}\right)\Big|_{C_1=\frac{2}{B+1}}$$

$$= \{(A + B^2)C_1^3 + 2(B^2 - 1)C_1^2 + 8\}\Big|_{C_1=\frac{2}{B+1}}$$

$$= 8\left(\frac{A + B^2}{(B + 1)^3} + \frac{2B}{B + 1}\right),$$

having made use of $B^2 - 1 = 24 \cos^2 \alpha > 0$, for $|\alpha| \neq \pi/2$.

Now we suppose that $\phi'(\rho)$ has its zero at

$$\rho_0 = \frac{C_1 B}{2 - C_1} > 1,$$

then

$$\frac{2}{B + 1} < C_1 \leqq 2.$$

Replacing ρ by 1 in (3.15), we see that our problem reduces to one of maximizing

$$(3.17) \quad \psi(C_1) = C_1((A - 2B - 1)C_1^2 + (8B + 4))$$

over the interval $\left(\frac{2}{B + 1}, 2\right]$. If $A - 2B - 1 \geqq 0$, then the maximum occurs at 2, and we conclude that

$$\frac{12|A_4|}{|1 - a|} \leqq 8A.$$

On the other hand, if $A - 2B - 1 < 0$, then the solution of $\psi'(C_1) = 0$ we are interested in is the (non-negative) solution of

$$C_1^2 = \frac{8B + 4}{3(1 + 2B - A)}$$

lying in the interval given above. These conditions on C_1 are equivalent to the statement

$$\frac{4}{(B + 1)^2} < \frac{8B + 4}{3(1 + 2B - A)} \leqq 4.$$

Then the maximum for ψ occurs at $C_1 = 2$, because $\psi' > 0$ over $\left(\frac{2}{B + 1}, 2\right]$. $\psi(2) = 8A$ and this is the upper bound when

$$\frac{A + B^2}{(1 + B)^3} + \frac{2B}{B + 1} < A,$$

which is equivalent to

$$2B^2 + 5B + 2 \leqq A(B^2 + 3B + 3).$$

The sharp upper bound corresponds to the example given above for (2.2) and (2.3).

4. Proofs of theorems 2 and 3. If $f(z)$ and $\check{f}(w)$ are as before, let $C(r)$ be the image of the circle $|z| = r$, $r < 1$, under $f(z)$, then

$$(4.1)\quad A_n = \frac{1}{2\pi i}\int_{C(r)} \frac{\check{f}(w)dw}{w^{n+1}} = \frac{1}{2\pi i}\int_{|z|=r} \frac{zf'(z)}{f(z)^{n+1}}dz$$
$$= \frac{1}{2\pi i n}\int_{|z|=r} \frac{dz}{f(z)^n},$$

and to bound A_n, using (4.1), we seek a bound for $|f(z)|^{-n}$.

Using the Stieltjes integral representation for $P(z)$ in $\mathcal{P}$, (see [3], for example) in (1.6), then performing an integration we have

$$(4.2)\quad \frac{z}{f(z)} = \exp\left\{2e^{-i\alpha}\cos\alpha \int_0^{2\pi} \log(1 - e^{it}z)\, d\mu(t)\right\},$$

for a non-decreasing $\mu(t)$ such that

$$\int_0^{2\pi} d\mu(t) = 1.$$

From (4.2) we get

$$(4.3)\quad \left|\frac{z}{f(z)}\right|^n$$
$$\leqq \exp\{n\pi \sin\alpha\cos\alpha\}\cdot\exp\left\{2n\cos^2\alpha \int_0^{2\pi} \log|1 - ze^{it}|d\mu(t)\right\}$$

and this, along with (4.1) yields

$$(4.4)\quad |A_n| \leqq \frac{1}{2\pi n}\int_{|z|=r} \frac{|dz|}{|f(z)|^n}$$
$$\leqq \frac{e^{\pi n\cos\alpha\,\sin\alpha}}{2\pi n r^n}\int_{|z|=r}\left(\exp\int_0^{2\pi}\log|1 - ze^{it}|^{2n\cos^2\alpha}\, d\mu(t)\right)|dz|$$
$$\leqq \frac{e^{\pi n\cos\alpha\,\sin\alpha}}{2\pi n r^n}\int_{|z|=r}\int_0^{2\pi}|1 - ze^{it}|^{2n\cos^2\alpha}d\mu(t)\,|dz|$$
$$= \frac{e^{\pi n\cos\alpha\,\sin\alpha}}{2\pi n r^{n-1}}\int_0^{2\pi}\int_0^{2\pi}|1 - re^{it}e^{i\theta}|^{2n\cos^2\alpha}d\theta d\mu(t),$$

having let $z = re^{i\theta}$, $0 < r < 1$. (Here we have used the integral generalization of the inequality between the arithmetic and geometric means. (see p. 110, [11], for example).)

