



# Li coefficients and the quadrilateral zeta function

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*Abstract.* In this note, we study the Li coefficients  $\lambda_{n,a}$  for the quadrilateral zeta function. Furthermore, we give an arithmetic and asymptotic formula for these coefficients. Especially, we show that for any fixed  $n \in \mathbb{N}$ , there exists  $a > 0$  such that  $\lambda_{2n-1,a} > 0$  and  $\lambda_{2n,a} < 0$ .

## 1 Introduction and statement of main results

### 1.1 Li coefficients

The Riemann hypothesis (RH) is a critical question in analytic number theory. As such, it is interesting to examine different ways to frame it, which may shed more light on its resolution. In 1997, Xian-Jin Li has discovered a new positivity criterion for the RH. In [10], he defined the Li coefficients for the Riemann zeta function as

$$\lambda_n = \frac{1}{(n-1)!} \frac{d^n}{ds^n} \left[ s^{n-1} \log \xi(s) \right]_{s=1},$$

where  $\xi$  is the completed Riemann zeta function defined by

$$\xi(s) = s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s),$$

which satisfies  $\xi(s) = \xi(1-s)$  and gave a simple equivalence criterion for the RH: RH is true if and only if these coefficients are nonnegative for every positive integer  $n$ . The Li coefficients  $\lambda_n$  can be written as follows:

$$\lambda_n = \sum_{\rho}^* \left[ 1 - \left( 1 - \frac{1}{\rho} \right)^n \right] = \lim_{T \rightarrow \infty} \sum_{\rho: |\operatorname{Im}(\rho)| \leq T} \left[ 1 - \left( 1 - \frac{1}{\rho} \right)^n \right],$$

where the sum runs over the nontrivial zeros of the Riemann zeta function counted with multiplicity. This criterion is generalized by Bombieri and Lagarias [4] for any arbitrarily multiset of numbers assuming certain convergence conditions. Voros [19, Section 3.3] has proved that the RH true is equivalent to the growth of  $\lambda_n$  as  $\frac{1}{2}n \log n$  determined by its archimedean part, while the RH false is equivalent to the oscillations of  $\lambda_n$  with exponentially growing amplitude, determined by its finite part. The Li

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Received by the editors May 21, 2023; revised March 25, 2024; accepted March 26, 2024.

Published online on Cambridge Core April 8, 2024.

The third author was partially supported by JSPS grant 22K03276.

AMS subject classification: 11M26, 11M35.

Keywords: Li's coefficients, the quadrilateral zeta function, Riemann Hypothesis.



coefficients were generalized in two ways: by generalizing these coefficients to various sets of functions (the Selberg class, the class of automorphic  $L$ -functions, zeta function on function fields, ... [8, 11, 17]) and by introducing new parameter in its definition (see [12]). The Li coefficients (and its generalizations) have generated a lot of research interest due to its applicability and simplicity.

## 1.2 Quadrilateral zeta function

Recall the definitions of Hurwitz and periodic zeta functions. The Hurwitz zeta function  $\zeta(s, a)$  is defined by the series

$$\zeta(s, a) := \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}, \quad \sigma > 1, \quad 0 < a \leq 1.$$

The function  $\zeta(s, a)$  is a meromorphic function with a simple pole at  $s = 1$  whose residue is 1 (see, for example, [1, Section 12]). The periodic zeta function  $\text{Li}_s(e^{2\pi i a})$  is defined by

$$\text{Li}_s(e^{2\pi i a}) := \sum_{n=1}^{\infty} \frac{e^{2\pi i n a}}{n^s}, \quad \sigma > 1, \quad 0 < a \leq 1$$

(see, for instance, [1, Exercise 12.2]). Note that the function  $\text{Li}_s(e^{2\pi i a})$  with  $0 < a < 1$  is analytically continuable to the whole complex plane since  $\text{Li}_s(e^{2\pi i a})$  does not have any pole, that is shown by the fact that the Dirichlet series of  $\text{Li}_s(e^{2\pi i a})$  converges uniformly in each compact subset of the half-plane  $\sigma > 0$  when  $0 < a < 1$  (see, for example, [9, p. 20]). For  $0 < a \leq 1/2$ , we define functions

$$\begin{aligned} Z(s, a) &:= \zeta(s, a) + \zeta(s, 1-a), & P(s, a) &:= \text{Li}_s(e^{2\pi i a}) + \text{Li}_s(e^{2\pi i(1-a)}), \\ 2Q(s, a) &:= Z(s, a) + P(s, a), & \xi_Q(s, a) &:= s(s-1)\pi^{-s/2}\Gamma(s/2)Q(s, a). \end{aligned}$$

