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Acyclic coefficient systems on buildings

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Abstract

For cohomological (respectively homological) coefficient systems $\mathcal{F}$ (respectively $\mathcal{V}$) on affine buildings X with Coxeter data of type $\widetilde{A}_d$, we give for any $k \geqslant 1$ a sufficient local criterion which implies $H^k(X, \mathcal{F}) = 0$ (respectively $H_k(X, \mathcal{V}) = 0$). Using this criterion we prove a conjecture of de Shalit on the acyclicity of coefficient systems attached to hyperplane arrangements on the Bruhat–Tits building of the general linear group over a local field. We also generalize an acyclicity theorem of Schneider and Stuhler on coefficient systems attached to representations.

Introduction

Let X be an affine building whose apartments are Coxeter complexes attached to Coxeter systems of type $\widetilde{A}_d$ and let $\mathcal{F}$ be a cohomological coefficient system (CCS) on X. The purpose of this paper is to give a *local* criterion which assures that for a given $k \geqslant 1$ the cohomology group $H^k(X, \mathcal{F})$ vanishes (similarly for homological coefficient systems) (HCSs).

For a sheaf $\mathcal{G}$ on a topological space Y it is well known that $H^k(Y, \mathcal{G}) = 0$ for all $k \geqslant 1$ if $\mathcal{G}$ is *flasque*, i.e. if all restriction maps $\mathcal{G}(U) \to \mathcal{G}(V)$ for open $V \subset U \subset Y$ are surjective. We are looking for an adequate notion of 'flasque' CCS on X.

If $d = 1$ the same condition works: if the restriction map $\mathcal{F}(\sigma) \to \mathcal{F}(\tau)$ for any 0-simplex σ contained in the 1-simplex τ is surjective, then $H^1(X, \mathcal{F}) = 0$. This is easily seen using the *contractibility* of X. However, if $d > 1$ the surjectivity of $\mathcal{F}(\sigma) \to \mathcal{F}(\tau)$ for any $(k-1)$-simplex σ contained in the k-simplex τ does not guarantee $H^k(X, \mathcal{F}) = 0$. The other naive transposition of the flasqueness concept from topological spaces to buildings would be to require for any $(k-1)$-simplex σ the surjectivity of $\mathcal{F}(\sigma) \to \prod_\tau \mathcal{F}(\tau)$, taking the product over *all* k-simplices τ containing σ. This would indeed force $H^k(X, \mathcal{F}) = 0$, but would also be a completely useless criterion: for example, it would not be satisfied by a constant CCS $\mathcal{F}$ (of which we know $H^k(X, \mathcal{F}) = 0$, by the contractibility of X).

Let us describe our criterion $\mathcal{S}(k)$. We fix an orientation of X. It defines a *cyclic* ordering on the set of vertices of any simplex, hence a true ordering on the set of vertices of any *pointed* simplex. To a pointed $(k-1)$-simplex $\widehat{\eta}$ we associate the set $N_{\widehat{\eta}}$ of all vertices z for which $(z, \widehat{\eta})$ (i.e. z as the first vertex) is an ordered k-simplex (in the previously qualified sense). We define what it means for a subset M_0 of $N_{\widehat{\eta}}$ to be *stable with respect to* $\widehat{\eta}$ (if, for example, $d = 1$ the condition is $|M_0| \leqslant 1$). Our criterion $\mathcal{S}(k)$ which assures $H^k(X, \mathcal{F}) = 0$ is then that for any such $\widehat{\eta}$ and for any subset M_0

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of $N_{\widehat{\eta}}$, stable with respect to $\widehat{\eta}$, the sequence

$$\mathcal{F}(\eta) \longrightarrow \prod_{z \in M_0} \mathcal{F}(\{z\} \cup \eta) \longrightarrow \prod_{z,z' \in M_0} \mathcal{F}(\{z, z'\} \cup \eta)$$

is exact (where in the target of the second arrow the product is over pairs $z, z' \in M_0$ of incident vertices). For example, a constant CCS $\mathcal{F}$ satisfies $\mathcal{S}(k)$.

Having fixed a vertex z_0, the central ingredient in the proof is then a certain function i on the set X^0 of vertices which measures the combinatorical (not Euclidean) distance from z_0; it depends on our chosen orientation.

Dual to $\mathcal{S}(k)$ we describe a criterion $\mathcal{S}^*(k)$ which guarantees $H_k(X, \mathcal{V}) = 0$ for HCSs $\mathcal{V}$ on X.

The basic example is, of course, the case where X is the Bruhat–Tits building of $\mathrm{PGL}_{d+1}(K)$ for a local field K. Our original motivation for developing the criterion $\mathcal{S}(k)$ was the following. In [Sch92], Schneider defined a certain class of $\mathrm{SL}_{d+1}(K)$-representations, the 'holomorphic discrete series representations', as the global sections of certain equivariant vector bundles on Drinfel'd's symmetric space $\Omega_K^{(d+1)}$ of dimension d over K. In a subsequent paper [Gro04], for any such vector bundle V, we will construct an integral model $\mathcal{V}$ as an equivariant coherent sheaf on the formal $\mathcal{O}_K$-scheme $\Omega_{\mathcal{O}_K}^{(d+1)}$ underlying $\Omega_K^{(d+1)}$. Using the criterion $\mathcal{S}(k)$ and the close relation between $\Omega_K^{(d+1)}$ and X, we will show that if V is strongly dominant (in a suitable sense), then $H^k(\Omega_{\mathcal{O}_K}^{(d+1)}, \mathcal{V}) = 0$ for all $k \geqslant 1$. Examples for such $\mathcal{V}$ are the terms of the logarithmic de Rham complex of $\Omega_{\mathcal{O}_K}^{(d+1)}$.