The integrals in (4.4) are bounded, consequently we let $r \to 1$ and obtain

$$(4.5)\quad |A_n| \leqq \frac{e^{\pi n\cos\alpha\,\sin\alpha}}{2\pi n}\int_0^{2\pi}|1 - e^{i\theta}|^{2n\cos^2\alpha}d\theta$$
$$= e^{\pi n\cos\alpha\,\sin\alpha}\cdot\left(\frac{2^{2n\cos^2\alpha}}{2\pi n}\int_{-\pi}^{\pi}\left|\cos\frac{\theta}{2}\right|^{2n\cos^2\alpha}d\theta\right)$$

$$= e^{\pi n \cos\alpha \sin\alpha} \cdot \frac{1}{n} \cdot \frac{\Gamma(2n\cos^2\alpha + 1)}{(\Gamma(n\cos^2\alpha + 1))^2},$$

having referred to standard tables ([**2**], or see the analogous form p. 108 [**6**]).

Our proof of Theorem 3 begins with an analysis of the relationships between coefficients C_k and A_k, all k, as given in (3.1), (3.2), (3.3), (3.4) and (3.5), but with $\alpha = 0$. Computation and rearrangement of terms yields the following relationships:

$$(4.6)\quad \begin{cases} A_2 = C_1 \\ 2A_3 = 2C_1^2 + C_2, \\ 3A_4 = 3C_1^3 + 4C_1C_2 + C_3, \\ 4!A_5 = 24C_1^4 + 58C_1^2C_2 + 28C_1C_3 + 9C_2^2 + 6C_4, \text{ and} \\ 5!A_6 = 120C_1^5 + 436C_1^3C_2 + 192C_1C_2^2 + 312C_1^2C_3 \\ \qquad + 108C_1C_4 + 72C_2C_3 + 24C_5. \end{cases}$$

Now, an examination of the way in which the coefficients in (4.6) are formed shows that

$$(4.7)\quad (k-1)!A_k = C_1^{k-1} + Q_k(C_1, C_2, \ldots, C_{k-1}),$$

for each k, and Q_k is a polynomial all of whose coefficients are non-negative. Consequently, a sharp upper bound for (4.7) is obtained by a direct application of the triangle inequality and the bounds $|C_k| \leqq 2$, all k. A function maximizing $|C_1|$ and all subsequent $|C_k|$ is $P_0(z)$, given above; the corresponding extremal in $\mathcal{S}^*$ is the inverse of a suitable rotation of the Koebe function, $K(z)$, see (1.2), and whose inverse has coefficients as in (1.4).

5. Remarks. (i) Replacing $P(z)$ in (1.6) by its reciprocal plays a significant role in all the above computations since doing so gives tractable representations for the A_k's in terms of coefficients of members of $\mathcal{P}$. This observation was made by Campschroer [**1**] in a similar situation.

(ii) The bound in (2.5) is not best possible. In particular, when $\alpha = 0$, (2.5) does not reduce to (2.6); however (2.5) is of the correct order.

(iii) The methods used to find bounds given in Theorem 1 appear too cumbersome to handle $|A_5|, |A_6|, \ldots$. For example,

$$4A_5 = (2-5a)C_1A_4 + (3-5a)C_2A_3 + (3-5a)C_2A_2^2 + (4-5a)C_3A_2 + (2-5a)C_1A_2A_3 + (1-a)C_4,$$

which upon elimination of C_k's reduces to

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$$4!A_5 = (1 - a)\{ (24 - 130a + 225a^2 - 125a^3)C_1^4 + 2(29 - 95a + 75a^2)C_1^2C_2 + 4(7 - 10a)C_1C_3 + 3(3 - 5a)C_2^2 + 6C_4\}.$$

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