We can see that  $Q(s, a)$  is meromorphic functions with a simple pole at  $s = 1$ . In addition, we have  $Q(0, a) = -1/2 = \zeta(0)$  and  $\xi_Q(s, a) = \xi_Q(1-s, a)$ , which is proved by

$$(1.1) \quad Q(1-s, a) = \Gamma_{\cos}(s)Q(s, a), \quad \Gamma_{\cos}(s) := \frac{2\Gamma(s)}{(2\pi)^s} \cos\left(\frac{\pi s}{2}\right)$$

(see [13, Theorem 1.1]). Moreover, the function  $Q(s, a)$  has the following properties. When  $a = 1/6, 1/4, 1/3$ , and  $1/2$ , the RH holds true if and only if all nonreal zeros of  $Q(s, a)$  are on the line  $\text{Re}(s) = 1/2$  (see [14, Proposition 1.3]). Let  $N_Q^{\text{CL}}(T)$  be the number of the zeros of  $Q(s, a)$  on the line segment from  $1/2$  to  $1/2 + iT$ . In [13, Theorem 1.2], the third author proved that for any  $0 < a \leq 1/2$ , there exist positive constants  $A(a)$  and  $T_0(a)$  such that

$$N_Q^{\text{CL}}(T) \geq A(a)T \quad \text{whenever} \quad T \geq T_0(a).$$

Next, let  $N_F(T)$  count the number of nonreal zeros of a function  $F(s)$  having  $|\text{Im}(s)| < T$ . Then, for any  $0 < a \leq 1/2$ ,

$$N_{\zeta}(T) - N_Q(T) = O_a(T),$$

and the third author [14, Proposition 1.8] proved that

$$N_Q(T) = \frac{T}{\pi} \log T - \frac{T}{\pi} \log(2e\pi a^2) + O_a(\log T).$$

Furthermore, he [14, Theorem 1.1] proved that there is a unique absolute  $a_0 \in (0, 1/2)$  such that

$$Q(1/2, a) > 0 \iff 0 < a < a_0.$$

In addition, it is proved in [14, Corollary 1.2] that all real zeros of  $Q(s, a)$  are simple and are located only at the negative even integers just like  $\zeta(s)$  if and only if  $a_0 < a \leq 1/2$ . Let us note by  $Z_Q$  the set of all nontrivial zeros  $\rho_a$  of  $\xi_Q(s, a)$ . Since it is an entire function of order 1, one has

$$(1.2) \quad \xi_Q(s, a) = e^{A+Bs} \prod_{\rho_a \in Z_Q} \left(1 - \frac{s}{\rho_a}\right) e^{\frac{s}{\rho_a}} = \xi_Q(0, a) \prod_{\rho_a \in Z_Q} \left(1 - \frac{s}{\rho_a}\right),$$

where  $e^A = 1/2$ ,  $B = \frac{Q'}{Q}(0, a) - 1 - \frac{\gamma + \log \pi}{2}$ , and  $\gamma$  denotes the Euler constant. Note that  $Q'(0, a)$  is given explicitly in [14, Theorem 1.5].

### 1.3 Main results

Recall that  $\zeta(1-s) = \Gamma_{\cos}(s)\zeta(s)$  and  $Q(1-s, a) = \Gamma_{\cos}(s)Q(s, a)$  by (1.1). However, the function  $Q(s, a)$  does not have an Euler product except for  $a = 1/6, 1/4, 1/3$ , and  $1/2$ . Hence, the function  $Q(s, a)$  is a suitable object to consider the influence of not Riemann's functional equation but an Euler product to zeros of zeta functions. We show a criterion for nonvanishing of  $Q(s, a)$  in terms of the positivity of the Li coefficients, an arithmetic and asymptotic formula for these coefficients in Theorems 1.1, 1.2, and 1.4, respectively. It should be emphasized that  $\lambda_{n,a}$  defined in (1.3) are the first Li coefficients that we can explicitly give  $n \in \mathbb{N}$  such that  $\lambda_{n,a} < 0$ . There is a possibility that this fact would give an idea to find negative Li coefficients for  $\zeta(s)$  if they would exist.

For  $n \neq 0$ , the Li coefficients attached to  $Q(s, a)$  nonvanishing at zero are defined by the sum

$$\lambda_{n,a} := \sum_{\rho_a \in Z_Q}^* \left(1 - \left(1 - \frac{1}{\rho_a}\right)^n\right) = \lim_{T \rightarrow \infty} \sum_{|\operatorname{Im}(\rho_a)| \leq T}^* \left(1 - \left(1 - \frac{1}{\rho_a}\right)^n\right).$$

The symmetry  $\rho_a \mapsto 1 - \rho_a$  in the set  $Z_Q$  of nontrivial zeros of  $Q(s, a)$  implies that  $\lambda_{-n,a} = \overline{\lambda_{n,a}} = \lambda_{n,a}$  for all  $n \in \mathbb{N}$ . So,  $\lambda_{n,a}$  are real. We have also

$$(1.3) \quad \lambda_{n,a} := \frac{1}{(n-1)!} \frac{d^n}{ds^n} [s^{n-1} \log \xi_Q(s, a)]_{s=1}.$$

Moreover, from (1.2), we have (see [4, Equations (2.3) and (2.4)] or [17, Appendix A])

$$\sum_{n=0}^{\infty} \lambda_{n+1,a} s^n = \frac{d}{ds} \log \left[ \xi_Q \left( \frac{1}{1-s}, a \right) \right].$$

As an analogue of Li coefficients for the Riemann zeta function, we have the following.

