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FINITELY GENERATED SUBGROUPS OF AMALGAMATED FREE PRODUCTS AND HNN GROUPS

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Abstract

The theory of groups acting on trees due to Bass and Serre (1969) is applied to simplify some results of Burns (1972, 1973) giving conditions under which an amalgamated free product or HNN extension has the properties that any finitely generated subgroup containing an infinite subnormal subgroup must have finite index and that the intersection of two finitely generated subgroups is finitely generated.

Burns (1972, 1973) gives sufficient conditions for an amalgamated free product $A *_U B$ or HNN group $\langle A, x; x^{-1}Ux = V\rangle$ to have the following properties: any two finitely generated subgroups intersect in a finitely generated subgroup, and any finitely generated subgroup containing a (sufficiently large) subnormal subgroup has finite index. His conditions are on the position of U in A, and include the case that A is free and U is cyclic generated by an element which is not a proper power. He observes that one can deduce from his results that a Fuchsian group has these properties.

In this paper we prove (and slightly extend) Burns's results using the Bass–Serre theory of groups acting on trees (Cohen (1974), Serre (1969)). To some extent this paper is propaganda for the view that the Bass–Serre theory leads to simpler proofs than more combinatorial methods (because the tree carries information that does not need to be mentioned explicitly). This paper is a sequel to Cohen (1974), to which the reader is referred for notation and many results.

1. Burns subgroups

The subgroup U of the group A is called a Burns subgroup if it has a left transversal T with $1 \in T$ which satisfies the following two conditions:

(1) there is a finite subset F of U such that $U(T-\{1\}) \subseteq TF$;

(2) for any finitely generated subgroup H of A and element $a \in A$ there is a finite subset F_1 of U (depending on H and a) such that $aH \subseteq TF_1(H \cap U)$.

These subgroups are introduced by Burns (1972, 1973) who calls them almost malnormal finitely involved, or AMFI, subgroups. Burns (1972) shows that when A is free and U is cyclic, generated by an element which is not a proper power, then U is a Burns subgroup of A. Plainly any finite subgroup is a Burns subgroup, so the results and methods of this paper extend those of Cohen (1974).

(2) is plainly equivalent to:

(2′) for any finitely generated subgroup H of A and element a there is a finite subset F_2 of A (depending on H and a) such that $Ha \subseteq TF_2(a^{-1}Ha \cap U)$.

(1) implies

(3) for any $a \notin U$, $U \cap aUa^{-1}$ is finite (which explains why U is referred to as almost malnormal).

For we may assume $a \in T - \{1\}$. If $u = tvt^{-1}$, where $u, v \in U$ and $t \in T - \{1\}$, then $tv = ut \in TF$ shows $v \in F$, as required.

From (2) follows

(4) for any finitely generated subgroups H, K of A, the intersection of a (H, U) double coset and a (K, U) double coset contains only finitely many $(H \cap K, U)$ double cosets (see the remark in section 4 of Burns (1972)).

Replacing H, K by $g^{-1}Hg, g^{-1}Kg$ if necessary, we need only consider $HU \cap KU$. Let $k = hu$, where $h \in H$, $k \in K$, and $u \in U$. There exist finite subsets F_1, F_2 of U with $H \subseteq TF_1(H \cap U)$ and $K \subseteq TF_2(K \cap U)$. Then $u \in (H \cap U)F_3(K \cap U)$ where $F_3 = F_1^{-1}F_2$ is a finite subset of U. Write $u = vcw$, where $v \in H \cap U, w \in K \cap U$, and $c \in F_3$. Then the $(H \cap K, U)$ double coset containing $k = hu$ will also contain $kw^{-1} = hvc$ which is an element of $K \cap Hc$. Now $K \cap Hc$ is a right $H \cap K$ coset. Hence every $(H \cap K, U)$ double coset contained in $HU \cap KU$ contains an element from one of finitely many right $H \cap K$ cosets, as needed.