Here we present two other applications. The first is a proof of a conjecture of de Shalit on p-adic CCSs of Orlik–Solomon algebras. For an arbitrary field K, the assignment of the Orlik–Solomon algebra A to a hyperplane arrangement W in $(K^{d+1})^*$, the complement of a *finite* set $\mathcal{A} \subset \mathbb{P}(K^{d+1})$ of hyperplanes in $(K^{d+1})^*$, is a classical theme. A is defined combinatorically in terms of the hyperplanes and turns out to be isomorphic with the cohomology ring of W. Of course K may also be a local field. However, de Shalit [deS01] discovered that one can go further and give the story a *genuinely p-adic* flavour. Namely, if K is a local field he allowed $\mathcal{A} \subset \mathbb{P}(K^{d+1})$ to be *infinite*. He did not assign a single Orlik–Solomon algebra to $\mathcal{A}$ but a CCS system $A = A(\cdot)$ of Orlik–Solomon algebras on the Bruhat–Tits building X of $\mathrm{PGL}_{d+1}(K)$ which then should play the role of a cohomology ring of the 'hyperplane arrangement' defined by $\mathcal{A}$. The algebra $A(\sigma)$ for a j-simplex σ is closely related to a suitable tensor product of Orlik–Solomon algebras for finite hyperplane arrangements in k-vector spaces, for k the residue field of K. de Shalit conjectures that these beautiful CCSs are acyclic in positive degrees, $H^k(X, A) = 0$ for all $k \geqslant 1$. He proved the conjecture for any $\mathcal{A}$ if $d \leqslant 2$. For arbitrary d he proved it if $\mathcal{A}$ is the full set $\mathbb{P}(K^{d+1})$ of all K-rational hyperplanes, and Alon [Alo] proved it if $\mathcal{A}$ is finite. Here we give a proof for all $\mathcal{A}$ and d by showing that A always satisfies $\mathcal{S}(k)$.

In fact we prove a version with arbitrary coefficient ring: while in [deS01] the coefficient ring of the CCS A is K, we allow an arbitrary coefficient ring R, e.g. also $R = \mathbb{Z}$ or $R = k$. While de Shalit's proof in the case $d \leqslant 2$ also works for arbitrary R, his proof in the case $\mathcal{A} = \mathbb{P}(K^{d+1})$ but d arbitrary, which is by reduction to the main result of [SS93], does not work for coefficient rings R other than characteristic zero fields. We explain why this improvement for $\mathcal{A} = \mathbb{P}(K^{d+1})$ and $R = \mathbb{Z}$ should have an application to a problem on p-adic Abel–Jacobi mappings raised in [RX03].

The second application we describe is concerned with the technique of Schneider and Stuhler to spread out representations of $\mathrm{GL}_{d+1}(K)$ as HCSs on the Bruhat–Tits building of $\mathrm{PGL}_{d+1}(K)$. For a $\mathrm{GL}_{d+1}(K)$-representation on a (not necessarily free) $\mathbb{Z}[\frac{1}{p}]$-module V (where $p = \mathrm{char}(k)$) which is generated by its vectors fixed under a principal congruence subgroup $U^{(n)}$ of some level $n > 1$, we prove that the chain complex of the corresponding HCS is a resolution of V. For fields of characteristic zero as coefficient ring (instead of $\mathbb{Z}[\frac{1}{p}]$) this is the main result of [SS93], where, however, $n = 1$ is allowed. While the proof in [SS93] uses the Bernstein–Borel–Matsumoto theory, we do not need any representation theoretic input whatsoever.

1. The criterion

Let $d \geqslant 1$ and let X be an affine building whose apartments are Coxeter complexes attached to Coxeter systems of type $\widetilde{A}_d$. We refer to [Bro89] for the basic definitions and properties of buildings. For $0 \leqslant j \leqslant d$ we denote by X^j the set of j-simplices. We generally identify a j-simplex with its set of vertices.

We fix an orientation of X. It distinguishes for any simplex a *cyclic* ordering on its set of vertices. A *pointed k-simplex* is an *enumeration* of the set of vertices of a k-simplex in its distinguished cyclic ordering; we write it as an ordered $(k+1)$-tuple of vertices.

For an apartment A in X we will slightly abuse notation by not distinguishing between A and its geometric realization $|A|$. There is (see [Bro89, p. 148]) an isomorphism of A with $\mathbb{R}^{d+1}/\mathbb{R}\cdot(1,\ldots,1)$, we view it here as an identification, such that, if $\{e_0,\ldots,e_d\}$ denotes the standard basis of $\mathbb{R}^{d+1}$ the following holds:

(a) the set of vertices in A is $\mathbb{Z}^{d+1}/\mathbb{Z}\cdot(1,\ldots,1)$;

(b) a $k+1$-tuple $(x_0,\ldots,x_k)$ of vertices in A is a pointed k-simplex if and only if there is a sequence
$$\emptyset \neq J_0 \subsetneq \cdots \subsetneq J_{k-1} \subsetneq \{0,\ldots,d\}$$
such that $\sum_{j\in J_t} e_j$ represents $x_t - x_k$ (formed with respect to the obvious group structure on $\mathbb{Z}^{d+1}/\mathbb{Z}\cdot(1,\ldots,1)$), for any $0 \leqslant t \leqslant k-1$.