**Theorem 1.1** *The function  $Q(s, a)$  does not vanish when  $\operatorname{Re}(s) > 1/2$  if and only if  $\lambda_{n,a} \geq 0$  for all  $n \in \mathbb{N}$ .*

An arithmetic formula for  $\lambda_{n,a}$  is stated in the following theorems.

**Theorem 1.2** *We have*

$$\lambda_{n,a} = 1 - \frac{n}{2}(\log(4\pi) + \gamma) + \sum_{k=2}^n (-1)^k \binom{n}{k} (1 - 2^{-k}) \zeta(k) + \sum_{k=1}^n \binom{n}{k} \gamma_Q(k-1),$$

where  $\gamma_Q(n)$  are defined as follows:

$$\frac{Q'}{Q}(s+1, a) + \frac{1}{s} = \sum_{n=0}^{\infty} \gamma_Q(n) s^n.$$

**Theorem 1.3** *For  $a = 1/2, 1/3, 1/4, 1/6$ , under the RH, we have*

$$\lambda_{n,a} = \frac{n}{2} \log n + \frac{n}{2} (\gamma - 1 - \log 2\pi) + O(\sqrt{n} \log n).$$

For a fixed  $l \in \mathbb{N}$ , we have the following asymptotic formula of  $\lambda_{l,a}$  when  $a \rightarrow +0$ . We can see that there exists  $n \in \mathbb{N}$  such that  $\lambda_{n,a} < 0$  by Theorem 1.1 and the fact that  $Q(s, a)$  does not satisfy an analogue of the RH when  $a \in \mathbb{Q} \cap (0, 1/2) \setminus \{1/6, 1/4, 1/3\}$  (see [14, Proposition 1.4]). Clearly, this argument gives no information on the frequency of  $n \in \mathbb{N}$ , the smallest  $n \in \mathbb{N}$  such that  $\lambda_{n,a} < 0$  and so on. However, the next theorem implies that  $\lambda_{2n,a} < 0$  if we fix any  $n \in \mathbb{N}$  and then we take  $a > 0$  sufficiently small.

**Theorem 1.4** *Fix  $l \in \mathbb{N}$ . Then it holds that*

$$\lambda_{l,a} = \frac{(-1)^{l+1}}{(2a)^l} + O_l(a^{1-l} |\log a|), \quad a \rightarrow +0.$$

*Especially, for any fixed  $n \in \mathbb{N}$ , there are  $a > 0$  such that*

$$\lambda_{2n-1,a} > 0 \quad \text{and} \quad \lambda_{2n,a} < 0.$$

## 2 Proofs

### 2.1 Proof of Theorem 1.1

Since  $\lambda_{-n,a} = \overline{\lambda_{n,a}} = \lambda_{n,a}$  for all  $n \in \mathbb{N}$ , then  $\operatorname{Re}(\lambda_{-n,a}) = \operatorname{Re}(\lambda_{n,a}) = \lambda_{n,a}$ . Using that  $\xi_Q(s, a)$  is an entire function of order 1, and its zeros lie in the critical strip  $0 < \operatorname{Re}(s) < 1$ , we obtain that the series  $\sum_{\rho \in Z_Q} \frac{1 + |\operatorname{Re}(\rho)|}{(1 + |\rho|)^2}$  is convergent. Application of [4, Theorem 1] to the multiset  $Z_Q$  of zeros of  $Q(s, a)$  gives that  $\operatorname{Re}(\rho) \leq 1/2$  if and only if  $\lambda_{n,a} \geq 0$  for all  $n \in \mathbb{N}$ . Now, the application of the same theorem to the multiset  $1 - Z_Q = Z_Q$  gives  $\operatorname{Re}(\rho) \geq 1/2$  if and only if  $\lambda_{n,a} \geq 0$ . This completes the proof.

Theorem 1.1 can also be proved by the same argument used in [5, Theorem 1], which is due to Oesterlé.