LEMMA 1. *Let H be a finitely generated subgroup of A. Let U be a subgroup of A which is either a Burns subgroup or is finitely generated and satisfies* (3) *and* (4). *Then the intersection of a (H, U) double coset and a (H, aUa^{-1}) double coset, where $a \notin U$, contains only finitely many right H cosets.*

PROOF. As usual, we need only consider $HU \cap HaUa^{-1}$. Choose an element $hu = ava^{-1}$ in some right H coset, where $h \in H$, $u, v \in U$.

Let U be a Burns subgroup of A. Then there is a finite subset F_1 of U with $a^{-1}H \subseteq TF_1(H \cap U)$ and (writing $a^{-1} = tw$, with $t \in T - \{1\}$, $w \in U$) a finite subset F_2 of U with $Ua^{-1} \subseteq TF_2$. As $a^{-1}hu = va^{-1} \in TF_1(H \cap U)u \cap TF_2$ we can write $u = kc$, where $k \in H \cap U$ and c is in the finite set $F_1^{-1}F_2$. Then the right H coset containing $hu = hkc$ also contains c, and so is one of finitely many cosets.

Suppose U is finitely generated and satisfies (3) and (4). Then by (4) there is a finite subset F_1 of U such that $hu = ava^{-1} \in HU \cap aUa^{-1}U \subseteq (H \cap aUa^{-1})F_1U$. Write $hu = kcw$ where $c \in F_1$, $w \in U$, and $k \in H \cap aUa^{-1}$. The right H coset containing hu will also contain $cw = k^{-1}hu = k^{-1}ava^{-1} = av'a^{-1}$ for some $v' \in U$. For given c, this element is in a unique left $(U \cap aUa^{-1})$ coset, so by (3) there are only finitely many possibilities for this element.

LEMMA 2. *Let U be a Burns subgroup of A, and let V be a subgroup of A such that there is a finite subset F_1 of U with $V(T-\{1\}) \subseteq TF_1$ (for instance, $V \subseteq U$). Let H and K be finitely generated subgroups of A and $c, d \in A$. Then there is a finite subset X of U such that if $u \in U$ satisfies $chu = vdk$ for some $h \in H, k \in K, v \in V$ with $dk \notin U$ then $u \in (H \cap U)X(K \cap U)$.*

PROOF. There is a finite subset F_2 of U such that $dk \in TF_2(K \cap U)$. As $dk \notin U$ we have $vdk \in TF_1F_2(K \cap U)$. There is also a finite subset F_3 of U such that $ch \in TF_3(H \cap U)$. Then

$$chu = vdk \in TF_3(H \cap U)u \cap TF_1F_2(K \cap U),$$

and so $F_3(H \cap U)u$ intersects $F_1F_2(K \cap U)$. Hence $u \in (H \cap U)F_3^{-1}F_1F_2(K \cap U)$.

2. Bass–Serre theory

If $G = A *_U B$ then G acts without inversions on a tree T. There is an edge e_0 (the initial edge), whose stabiliser, stab e_0, is U. One vertex of e_0, the origin O, has stabiliser A, the other has stabiliser B. There is one orbit of edges and two of vertices, corresponding to e_0 and its two vertices. If e is an edge with vertex O then $e \in Ae_0$.

If $G = \langle A, x; x^{-1}Ux = V \rangle$, an HNN extension of A, then G acts without inversions on a tree T. There is an edge e_0 (the initial edge) with stab $e_0 = U$. One vertex of e_0, the origin O, has stabiliser A, the other is xO with stabiliser xAx^{-1}. There is one orbit of edges and one of vertices. The edge $x^{-1}e_0$ has O as a vertex and stab $x^{-1}e_0 = x^{-1}Ux = V$. If e is an edge with vertex O then either $e \in Ae_0$ or $e \in Ax^{-1}e_0$ but not both.

In either case if e has vertex O and f is an edge with $f \in Ae$, then any $h \in G$ with $f = he$ must be in A. For let $f = ae$ with $a \in A$. Then $a^{-1}he = e$ and as G acts without inversions we must have $a^{-1}hO = O$, whence $a^{-1}h \in A$. For the same reason, if e is an edge with vertex P and $he = e_0$ with $hP = O$ for some $h \in G$, then any g with $ge = e_0$ has $gP = O$.