If $(x_0,\ldots,x_k)$ is a pointed k-simplex, we define $\ell((x_0,\ldots,x_k))$ to be the maximal number r such that there exists a pointed r-simplex $(y_0,\ldots,y_r)$ with $x_0 = y_0$ and $x_k = y_r$. For a pointed $(k-1)$-simplex $\widehat{\eta} = (x_1,\ldots,x_k)$ we define the set
$$N_{\widehat{\eta}} = \{z \in X^0 \mid (z,x_1,\ldots,x_k) \text{ is a pointed } k\text{-simplex}\}.$$
For $z \in N_{\widehat{\eta}}$ we write $(z,\widehat{\eta})$ for the pointed k-simplex $(z,x_1,\ldots,x_k)$. We define a partial ordering $\leqslant$ on $N_{\widehat{\eta}}$ by
$$u_1 \leqslant u_2 \iff [u_1 = u_2 \text{ or } (u_1,u_2,\widehat{\eta}) \text{ is a pointed } (k+1)\text{-simplex}].$$

Lemma 1.1. *For any $u_1, u_2 \in N_{\widehat{\eta}}$ the set*
$$W^{\widehat{\eta}}_{u_1,u_2} = \{u \in N_{\widehat{\eta}} \mid u \leqslant u_1 \text{ and } u \leqslant u_2\}$$
is empty or it contains an element u such that $\ell((u,\widehat{\eta})) < \ell((w,\widehat{\eta}))$ for all $w \in W^{\widehat{\eta}}_{u_1,u_2} - \{u\}$.

Proof. Suppose we have two such candidates u, u'. We can find an apartment A which contains η, u_1, u_2, u and u' (for example, because $\eta \cup \{u_1,u\}$ and $\{u',u_2\}$ are simplices). We identify A with $\mathbb{R}^{d+1}/\mathbb{R}\cdot(1,\ldots,1)$ as above. There exist subsets $\emptyset \neq J_t \subsetneq \{0,\ldots,d\}$ for $1 \leqslant t \leqslant k-1$ and $\emptyset \neq I_s \subsetneq \{0,\ldots,d\}$ for $s = 1,2$ such that $x_t - x_k$ is represented by $\sum_{j\in J_t} e_j$ and $u_s - x_k$ is represented by $\sum_{j\in I_s} e_j$. We have $I_s \subsetneq J_1 \subsetneq \cdots \subsetneq J_{k-1}$ for $s = 1,2$. Hence, both $u - x_k$ and $u' - x_k$ are represented by $\sum_{j\in I_1\cap I_2} e_j$. ☐

If $W^{\widehat{\eta}}_{u_1,u_2} \neq \emptyset$ we denote the element $u \in W^{\widehat{\eta}}_{u_1,u_2}$ from Lemma 1.1 by $[\eta|u_1,u_2]$. If $W^{\widehat{\eta}}_{u_1,u_2} = \emptyset$ then $[\eta|u_1,u_2]$ is undefined. A subset M_0 of $N_{\widehat{\eta}}$ is called *stable with respect to* $\widehat{\eta}$ if for any two vertices $u_1, u_2 \in M_0$ the vertex $[\widehat{\eta}|u_1,u_2]$ is defined and belongs to M_0. (See Lemma 2.2 below for what this means on Bruhat–Tits buildings. When working out the applications described later, we became fully convinced that stability is a very natural condition.) Note that M_0 is stable with respect

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to $\widehat{\eta} = (x_1, \dots, x_k)$ if and only if it is stable with respect to the pointed 0-simplex x_k; this is because of $[x_k|u_1, u_2] = [\eta|u_1, u_2]$, as we saw in the proof of Lemma 1.1.

A CCS $\mathcal{F}$ on X is the assignment of an abelian group $\mathcal{F}(\tau)$ to every simplex τ of X, and a homomorphism $r_\sigma^\tau : \mathcal{F}(\tau) \to \mathcal{F}(\sigma)$ to every face inclusion $\tau \subset \sigma$, such that $r_\rho^\sigma \circ r_\sigma^\tau = r_\rho^\tau$ whenever $\tau \subset \sigma \subset \rho$, and r_τ^τ is the identity.

Given a CCS $\mathcal{F}$, define the group $C^k(X, \mathcal{F})$ of k-cochains ($0 \leqslant k \leqslant d$) to consist of the maps c, assigning to each k-simplex τ an element $c_\tau \in \mathcal{F}(\tau)$. Define

$$\partial = \partial^{k+1} : C^k(X, \mathcal{F}) \longrightarrow C^{k+1}(X, \mathcal{F})$$

by the rule

$$(\partial c)_\tau = \sum_{\tau' \subset \tau} [\tau : \tau'] r_\tau^{\tau'}(c_{\tau_i})$$

where $[\tau : \tau'] = \pm 1$ is the incidence number (with respect to a fixed labeling of X as in [Bro89, p. 30]). Then $(C^\bullet(X, \mathcal{F}), \partial)$ is a complex ($\partial^2 = 0$), and its cohomology groups are denoted $H^k(X, \mathcal{F})$.

Consider for $1 \leqslant k \leqslant d$ the following condition $\mathcal{S}(k)$ for a CCS $\mathcal{F}$ on X: for any pointed $(k-1)$-simplex $\widehat{\eta}$ with underlying $(k-1)$-simplex η and for any subset M_0 of $N_{\widehat{\eta}}$ which is stable with respect to $\widehat{\eta}$, the following subquotient complex of $C^\bullet(X, \mathcal{F})$ is exact:

$$\mathcal{F}(\eta) \xrightarrow{\partial^k} \prod_{z \in M_0} \mathcal{F}(\{z\} \cup \eta) \xrightarrow{\partial^{k+1}} \prod_{\substack{z, z' \in M_0 \\ \{z, z'\} \in X^1}} \mathcal{F}(\{z, z'\} \cup \eta).$$

(We regard the first term as a subgroup of $C^{k-1}(X, \mathcal{F})$, the second as a direct summand of $C^k(X, \mathcal{F})$, and the third term as a quotient of $C^{k+1}(X, \mathcal{F})$.) Note that $\mathcal{S}(k)$ depends on the chosen orientation of X.