## 2.2 Proof of Theorem 1.2

From the expression of  $\xi_Q(s, a)$ , one has

$$\frac{\xi'_Q}{\xi_Q}(s, a) = \frac{1}{s} + \frac{1}{s-1} - \frac{1}{2} \log \pi + \frac{1}{2} \frac{\Gamma'}{\Gamma}(s/2) + \frac{Q'}{Q}(s, a),$$

which is rewritten as

$$(2.1) \quad \frac{\xi'_Q}{\xi_Q}(s+1, a) = \frac{1}{s+1} + \frac{1}{s} - \frac{1}{2} \log \pi + \frac{1}{2} \frac{\Gamma'}{\Gamma}((s+1)/2) + \frac{Q'}{Q}(s+1, a).$$

Note that  $Q(s, a)$  is a meromorphic function on the whole complex plane, which is holomorphic everywhere except for a simple pole at  $s = 1$  with residue 1 (see [13, Section 2.1]). Let us define the coefficients  $\gamma_Q(n)$  and  $\tau_Q(n)$  as follows:

$$(2.2) \quad \frac{Q'}{Q}(s+1, a) + \frac{1}{s} = \sum_{n=0}^{\infty} \gamma_Q(n) s^n$$

and

$$(2.3) \quad -\frac{1}{2} \log \pi + \frac{1}{2} \frac{\Gamma'}{\Gamma}((s+1)/2) = \sum_{n=0}^{\infty} \tau_Q(n) s^n.$$

By Equation (1.2), one has

$$\log \xi_Q(s, a) = \log \xi_Q(0, a) - \sum_{\rho_a \in Z_Q} \sum_{m=1}^{\infty} \frac{1}{m \rho^m} s^m.$$

From the functional equation for the function  $\xi_Q(s, a)$ , in the neighborhood of  $s = 0$ , we have

$$(2.4) \quad \frac{\xi'_Q}{\xi_Q}(s+1, a) = -\frac{\xi'_Q}{\xi_Q}(-s, a) = \sum_{m=0}^{\infty} (-1)^m \sum_{\rho_a \in Z_Q} \frac{1}{\rho^{m+1}} s^m.$$

Comparing Equations (2.1)–(2.4), we get

$$(-1)^m \sum_{\rho_a \in Z_Q} \frac{1}{\rho^{m+1}} = (-1)^m + \gamma_Q(m) + \tau_Q(m),$$

for  $m \geq 0$ . Hence, the definition of  $\lambda_{n,a}$  yields

$$\lambda_{n,a} = \sum_{k=1}^n (-1)^{k-1} \binom{n}{k} \sum_{\rho_a \in Z_Q} \frac{1}{\rho^k} = 1 + \sum_{k=1}^n \binom{n}{k} \gamma_Q(k-1) + \sum_{k=1}^n \binom{n}{k} \tau_Q(k-1),$$

where

$$\tau_Q(0) = -\frac{1}{2} \log \pi + \frac{1}{2} \psi(1/2) \text{ and } \tau_Q(k-1) = (-1)^k \sum_{m=0}^{\infty} \frac{1}{(2m+1)^k}$$

using that  $\psi(z) = -\gamma - \frac{1}{z} + \sum_{k=1}^{\infty} \frac{z}{k(k+z)}$ . Here,  $\psi(s) = \frac{\Gamma'}{\Gamma}(s)$  is the logarithmic derivative of the Gamma function. Since  $\psi(1/2) = -\gamma - 2\log 2$ , we obtain

$$\begin{aligned}\lambda_{n,a} &= 1 - \frac{n}{2}(\log(4\pi) + \gamma) + \sum_{k=2}^n (-1)^k \binom{n}{k} \sum_{m=0}^{\infty} \frac{1}{(2m+1)^k} + \sum_{k=1}^n \binom{n}{k} \gamma_Q(k-1) \\ &= 1 - \frac{n}{2}(\log(4\pi) + \gamma) + \sum_{k=2}^n (-1)^k \binom{n}{k} (1-2^{-k}) \zeta(k) + \sum_{k=1}^n \binom{n}{k} \gamma_Q(k-1).\end{aligned}$$

The equality above implies Theorem 1.2.

### 2.3 Proof of Theorem 1.3

Let us note that

$$\sum_{k=2}^n (-1)^k \binom{n}{k} (1-2^{-k}) \zeta(k) = \sum_{k=2}^n (-1)^l \binom{n}{k} \frac{\zeta(k, 1/2)}{2^k},$$

where  $\zeta(s, a)$  is the Hurwitz zeta function defined in Section 1.2. With the notation of Flajolet and Vespas [7, Lines 2–4, p. 70], this is  $A_n(1, 2)$  and which is equal to

$$\frac{n}{2} \psi(n) + n \left( \gamma - \frac{1}{2} + \frac{1}{2} \log 2 \right) + o(1),$$

where the  $o(1)$  error term above is exponentially small and oscillating and equal to

$$\frac{1}{2} \left( \frac{n}{\pi} \right)^{1/4} \exp(-\sqrt{2\pi n}) \cos \left( \sqrt{2\pi n} - \frac{5\pi}{8} \right) + O \left( n^{-1/4} e^{-\sqrt{2\pi n}} \right).$$