In either case we call an edge *special* if the following two conditions hold:

(i) the initial vertex of e (orienting outwards from O) is in GO, but is not O;

(ii) if P is the initial vertex of e, and e' the unique edge ending at P, then $e \in (\operatorname{stab} P)\, e'$.

Note that if $P \neq O$, (i) always holds in the HNN case, while (ii) always holds for the amalgamated free product.

Lemma 3 below is essentially the same as Lemma 2.6 of Burns (1973), a reversing H-orbit containing a special edge being exactly what Burns calls a double-ended coset. For the definition and significance of reversing orbits see Cohen (1974). The reader is advised to draw parts of the relevant trees to help visualise the proofs of Lemma 3 and later results.

LEMMA 3. *Let G be either an amalgamated free product or an HNN group, T the tree described above on which G acts. Let H be a subgroup of G. Then H has only finitely many reversing orbits provided only finitely many of its reversing orbits contain a special edge.*

PROOF. If G is an amalgamated free product then any reversing H-orbit contains an edge starting at a vertex of GO, which is special unless it starts at O. Suppose a reversing H-orbit contains an edge e_1 starting at O, and let h be negative for e_1. Then if $e \neq e_1$ is an edge starting at O, He contains he, which is special since he_1 is the only edge ending at hO. Hence every reversing H-orbit except perhaps He_1 contains a special edge.

Now let G be an HNN group. We first show that only finitely many reversing H-orbits contain an edge e starting at $P \neq O$ (for some P) and such that another edge f starting at P is also in a reversing H-orbit.

For either $f \in (\operatorname{stab} P)\, e$ or $e \in (\operatorname{stab} P)\, e'$ or $f \in (\operatorname{stab} P)\, e'$, since there are only two G-orbits of edges at each point. In the first case choose $h \in H$ so that hf ends at hP (by choosing h negative for f). As the edge he lies in $(\operatorname{stab} hP)\, hf$, it is special, and so e lies on one of finitely many orbits. The same holds in the second case, as e is special.

In the third case f, being special, lies in one of finitely many orbits, hence so does P (which is a vertex of f). In each orbit choose a vertex $P_i \neq O$ and (if possible) choose an edge e_i starting at P_i with e_i in a reversing orbit and $e_i \notin (\operatorname{stab} P_i)\, e'_i$, where e'_i is the edge ending at P_i. Let $P_i = hP$. Then either $he = e'_i$ or he starts at P_i and lies in either $(\operatorname{stab} P_i)e'_i$ or $(\operatorname{stab} P_i)e_i$ (for he is in a reversing orbit, so if $he \notin (\operatorname{stab} P_i)e'_i$ an edge e_i exists and every edge with vertex P_i is in the $(\operatorname{stab} P_i)$-orbit of either e_i or e'_i). Thus either $he = e'_i$ or he is special or (as in the first case of the previous paragraph) he can be shown to be in one of finitely many orbits.

Also only finitely many reversing orbits contain an edge starting at O. For let e_1, e_2, and e start at O and lie in reversing orbits. Let h be negative for e_2. Then he_2 ends at hO, so $hO \neq O$, and both he and he_1 start at hO. Hence he is one of the edges previously considered.

Choosing (at most) two suitable edges from each of the orbits we have considered, we obtain a finite set X such that if e is in such an orbit there is an h positive for e with $he \in X$. Let Y be the finite set consisting of all edges in the irreducible paths from O to the edges of X. We show that any reversing orbit meets Y.

Note that if e is in a reversing orbit either $e \in HX$ or there is an edge f, starting where e ends, in a reversing orbit. For let e end at Q. Choose h negative for e, so that he starts at hQ. If $hQ = O$ then $he \in HX$. Otherwise take f to be the edge such that hf ends at hQ.

Hence if e_1 is in a reversing orbit there is either an infinite sequence of edges $e_1, e_2, \cdots$, each in a reversing orbit, none in HX, with e_i starting at P_{i-1} and ending at P_i for all i, or else there is a finite sequence of edges $e_1, \cdots, e_n$, each in a reversing orbit, with e_i starting at P_{i-1} and ending at P_i, with $e_n \in HX$.