Theorem 1.2. *Let $\mathcal{F}$ be a CCS on X. Let $1 \leqslant k \leqslant d$ and suppose $\mathcal{S}(k)$ holds true. Then $H^k(X, \mathcal{F}) = 0$.*

We fix once and for all a vertex $z_0 \in X^0$. Given an arbitrary vertex $x \in X^0$, choose an apartment A containing z_0 and x. Choose an identification of A with $\mathbb{R}^{d+1}/\mathbb{R} \cdot (1, \dots, 1)$ as before, but now require in addition that $z_0 \in A$ corresponds to the class of the origin in $\mathbb{R}^{d+1}/\mathbb{R} \cdot (1, \dots, 1)$. Let $\sum_{j=0}^d m_j e_j$ be the unique representative of x for which $m_j \geqslant 0$ for all j, and $m_j = 0$ for at least one j. Let π be a permutation of $\{0, \dots, d\}$ such that $0 = m_{\pi(d)} \geqslant \cdots \geqslant m_{\pi(0)}$ and set

$$i(x) = (m_{\pi(d)}, \dots, m_{\pi(1)}) \in \mathbb{N}_0^d.$$

Lemma 1.3. *The d-tuple $i(x)$ is independent of the choice of A.*

Proof. Let us write $i_A(x)$ instead of $i(x)$ in order to indicate the reference to A in the above definition. Suppose the apartment A' also contains z_0 and x. Choose a chamber (d-simplex) C in A containing x, and a chamber C' in A' containing z_0. Choose an apartment A'' in X containing C and C', and let $\pi : X \to A''$, respectively $\pi' : X \to A''$, be the retraction from X to A'' centered in C, respectively centered in C' (see [Bro89, p. 86]). Then π, respectively π', induces an isomorphism of oriented chamber complexes $A \cong A''$, respectively $A' \cong A''$. Hence $i_A(x) = i_{A''}(x) = i_{A'}(x)$. □

Here is another, equivalent but more intrinsic definition of $i(x)$ (we do not need it). For $x, y \in X^0$ let $d(x, y) \in \mathbb{Z}_{\geqslant 0}$ be the minimal number t such that there exists a sequence $x_0, \dots, x_t$ in X^0 with $x = x_0$, $y = x_t$ and $\{x_{r-1}, x_r\} \in X^1$ for all $1 \leqslant r \leqslant t$. For $x \in X^0$ and a subset $W \subset X^0$ let $d(x, W) = \min\{d(x, y) \mid y \in W\}$. For $1 \leqslant i \leqslant d$ define the subset W_i of X^0 inductively as follows: $W_1 = \{z_0\}$ and

$$W_i = \left\{ z \in X^0 \;\middle|\; \left\{ \begin{array}{c} \text{there exist elements } z_0, \dots, z_r \in X^0 \text{ such that } r = d(z, W_{i-1}), \\ z_0 \in W_{i-1}, z_r = z \text{ and } \ell((z_{\ell-1}, z_\ell)) = 1 \text{ for all } 1 \leqslant \ell \leqslant r \end{array} \right\} \right\}.$$

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In particular $W_1 \subset W_2 \subset \cdots \subset W_d$. For $x \in X^0$ we then have

$$i(x) = (d(x, W_1), d(x, W_2), \ldots, d(x, W_d)).$$

Yet another equivalent definition of $i(x)$ (which we do not need either) results from the fact that the type of a minimal chamber-gallery connecting x and z_0 encodes $i(x)$ if x and z_0 are *not* incident.

On the set of ordered d-tuples $(n_1, \ldots, n_d) \in \mathbb{Z}^d_{\geqslant 0}$ (and hence on the set of d-tuples $i(x)$ for $x \in X^0$) we use the lexicographical ordering:

$$(n_1, \ldots, n_d) < (n'_1, \ldots, n'_d) \iff \left\{ \begin{array}{c} \text{there is a } 1 \leqslant r \leqslant d \text{ such that} \\ n_j = n'_j \text{ for } 1 \leqslant j \leqslant r-1 \text{ and } n_r < n'_r \end{array} \right\}.$$

Lemma 1.4. *Let η be a $(k-1)$-simplex and let $x_1, \ldots, x_k$ be an enumeration of its vertices which satisfies $i(x_1) \leqslant \cdots \leqslant i(x_k)$. Then in fact $i(x_1) < \cdots < i(x_k)$ and $\widehat{\eta} = (x_1, \ldots, x_k)$ is a pointed $(k-1)$-simplex.*

Proof. Choose an apartment A containing z_0 and η, and choose an identification of A with $\mathbb{R}^{d+1}/\mathbb{R} \cdot (1, \ldots, 1)$ as before, with $z_0 \in A$ corresponding to the class of the origin in $\mathbb{R}^{d+1}/\mathbb{R} \cdot (1, \ldots, 1)$. Then the claims follow easily from our description of the simplicial structure of $\mathbb{Z}^{d+1}/\mathbb{Z} \cdot (1, \ldots, 1)$. $\square$

Lemma 1.5. *For any $x \in X^0$, $x \neq z_0$, there is among the vertices incident to x a* unique *vertex $\nu(x)$ with minimal i-value: for all other vertices z incident to x we have $i(\nu(x)) < i(z)$. If z is incident to x, different from $\nu(x)$ and satisfies $i(z) < i(x)$, then $\nu(x)$ and z are incident and $\ell((\nu(x), z)) \leqslant \ell((x, z))$.*