Then we have

$$\lambda_{n,a} = \frac{n}{2} \log n + \frac{n}{2} (\gamma - 1 - \log 2\pi) + \sum_{k=1}^n \binom{n}{k} \gamma_Q(k-1) + O(1).$$

It remains to prove that

$$(2.5) \quad \sum_{k=1}^n \binom{n}{k} \gamma_Q(k-1) = O(\sqrt{n} \log n).$$

To do so, we follow very closely the lines of the proof of the corresponding result in [8, Theorem 6.1] or [16, Lemma 3.3] and it will be shortened. We use the following kernel function:

$$k_n(s) := \left(1 + \frac{1}{s}\right)^n - 1 = \sum_{k=1}^n \binom{n}{k} \frac{1}{s^k}.$$

The residue theorem gives

$$\sum_{k=1}^n \binom{n}{k} \gamma_Q(k-1) = \frac{1}{2i\pi} \int_C k_n(s) \left( -\frac{Q'}{Q}(s+1, a) \right) ds,$$

where  $C$  is a contour enclosing the point  $s = 0$  counterclockwise on a circle of small enough positive radius. The residue comes entirely from the singularity at  $s = 0$ , as no other singularities lie inside the contour. Let  $T = \sqrt{n} + \varepsilon_n$ , for some  $0 < \varepsilon_n < 1$ .

Now we follow very closely the lines in [16, pp. 1106–1107] using that the function  $\frac{Q'}{Q}(s, a)$  satisfies the properties<sup>1</sup>

$$\frac{Q'}{Q}(s, a) = \sum_{\rho_a; |\operatorname{Im}(\rho_a - s)| < 1} \frac{1}{s - \rho_a} + O(\log(1 + |s|)),$$

for  $-2 < \operatorname{Re}(s) < 2$  and

$$\left| \frac{Q'}{Q}(s + 1, a) \right| = O(\log^2 T),$$

for  $-2 \leq \operatorname{Re}(s) \leq 2$ , and we get

$$\sum_{k=1}^n \binom{n}{k} \gamma_Q(k-1) = \lambda_{-n, a, T} + O(\sqrt{n} \log n),$$

where

$$\lambda_{-n, a, T} = \sum_{\rho_a \in Z_Q; |\operatorname{Im}(\rho_a)| \leq T}^* \left( 1 - \left( 1 - \frac{1}{\rho_a} \right)^n \right),$$

with  $T = \sqrt{n} + \varepsilon_n$ . For  $a = 1/2, 1/3, 1/4, 1/6$ , under the RH, since  $\left| 1 - \frac{1}{\rho_a} \right| = 1$  and using formula of  $N_Q(T)$  given in Section 1.2, we obtain  $\lambda_{-n, a, T} = O(T \log T + 1)$ . Therefore, Equation (2.5) follows from that  $\lambda_{-n, a, \sqrt{n}} = \lambda_{-n, a, \sqrt{n}} = O(\sqrt{n} \log n)$ .

**Remark** Since  $2Q(s, a) := Z(s, a) + P(s, a)$ , from Corollary 2.3 below and [6, Equation (1.18)], we obtain

$$\gamma_Q(n) = \frac{1}{2} \left( \delta_n(a) + \frac{(-1)^n}{n!} (l_n(a) + l_n(1-a)) \right),$$

where  $\delta_n(a) = \frac{|\log a|^n}{an!} + O(1)$  and  $l_n(a)$  are the coefficients in the expansion of  $\operatorname{Li}_s(e^{2\pi i a})$  at  $s = 1$ ; for  $a \notin \mathbb{Z}$ , one has

$$\operatorname{Li}_s(e^{2\pi i a}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} l_n(a) (s-1)^n.$$

## 2.4 Proof of Theorem 1.4

To show Theorem 1.4, we quote the following lemmas from [2, 3].

**Lemma 2.1** [3, Theorem 1] *We set*

$$(s-1)\zeta(s, a) = 1 + \sum_{n=0}^{\infty} \gamma_n(a) (s-1)^{n+1}, \quad 0 < a \leq 1.$$

<sup>1</sup>These properties are well known for the Riemann zeta function. The proof for the function  $Q(s, a)$  is exactly the same since the Riemann–von Mangoldt formula holds for  $Q(s, a)$  (see [14, Proposition 2.5] or [18, p. 217]).