In the first case choose h so that he_1 ends at hP_o. We cannot have he_i ending at hP_{i-1} for all i, as this would require the path from O to hP_o to contain he_i for all i. Hence we can choose r so that he_r ends at hP_{r-1} but he_{r+1} does not end at hP_r. Then both he_r and he_{r+1} start at hP_r. By definition this gives $e_r \in HX$. This contradicts the definition of the sequence, so this case does not occur.

In the second case take n minimal. Choose h with $he_n \in X$ and he_n ending at hP_n. If he_i ends at hP_i for all i the definition of Y (since we are orienting outwards from O) gives $he_1 \in Y$. If not we can choose r so that he_r ends at hP_r while he_{r-1} ends at hP_{r-2}. Then both he_r and he_{r-1} start at hP_{r-1}. By definition of X this gives $he_{r-1} \in HX$. As this contradicts the minimality of n, this case cannot occur. Hence we must have $he_1 \in Y$, so $e_1 \in HY$, as required.

3. The main theorems

In the theorems below we consider a group A and a Burns subgroup U. It is easy to see that we need only require the conclusions of Lemma 1 (or Lemma 2) to hold. Also we require certain subgroups of U to be finitely generated. The easiest way to ensure this is to require that all subgroups of U are finitely generated (the conditions stated require us to look outside U).

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THEOREM 1. *Let U be a Burns subgroup of A and let G be either $A *_U B$ or an HNN group $\langle A, x; U^x = V\rangle$. Let H be a finitely generated subgroup of G containing an infinite subnormal subgroup of G. If $H^g \cap U$ is infinitely generated for all $g \in G$ and $U \neq A$ (and $U \neq B$ also, in the first case) then H has finite index in G.*

PROOF. We first show that any subnormal subgroup N of G contained in a conjugate of A or B must be finite. For it is easy to see that if A_1 is a subgroup of A its normaliser will lie in A unless it has a conjugate in U or V. Hence we may assume $N \subseteq U$. Now N is also subnormal in A. But any infinite subgroup of U has A-normaliser in U, since $U \cap U^a$ is finite for $a \notin U$. Hence N must be finite.

Theorem 8 of Cohen (1974) now tells us that there are only finitely many (H, U) double cosets. For the same reason there are only finitely many (H^a, U) double cosets, and so only finitely many (H, aUa^{-1}) double cosets, where $a \in A - U$.

So it is enough to show that the intersection of any (H, U) double coset and any (H, aUa^{-1}) double coset contains only finitely many right H cosets. As usual, it is enough to prove this for $HU \cap HaHa^{-1}$.

Any H coset in $HU \cap HaUa^{-1}$ can be represented by $hu = ava^{-1}$, $h \in H$, $u, v \in U$. Plainly $h \in H \cap A$. But (Lemma 2 of Cohen (1974)) $H \cap A$ is finitely generated. Hence Lemma 1 gives the result.

THEOREM 2. *Let A and B have the property that the intersection of any two finitely generated subgroups is finitely generated. Then the same property holds for $A *_U B = G$ if U is a Burns subgroup of A and $H \cap U$ is finitely generated for all finitely generated subgroups H of G.*

THEOREM 3. *Let A have the above property, and let G be the HNN group $\langle A, x : U^x = V\rangle$. Then G has the same property if both U and V are Burns subgroups of A, and $H \cap U$ is finitely generated for all finitely generated subgroups H of G.*

PROOFS. The two cases may be considered together, using the action of G on the tree T considered in Section 2. As in Theorem 7 of Cohen (1974), we know that $H \cap \operatorname{stab} P$ and $K \cap \operatorname{stab} P$ are finitely generated for any vertex P of T, and that there are only finitely many reversing H-orbits and reversing K-orbits. As $H \cap K \cap \operatorname{stab} P$ will be finitely generated for any P, it will be enough to prove that there are only finitely many reversing $(H \cap K)$-orbits. Hence, by Lemma 3, it is enough to show that for any edge f only finitely many reversing $(H \cap K)$-orbits in $Hf \cap Kf$ contain a special edge.