Proof. Let $[z_0, x]$ be the geodesic (with respect to the Euclidean distance function on the geometric realization $|X|$ of X) between z_0 and x. Let τ be the minimal simplex which contains x and whose open interior (viewed as a subset of $|X|$) contains a point of $[z_0, x]$. We assert that the vertex $\nu(x)$ of τ with minimal i-value is as claimed. To see this, let A be an arbitrary apartment containing x and z_0. Then τ is contained in A because this is true for $[z_0, x]$. Explicitly it can be described as follows. Choose an identification of A with $\mathbb{R}^{d+1}/\mathbb{R} \cdot (1, \ldots, 1)$ as before, with $z_0 \in A$ corresponding to the class of the origin in $\mathbb{R}^{d+1}/\mathbb{R} \cdot (1, \ldots, 1)$. After reindexing the basis if necessary there are sequences $0 \leqslant r_0 < r_1 < \cdots < r_s = d$ and $0 < m_1 < \cdots < m_s$ (some $1 \leqslant s \leqslant d$) such that the vertex x is represented by $y_s = \sum_{i=1}^{s} \sum_{j=r_{i-1}+1}^{r_i} m_i e_j$. Then $\{h y_s \mid 0 \leqslant h \leqslant 1\}$ represents $[z_0, x]$, and τ is a s-simplex, the other vertices are represented by $y_t = \sum_{i=1}^{t} \sum_{j=r_{i-1}+1}^{r_i} m_i e_j + \sum_{i=t+1}^{s} \sum_{j=r_{i-1}+1}^{r_i} (m_i - 1) e_j$ for $t = 0, \ldots, s-1$. In particular, $\nu(x)$ is represented by y_0 and it is clear that it has minimal i-value among the vertices of A incident to x. Since any vertex in X incident to x lies in such an A the assertion follows. Also the other claims can immediately be read off from this analysis on an apartment. $\square$

Proposition 1.6. *Let $\widehat{\eta} = (x_1, \ldots, x_k)$ be as in Lemma 1.4 and let $\eta = \{x_1, \ldots, x_k\}$. Then*

$$M_0 = \{u \in X^0 \mid \{u\} \cup \eta \text{ is a } k\text{-simplex and } i(u) < i(x_1)\}$$

is contained in $N_{\widehat{\eta}}$ and stable with respect to $\widehat{\eta}$.

Proof. The containment $M_0 \subset N_{\widehat{\eta}}$ follows from Lemma 1.4 with k instead of $(k-1)$. To prove that M_0 is stable with respect to $\widehat{\eta}$ let $u_1, u_2 \in M_0$. First it follows from Lemma 1.4 (with k and $k+1$ instead of $(k-1)$) and Lemma 1.5 that $\nu(x_k) \in W^{\widehat{\eta}}_{u_1, u_2}$, hence $[\widehat{\eta}|u_1, u_2]$ is defined. Since $([\widehat{\eta}|u_1, u_2], x_1, \ldots, x_k)$ is a pointed k-simplex and since for any pointed k-simplex the underlying *cyclic* ordering of the vertices is independent of the pointing, there are, in view of Lemma 1.4 with k instead of $(k-1)$, only the two possibilities $i([\widehat{\eta}|u_1, u_2]) < i(x_1)$ and $i([\widehat{\eta}|u_1, u_2]) > i(x_k)$. If $i([\widehat{\eta}|u_1, u_2]) < i(x_1)$ then $[\widehat{\eta}|u_1, u_2] \in M_0$ and we are done. If $i([\widehat{\eta}|u_1, u_2]) > i(x_k)$ then $\nu([\widehat{\eta}|u_1, u_2]) \in W^{\widehat{\eta}}_{u_1, u_2}$ by Lemmas 1.4 and 1.5. Moreover

$$\ell((\nu([\widehat{\eta}|u_1, u_2]), \widehat{\eta})) \leqslant \ell(([\widehat{\eta}|u_1, u_2], \widehat{\eta})),$$

also by Lemma 1.5. Since $\nu([\widehat{\eta}|u_1, u_2]) \neq [\widehat{\eta}|u_1, u_2]$ this contradicts the definition of $[\widehat{\eta}|u_1, u_2]$. Hence $i([\widehat{\eta}|u_1, u_2]) > i(x_k)$ cannot happen and the proof is finished. □

Proof of Theorem 1.2. Given $\eta, \eta' \in X^{k-1}$ let $x_1, \dots, x_k$, respectively $x'_1, \dots, x'_k$, be that enumeration of the vertices of η, respectively of η', which satisfies $i(x_1) < \cdots < i(x_k)$, respectively $i(x'_1) < \cdots < i(x'_k)$. We define

$$\eta \tilde{<} \eta' \iff \left\{ \begin{array}{c} \text{there is a } 1 \leqslant q \leqslant k \text{ such that} \\ i(x_t) = i(x'_t) \text{ for } 1 \leqslant t \leqslant q-1 \text{ and } i(x_q) < i(x'_q) \end{array} \right\}.$$

We define $\eta \cong \eta'$ if $i(x_t) = i(x'_t)$ for all $1 \leqslant t \leqslant k$. Let $\nabla : X^{k-1} \longrightarrow \mathbb{N}$ be the surjective map with

$$\nabla(\eta) < \nabla(\eta') \iff \eta \tilde{<} \eta'$$
$$\nabla(\eta) = \nabla(\eta') \iff \eta \cong \eta'.$$