Then it holds that

$$\gamma_n(a) = \frac{(-1)^n}{n!} \lim_{m \rightarrow \infty} \left( \sum_{k=0}^m \frac{\log^n(k+a)}{k+a} - \frac{\log^{n+1}(m+a)}{n+1} \right).$$

**Lemma 2.2** [2, Equation (26)] Let  $0 < a \leq 1$ , and let  $n$  be a nonnegative integer. Then one has

$$\begin{aligned} \zeta^{(n)}(0, a) &= \left( \frac{1}{2} - a \right) |\log a|^n - n! + n! a \sum_{m=n}^{\infty} \frac{|\log a|^m}{m!} \\ &\quad + (-1)^n n \int_0^{\infty} \frac{\varphi(x) \log^{n-1}(x+a)}{(x+a)^2} dx \\ &\quad - (-1)^n n(n-1) \int_0^{\infty} \frac{\varphi(x) \log^{n-2}(x+a)}{(x+a)^2} dx, \end{aligned}$$

where  $\varphi(x) = \int_0^x (y - \lfloor y \rfloor - 1/2) dy$  is periodic with period 1 and satisfies  $2\varphi(x) = x(x-1)$  if  $0 \leq x \leq 1$ .

By using the lemmas above, we immediately obtain the following.

**Corollary 2.3** When  $a > 0$  is sufficiently small,

$$\begin{aligned} (s-1)Z(s, a) &= 2 + \sum_{n=0}^{\infty} \delta_n(a) (s-1)^{n+1}, \quad \delta_n(a) = \frac{|\log a|^n}{an!} + O(1), \\ Z(s, a) &= \sum_{n=1}^{\infty} \varepsilon_n(a) s^n, \quad \varepsilon_n(a) = O(|\log a|^n). \end{aligned}$$

**Proof** The first formula and estimation are easily proved by Lemma 2.1 (see also [3, Theorem 2]). For the first integral in Lemma 2.2, one has

$$\begin{aligned} \int_0^1 \frac{\varphi(x) \log^{n-1}(x+a)}{(x+a)^2} dx &\ll \int_0^1 \frac{\log^{n-1}(x+a)}{x+a} dx = O(|\log a|^n), \\ \int_1^{\infty} \frac{\varphi(x) \log^{n-1}(x+a)}{(x+a)^2} dx &\ll \int_1^{\infty} \frac{\log^{n-1}(x+a)}{(x+a)^2} dx = O(1) \end{aligned}$$

from  $x < x+a$  when  $x, a > 0$ . In addition, we have

$$a \sum_{m=n}^{\infty} \frac{|\log a|^m}{m!} \leq a \sum_{m=0}^{\infty} \frac{|\log a|^m}{m!} = ae^{|\log a|} = ae^{-\log a} = 1, \quad 0 < a < 1/2.$$

Hence, we obtain

$$\zeta(s, a) = \sum_{n=0}^{\infty} \frac{\zeta^{(n)}(0, a)}{n!} s^n, \quad \zeta^{(n)}(0, a) = O(|\log a|^n).$$

Therefore, we have  $\varepsilon_n(a) = O(|\log a|^n)$  and the second formula in this corollary by the definition of  $Z(s, a)$  and  $Z(0, a) = \zeta(0, a) + \zeta(0, 1-a) = 0$  (see [15, Equation (4.11)]). ■



**Proof of Theorem 1.4** Recall the functional equation

$$Z(1-s, a) = \Gamma_{\cos}(s)P(s, a), \quad \Gamma_{\cos}(s) := \frac{2\Gamma(s)}{(2\pi)^s} \cos\left(\frac{\pi s}{2}\right)$$

(see [15, Lemma 4.11]). By using  $\Gamma_{\cos}(s)\Gamma_{\cos}(1-s) = 1$ , we have

$$2Q(s, a) = Z(s, a) + P(s, a) = Z(s, a) + \Gamma_{\cos}(1-s)Z(1-s, a).$$

Let  $|s-1|$  be sufficiently small. Then, by  $\lim_{s \rightarrow 1}(s-1)Q(s, a) = 1$ , the equation above, and the definitions of  $Q(s, a)$  and  $\xi_Q(s, a)$ , we have

$$\begin{aligned} \frac{d^l}{ds^l} [s^{l-1} \log \xi_Q(s, a)]_{s=1} &= \frac{d^l}{ds^l} [s^{l-1} \log((s-1)Q(s, a)) + s^{l-1} \log(sp^{-s/2}\Gamma(s/2))]_{s=1} \\ &= \frac{d^l}{ds^l} \left[ s^{l-1} \log\left(\frac{s-1}{2} (Z(s, a) + \Gamma_{\cos}(1-s)Z(1-s, a))\right) \right]_{s=1} + O_l(1) \\ &= \frac{d^l}{ds^l} \left[ s^{l-1} \log\left(1 + \sum_{n=0}^{\infty} (\delta'_n(a) + \varepsilon'_n(a))(s-1)^{n+1}\right) \right]_{s=1} + O_l(1), \end{aligned}$$

where  $\delta'_n(a)$  and  $\varepsilon'_n(a)$  are defined by

$$\delta'_n(a) := \frac{\delta_n(a)}{2}, \quad (s-1)\Gamma_{\cos}(1-s)Z(1-s, a) = 2 \sum_{n=0}^{\infty} \varepsilon'_n(a)(s-1)^{n+1}.$$