Take a special edge e in a reversing $(H \cap K)$-orbit in $Hf \cap Kf$. We may assume there is a g with $gO = P$ (the start of e) and $ge_o = e$. This is certainly possible for the amalgamated free product; for the HNN case if this does not hold we can find g with $gO = P$ and $gx^{-1}e_o = e$. In the latter case we simply replace e_o and U by $x^{-1}e_o$ and V in the rest of the proof.

We may also assume (replacing f by another edge, for instance e itself, if necessary) that we can find g_1 with $g_1e_o = f$ and g_1O the start of f. Then we may take $f = e_o$; if this does not hold we replace A and U by A^{g_1} and U^{g_1} but need make no other changes.

Since there are only finitely many reversing H-orbits, the edges which start at O and lie in a reversing H-orbit will lie in finitely many $(H \cap A)$-orbits. Hence there is a finite subset C of A such that all these edges lie in $(H \cap A)Ce_o$ if they are in Ae_o. Similarly there is a finite subset D of A such that any edge in Ae_0 which lies in a reversing K-orbit lies in $(K \cap A)De_o$.

Now consider a special edge e with $e = he_o = ke_o$, where $h \in H$, $k \in K$, and let e' be the edge ending at P. As remarked at the beginning of Section 2 we must have $hO = P = kO$. Also $k = hu$ for some $u \in U$.

If g is negative for e, then ge ends at gP and so ge' starts at gP, whence g is negative for e'. Hence e' is also in a reversing $(H \cap K)$-orbit. As e is special we have $e' \in (\operatorname{stab} P)e$, and so $h^{-1}e' \in Ae_o$. Hence $h^{-1}e' = yce_o$ for some $y \in H \cap A$, $c \in C$; similarly $k^{-1}e' = zde_o$ for some $z \in K \cap A$, $d \in D$.

Then $uzde_o = yce_o$, so that $uzd = ycw$ for some $w \in U$. This equation can be written as $c^{-1}y^{-1}u = wd^{-1}z^{-1}$. Also $zd \notin U$ as $k^{-1}e' \neq e_o = k^{-1}e$. Consequently by Lemma 2 there is a finite subset S of U such that $u \in (H \cap U)S(K \cap U)$ for all choices of special edge e in $He_o \cap Ke_o$.

Taking suitable $h' \in hU$, $k' \in kU$, we can write $e = h'e_o = k'e_o$ with $k' \in h'S$. If another edge e_1 corresponds to the same element of S, say $e_1 = h'_1e_o = k'_1e_o$ then e_1 belongs to $(H \cap K)e$, since $k'_1k^{-1} = h'_1h^{-1} \in H \cap K$. Thus the special edges in $He_o \cap Ke_o$ lie in finitely many $(H \cap K)$-orbits, as required.

We conclude by discussing the group $G = \langle x, y; y^x = y^k, k \text{ an integer}\rangle$. Moldavanskiĭ (1968) gave a short direct proof that the intersection of two finitely generated subgroups of G is finitely generated. G is an HNN group with $A = U = \langle y \rangle$ and $V = \langle y^k \rangle$, and so does not satisfy the hypotheses of Theorem 3. We indicate how the previous proof can be modified.

Let H and K be finitely generated subgroups of G and suppose $H \cap K \neq \{1\}$. Then we can find $g = x^r a x^{-s} \in H \cap K$ where $1 \neq a \in A$ and $r \geqq s \geqq 0$. If $r = s$ then $1 \neq g^{k^r} \in H \cap K \cap A$, since $g^{k^r} = x^r a^{k^r} x^{-r} = a$. Then any (H, A) double coset contains only finitely many H cosets, whence the intersection of a (H, U) double coset and a (K, U) double coset contains only finitely many $(H \cap K)$ cosets. If $r > s$, then $x^{n+r}e_0 \in (H \cap K)x^{n+s}e_0$ for any $n \geqq 0$ and any reversing $(H \cap K)$ orbit will contain $x^n e_0$ for some n with $0 \leqq n \leqq r$, so there are only finitely many such orbits.

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