For a k-simplex $\sigma \in X^k$ let $\sigma^- \in X^{k-1}$ be the $(k-1)$-simplex obtained from σ by omitting the vertex $x \in \sigma$ for which $i(x)$ is minimal. We need to show that

$$\prod_{\eta \in X^{k-1}} \mathcal{F}(\eta) \xrightarrow{\partial^k} \prod_{\sigma \in X^k} \mathcal{F}(\sigma) \xrightarrow{\partial^{k+1}} \prod_{\tau \in X^{k+1}} \mathcal{F}(\tau)$$

is exact. So let a k-cocycle $c = (c_\sigma)_{\sigma \in X^k} \in \operatorname{Ker}(\partial^{k+1})$ be given. It suffices to show that there is a sequence of $(k-1)$-cochains $(b_n)_{n \in \mathbb{N}} = ((b_{n,\eta})_{\eta \in X^{k-1}})_{n \in \mathbb{N}}$ with $b_{n,\eta} \in \mathcal{F}(\eta)$ satisfying the following properties:

(i) $b_{n,\eta} = b_{\nabla(\eta),\eta}$ for all $n \geqslant \nabla(\eta)$;

(ii) $b_{n,\eta} = 0$ for all $\eta \in X^{k-1}$ with $\nabla(\eta) > n$;

(iii) for all $\sigma \in X^k$ with $\nabla(\sigma^-) \leqslant n$ we have $(\partial^k b_n - c)_\sigma = 0$.

Then the cochain $b_\infty = (b_{\infty,\eta})_{\eta \in X^{k-1}}$ defined by $b_{\infty,\eta} = b_{\nabla(\eta),\eta}$ will be a preimage of c, as follows from (i) and (iii).

We construct $(b_n)_{n \in \mathbb{N}}$ inductively. Suppose b_{n-1} has been constructed. We set $b_{n,\eta} = b_{n-1,\eta}$ for all $\eta \in X^{k-1}$ with $\nabla(\eta) < n$, and $b_{n,\eta} = 0$ for all $\eta \in X^{k-1}$ with $\nabla(\eta) > n$. Now suppose we have $\eta \in X^{k-1}$ with $\nabla(\eta) = n$. Let $x_1, \dots, x_k$ be that enumeration of the vertices of η which satisfies $i(x_1) < \cdots < i(x_k)$. Consider the set

$$M_0 = \{z \in X^0 \mid \{z\} \cup \eta \text{ is a } k\text{-simplex and } i(z) < i(x_1)\}. \tag{1}$$

We know from Lemma 1.4 and Proposition 1.6 that $\widehat{\eta} = (x_1, \dots, x_k)$ is a pointed $(k-1)$-simplex and that M_0 is stable with respect to $\widehat{\eta}$ and contained in $N_{\widehat{\eta}}$. If $M_0 = \emptyset$ we put $b_{n,\eta} = 0$. So assume now that $M_0 \neq \emptyset$. Let $z,\ z' \in M_0$ with $\{z, z'\} \in X^1$. We compute (with $\pm$, respectively r, denoting the respective incidence numbers, respectively restriction maps):

$$\begin{aligned} &\partial^{k+1}(((\partial^k b_{n-1} - c)_{z'' \cup \eta})_{z'' \in M_0})_{\{z,z'\} \cup \eta} \\ &\quad = \pm r((\partial^k b_{n-1} - c)_{z \cup \eta}) + \pm r((\partial^k b_{n-1} - c)_{z' \cup \eta}) \\ &\quad = \pm r((\partial^k b_{n-1} - c)_{z \cup \eta}) + \pm r((\partial^k b_{n-1} - c)_{z' \cup \eta}) + \sum_{\{z,z'\} \subset \sigma \subset \{z,z'\} \cup \eta} \pm r((\partial^k b_{n-1} - c)_\sigma) \\ &\quad = \sum_{\sigma \subset \{z,z'\} \cup \eta} \pm r((\partial^k b_{n-1} - c)_\sigma) \\ &\quad = (\partial^{k+1}(\partial^k b_{n-1} - c))_{\{z,z'\} \cup \eta} \end{aligned}$$

and this is zero because c is a cocycle. For the second equality note that for all $\sigma \in X^k$ with $\{z, z'\} \subset \sigma \subset \{z, z'\} \cup \eta$ we have $\nabla(\sigma^-) < n$ which by the induction hypothesis implies $(\partial^k b_{n-1} - c)_\sigma = 0$.

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We have seen that $((\partial^k b_{n-1} - c)_{\{z\}\cup\eta})_{z\in M_0}$ lies in

$$\mathrm{Ker}\Bigg[\prod_{z\in M_0} \mathcal{F}(\{z\}\cup\eta) \xrightarrow{\partial^{k+1}} \prod_{\substack{z,z'\in M_0\\ \{z,z'\}\in X^1}} \mathcal{F}(\{z,z'\}\cup\eta)\Bigg].$$

We can therefore define $b_{n,\eta} \in \mathcal{F}(\eta)$ as a preimage of $[\sigma:\sigma^-]((c-\partial^k b_{n-1})_{\{z\}\cup\eta})_{z\in M_0}$, by hypothesis $\mathcal{S}(k)$. To see that b_n satisfies (iii) for $\sigma \in X^k$ with $\nabla(\sigma^-) = n$ we compute

$$\begin{aligned}
(\partial b_n - c)_\sigma &= \Bigg(\sum_{\eta\subset\sigma}[\sigma:\eta] r^\eta_\sigma(b_{n,\eta})\Bigg) - c_\sigma \\
&= \Bigg(\sum_{\substack{\eta\subset\sigma\\ \nabla(\eta)<n}} [\sigma:\eta] r^\eta_\sigma(b_{n,\eta})\Bigg) + [\sigma:\sigma^-] r^{\sigma^-}_\sigma(b_{n,\sigma^-}) - c_\sigma \\
&= \Bigg(\sum_{\substack{\eta\subset\sigma\\ \nabla(\eta)<n}} [\sigma:\eta] r^\eta_\sigma(b_{n-1,\eta})\Bigg) + (c-\partial^k b_{n-1})_\sigma - c_\sigma \\
&= \Bigg(\sum_{\substack{\eta\subset\sigma\\ \nabla(\eta)<n}} [\sigma:\eta] r^\eta_\sigma(b_{n-1,\eta})\Bigg) - \Bigg(\sum_{\eta\subset\sigma}[\sigma:\eta] r^\eta_\sigma(b_{n-1,\eta})\Bigg)
\end{aligned}$$

and this is zero because we have $b_{n-1,\sigma^-} = 0$ by induction hypothesis (ii). □