Clearly, the second estimation in Corollary 2.3 implies

$$Z(1-s, a) = \sum_{n=1}^{\infty} \varepsilon_n(a)(1-s)^n, \quad \varepsilon_n(a) = O(|\log a|^n).$$

Thus, we can see that  $\varepsilon'_n(a) = O(|\log a|^{n+1})$  from  $\lim_{s \rightarrow 1}(s-1)\Gamma_{\cos}(1-s) = -2$  and the fact that the function  $(s-1)\Gamma_{\cos}(1-s)$  does not depend on  $a$ . Put  $\eta_n(a) := \delta'_n(a) + \varepsilon'_n(a)$ . Then, for  $n \geq 0$ , we have

$$(2.6) \quad \eta_n(a) = \frac{1}{n!} \frac{|\log a|^n}{2a} + O(|\log a|^{n+1}), \quad a \rightarrow +0$$

by Corollary 2.3. By virtue of

$$\begin{aligned} (a_0x + a_1x^2 + a_2x^3 + \cdots)^m &= a_0^m x^m + \binom{m}{1} a_0^{m-1} a_1 x^{m+1} + \cdots \\ (a_0x + a_1x^2 + a_2x^3 + \cdots)^{m-1} &= a_0^m x^{m-1} + \binom{m-1}{1} a_0^{m-2} a_1 x^m + \cdots \\ &\vdots \\ (a_0x + a_1x^2 + a_2x^3 + \cdots)^1 &= \cdots + a_m x^m + \cdots, \end{aligned}$$

where  $m \in \mathbb{N}$  and  $a_m, x \in \mathbb{C}$ , the coefficient of  $(s-1)^l$  in the function

$$f(s, a) := \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \left( \sum_{n=0}^{\infty} \eta_n(a)(s-1)^{n+1} \right)^m$$

is expressed as

$$(2.7) \quad \frac{(-1)^{l+1}}{l} (\eta_0(a))^l + \frac{(-1)^l}{l-1} \binom{l-1}{1} \eta_0(a)^{l-2} \eta_1(a) + \cdots + \frac{(-1)^{1+1}}{1} \eta_{l-1}(a).$$

Note that the function above is estimated by

$$(2.8) \quad \frac{(-1)^{l+1}}{l} (\eta_0(a))^l + O_l(\eta_0(a)^{l-2} \eta_1(a)) = \frac{(-1)^{l+1}}{l} (2a)^{-l} + O_l(a^{1-l} |\log a|)$$

from (2.6) when  $a \rightarrow +0$ . We can find that

$$(s-1) \left( Z(s, a) + \Gamma_{\cos}(1-s) Z(1-s, a) \right) = 1 + \sum_{n=0}^{\infty} \eta_n(a) (s-1)^{n+1}$$

is analytic when  $|s-1| < 1$  from the poles of  $Z(s, a)$  and  $\Gamma_{\cos}(1-s)$ . So we can choose  $|s-1| > 0$  such that

$$\sum_{n=0}^{\infty} |\eta_n(a)| |s-1|^{n+1} < \frac{1}{2}.$$

Then, from (2.7), the Leibniz product rule, the definition of  $\eta_n(a)$ , and the Taylor expansion of  $\log(1+x)$  with  $|x| < 1$ , one has

$$\begin{aligned} \frac{d^l}{ds^l} \left[ s^{l-1} \log \xi_Q(s, a) \right]_{s=1} &= \frac{d^l}{ds^l} \left[ s^{l-1} f(s, a) \right]_{s=1} + O_l(1) \\ (b) \quad &= \binom{l}{l} \frac{(-1)^{l+1}}{l} l! (\eta_0(a))^l + O_l(\eta_0(a)^{l-2} \eta_1(a)) \\ (h) \quad &+ \binom{l}{l-1} (l-1) \frac{(-1)^l}{l-1} (l-1)! (\eta_0(a))^{l-1} + O_l(\eta_0(a)^{l-3} \eta_1(a)) \\ (\sharp) \quad &+ \cdots + \binom{l}{1} (l-1)! \frac{(-1)^{1+1}}{1} (\eta_0(a))^1 + O_l(1). \end{aligned}$$