The reader will have observed that any $\mathbb{N}$-valued function i on X^0 which takes different values on incident vertices gives rise to a local vanishing criterion like $\mathcal{S}(k)$, by the same formal proof above. However, the applicability of the resulting criterion depends on the control one gets over the corresponding sets M_0 defined analogously through formula (1). In this optic, the virtue of our particular choice of i lies in the fact that we can control the corresponding sets M_0: they satisfy the *local* (no reference to the global function i) property of being stable with respect to $\widehat{\eta}$; hence our vanishing criterion $\mathcal{S}(k)$, expressed entirely in local terms.

A HCS $\mathcal{V}$ of abelian groups on a building X is the assignment of an abelian group $\mathcal{V}(\tau)$ to every simplex τ of X, and a homomorphism $r^\tau_\sigma : \mathcal{V}(\tau) \to \mathcal{V}(\sigma)$ to every face inclusion $\sigma \subset \tau$, such that $r^\tau_\rho \circ r^\sigma_\tau = r^\sigma_\rho$ whenever $\rho \subset \tau \subset \sigma$, and r^τ_τ is the identity.

The group $C_k(X,\mathcal{V})$ of k-chains ($0 \leqslant k \leqslant d$) consists of all the finitely supported maps c which assign to each k-simplex τ an element $c_\tau \in \mathcal{V}(\tau)$. Define

$$\partial = \partial_k : C_{k+1}(X,\mathcal{V}) \longrightarrow C_k(X,\mathcal{V})$$

by the rule

$$(\partial c)_\tau = \sum_{\tau'\supset\tau} [\tau':\tau] r^{\tau'}_\tau(c_{\tau'}).$$

Then $(C_\bullet(X,\mathcal{V}),\partial)$ is a complex ($\partial^2 = 0$), and its homology groups are denoted $H_k(X,\mathcal{V})$.

Consider for $1 \leqslant k \leqslant d$ the following condition $\mathcal{S}^*(k)$ for a HCS $\mathcal{V}$ on X: for any pointed $(k-1)$-simplex $\widehat{\eta}$ with underlying $(k-1)$-simplex η and for any subset M_0 of $N_{\widehat{\eta}}$ which is stable with respect to $\widehat{\eta}$, the following subquotient complex of $(C_\bullet(X,\mathcal{V}),\partial)$ is exact:

$$\bigoplus_{\substack{z,z'\in M_0\\ \{z,z'\}\in X^1}} \mathcal{V}(\{z,z'\}\cup\eta) \xrightarrow{\partial_k} \bigoplus_{z\in M_0} \mathcal{V}(\{z\}\cup\eta) \xrightarrow{\partial_{k-1}} \mathcal{V}(\eta).$$

THEOREM 1.7. *Let $\mathcal{V}$ be a HCS on X. Let $1 \leqslant k \leqslant d$ and suppose $\mathcal{S}^*(k)$ holds true. Then $H_k(X,\mathcal{V}) = 0$.*

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Proof. We need to show that

$$\bigoplus_{\tau\in X^{k+1}}\mathcal{V}(\tau)\xrightarrow{\partial_k}\bigoplus_{\sigma\in X^k}\mathcal{V}(\sigma)\xrightarrow{\partial_{k-1}}\bigoplus_{\eta\in X^{k-1}}\mathcal{V}(\eta)$$

is exact. We use notations from the proof of Theorem 1.2. For $n\in\mathbb{Z}_{\geqslant 0}$ and elements $c=(c_\sigma)_\sigma\in\oplus_{\sigma\in X^k}\mathcal{V}(\sigma)$ consider the condition

$$C(n):\ \text{for all } \sigma\in X^k \text{ with } \nabla(\sigma^-)\geqslant n \text{ we have } c_\sigma=0.$$

Similarly as in the proof of Theorem 1.2 one shows by induction on n: all elements $c\in\operatorname{Ker}(\partial_{k-1})$ which satisfy $C(n)$ lie in $\operatorname{Im}(\partial_k)$. Indeed, given such an element c one uses $\mathcal{S}^*(k)$ in order to modify c by an element of $\operatorname{Im}(\partial_k)$ in such a way that it even satisfies $C(n-1)$, and then the induction hypothesis applies. □

2. p-adic hyperplane arrangements

Let K denote a non-archimedean locally compact field, $\mathcal{O}_K$ its ring of integers, $\pi\in\mathcal{O}_K$ a fixed prime element and k the residue field. Let X be the Bruhat–Tits building of PGL_{d+1}/K; it has Coxeter data of type $\widetilde{A}_d$. A concrete description of X is the following. A lattice in the K-vector space K^{d+1} is a free $\mathcal{O}_K$-submodule of K^{d+1} of rank $d+1$. Two lattices L, L' are called homothetic if $L'=\lambda L$ for some $\lambda\in K^\times$. We denote the homothety class of L by $[L]$. The set of vertices of X is the set of the homothety classes of lattices (always in K^{d+1}). For a lattice chain

$$\pi L_k\subsetneq L_0\subsetneq L_1\subsetneq\cdots\subsetneq L_k$$

we declare $([L_0],\ldots,[L_k])$ to be a pointed k-simplex. This defines a simplicial structure with orientation.