Note that (b) comes from  $f^{(l)}(s, a)$ , (h) is deduced by  $f^{(l-1)}(s, a)$ , and ( $\sharp$ ) derives from  $f^{(1)}(s, a)$ ,  $f^{(0)}(s, a)$ , and  $O_l(1)$  in the left-hand side of the formula above. Therefore, by (2.8), we obtain

$$\begin{aligned} \frac{d^l}{ds^l} \left[ s^{l-1} \log \xi_Q(s, a) \right]_{s=1} &= (-1)^{l+1} (l-1)! (\eta_0(a))^l + O_l(\eta_0(a)^{l-2} \eta_1(a)) \\ &= (-1)^{l+1} \frac{(l-1)!}{(2a)^l} + O_l(a^{1-l} |\log a|), \end{aligned}$$

which implies Theorem 1.4. ■

At the end of the paper, we give numerical computation for  $\lambda_{n,a}$  by Mathematica 13.0. Let

$$\lambda_{n,a}^{[k]} := \frac{1}{(n-1)!} \frac{d^n}{ds^n} \left[ s^{n-1} \log \xi_Q(s, a) \right]_{s=1-10^{-k}}, \quad \lambda_{n,a}^* := \frac{(-1)^{n+1}}{(2a)^n}.$$

Then, we have the following.

For  $n = 1$ , we have

$$\begin{array}{lll}
 a := 2^{-17} & \lambda_{1,a}^{[10]} = 65, 537... & \lambda_{1,a}^{[10]} / \lambda_{1,a}^* = 1.00001... \\
 a := 2^{-18} & \lambda_{1,a}^{[10]} = 131, 074... & \lambda_{1,a}^{[10]} / \lambda_{1,a}^* = 1.00002... \\
 a := 2^{-19} & \lambda_{1,a}^{[10]} = 262, 151... & \lambda_{1,a}^{[10]} / \lambda_{1,a}^* = 1.00003... \\
 a := 2^{-17} & \lambda_{1,a}^{[11]} = 65, 536.6... & \lambda_{1,a}^{[11]} / \lambda_{1,a}^* = 1.00001... \\
 a := 2^{-18} & \lambda_{1,a}^{[11]} = 131, 073... & \lambda_{1,a}^{[11]} / \lambda_{1,a}^* = 1.00001... \\
 a := 2^{-19} & \lambda_{1,a}^{[11]} = 262, 145... & \lambda_{1,a}^{[11]} / \lambda_{1,a}^* = 1.00000... \\
 a := 2^{-17} & \lambda_{1,a}^{[12]} = 655, 365... & \lambda_{1,a}^{[12]} / \lambda_{1,a}^* = 1.00001... \\
 a := 2^{-18} & \lambda_{1,a}^{[12]} = 131, 073... & \lambda_{1,a}^{[12]} / \lambda_{1,a}^* = 1.00000... \\
 a := 2^{-19} & \lambda_{1,a}^{[12]} = 262, 145... & \lambda_{1,a}^{[12]} / \lambda_{1,a}^* = 1.00000...
 \end{array}$$

For  $n = 2$ , we have

$$\begin{array}{lll}
 a := 2^{-17} & \lambda_{2,a}^{[10]} = -4.29352... \times 10^9 & \lambda_{2,a}^{[10]} / \lambda_{2,a}^* = 0.999663... \\
 a := 2^{-18} & \lambda_{2,a}^{[10]} = -1.7177... \times 10^{10} & \lambda_{2,a}^{[10]} / \lambda_{2,a}^* = 0.999836... \\
 a := 2^{-19} & \lambda_{2,a}^{[10]} = -6.87162... \times 10^{10} & \lambda_{2,a}^{[10]} / \lambda_{2,a}^* = 0.999952... \\
 a := 2^{-17} & \lambda_{2,a}^{[11]} = -4.29478... \times 10^9 & \lambda_{2,a}^{[11]} / \lambda_{2,a}^* = 0.999956... \\
 a := 2^{-18} & \lambda_{2,a}^{[11]} = -1.71753... \times 10^{10} & \lambda_{2,a}^{[11]} / \lambda_{2,a}^* = 0.999736... \\
 a := 2^{-19} & \lambda_{2,a}^{[11]} = -6.87149... \times 10^{10} & \lambda_{2,a}^{[11]} / \lambda_{2,a}^* = 0.999933... \\
 a := 2^{-17} & \lambda_{2,a}^{[12]} = -4.29477... \times 10^9 & \lambda_{2,a}^{[12]} / \lambda_{2,a}^* = 0.999955... \\
 a := 2^{-18} & \lambda_{2,a}^{[12]} = -1.6911... \times 10^{10} & \lambda_{2,a}^{[12]} / \lambda_{2,a}^* = 0.984353... \\
 a := 2^{-19} & \lambda_{2,a}^{[12]} = -6.87187... \times 10^{10} & \lambda_{2,a}^{[12]} / \lambda_{2,a}^* = 0.999989...
 \end{array}$$

**Acknowledgements** The authors want to thank the anonymous referees for their many insightful comments and suggestions.

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