The following definitions are due to de Shalit [deS01] (who takes $R=K$, $\mathrm{char}(K)=0$ below). Let $\mathcal{A}$ be a non empty subset of $\mathbb{P}(K^{d+1})$. We view $\mathcal{A}$ as a set of lines in K^{d+1}, or hyperplanes in $(K^{d+1})^*$. We write e_a for the line represented by $a\in K^{d+1}-\{0\}$, so that $e_{\lambda a}=e_a$ for any $\lambda\in K^\times$.

Let R be a commutative ring and let $\widetilde{E}$ be the free exterior algebra over R, on the set $\mathcal{A}$. It is graded and anti-commutative. There is a unique derivation $\delta:\widetilde{E}\to\widetilde{E}$, homogeneous of degree -1, mapping each $e\in\mathcal{A}$ to 1. It satisfies $\delta^2=0$ and

$$\delta(e_0\wedge\cdots\wedge e_k)=\sum_{i=0}^{k}(-1)^i e_0\wedge\cdots\wedge\widehat{e_i}\wedge\cdots\wedge e_k$$
$$\operatorname{Im}(\delta)=\operatorname{Ker}(\delta).$$

The subalgebra $E=\operatorname{Im}(\delta)=\operatorname{Ker}(\delta)$ of $\widetilde{E}$ is generated by all elements $e-e'$ for $e,e'\in\mathcal{A}$. There is an exact sequence

$$0\longrightarrow E\longrightarrow\widetilde{E}\xrightarrow{\delta}E[1]\longrightarrow 0\tag{2}$$

and any $e\in\mathcal{A}$ supplies a splitting $E[1]\to\widetilde{E}$, $x\mapsto e\wedge x$.

Let $x\in X^0$ be a vertex. We say that an element $g\in\widetilde{E}$ is a *standard generator of* $I(x)$ if there are a lattice L_x representing x and elements $\{a_0,\ldots,a_m\}$ of $\mathcal{A}\cap L_x-\pi L_x$, linearly dependent modulo πL_x, such that $g=\delta(e_{a_0}\wedge\cdots\wedge e_{a_m})$. We define the ideal $I(x)$ in $\widetilde{E}$ as that generated by the standard generators of $I(x)$. For an arbitrary simplex σ we define the ideal $I(\sigma)=\sum_{x\in\sigma}I(x)$. We set

$$\widetilde{A}(\sigma)=\frac{\widetilde{E}}{I(\sigma)},\quad A(\sigma)=\frac{E}{E\cap I(\sigma)}.$$

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The split exact sequence (2) provides us with a split exact sequence

$$0 \longrightarrow A(\sigma) \longrightarrow \widetilde{A}(\sigma) \xrightarrow{\delta} A(\sigma)[1] \longrightarrow 0. \tag{3}$$

The ideal $I(\sigma)$ is homogeneous, hence there is a natural grading on $\widetilde{A}(\sigma)$ and on $A(\sigma)$. We denote by $\widetilde{A}^q(\sigma)$ respectively by $A^q(\sigma)$ the qth graded piece. For varying σ the $\widetilde{A}(\sigma)$ and $A(\sigma)$ form CCSs $\widetilde{A}$ and A on X.

Suppose that we are given a lattice chain

$$\pi L_k \subsetneq L_1 \subsetneq L_2 \subsetneq \cdots \subsetneq L_k.$$

Let $x_j \in X^0$ be the vertex defined by L_j; then $\widehat{\eta} = (x_1, \ldots, x_k)$ is a pointed $(k-1)$-simplex; we denote by η the underlying non pointed $(k-1)$-simplex. We write $L_0 = \pi L_k$. As long as L_k is fixed we abuse notation in that we identify an element $e \in \mathcal{A}$ with an element $a \in L_k - L_0$ for which $e = e_a$; such an a is unique up to a unit in $\mathcal{O}_K$. Thus we regard $\mathcal{A}$ as a subset of $L_k - L_0$.

Assume that $\mathcal{A}$ is finite and fix a linear ordering $\prec$ on $\mathcal{A}$ which is *adapted to* $\widehat{\eta}$. By definition, this means $\max(\mathcal{A} \cap L_j - L_{j-1}) \prec \min(\mathcal{A} \cap L_{j+1} - L_j)$ for all $1 \leqslant j \leqslant k-1$. For $S \subset \mathcal{A}$ and $e \in S$ we define the $\mathcal{O}_K$-submodule $L(\widehat{\eta}, \prec, S, e)$ of K^{d+1} as follows: let $1 \leqslant j \leqslant k$ be the number for which $e \in L_j - L_{j-1}$; then

$$L(\widehat{\eta}, \prec, S, e) = \langle L_{j-1}, \{e' \in S \mid e' \prec e \text{ or } e' = e\}\rangle_{\mathcal{O}_K}.$$

We say that e *is* $(S, \widehat{\eta})$*-special with respect to* $\prec$ if

$$e = \max_{\prec}(\mathcal{A} \cap L(\widehat{\eta}, \prec, S, e)).$$

(Here $\max_{\prec}(Q)$ for a subset Q of $\mathcal{A}$ means the maximal element of Q with respect to the fixed ordering $\prec$. The subscript $\prec$ does *not* indicate that $\prec$ is a running parameter.)

Fix another linear ordering $<$ on $\mathcal{A}$ and for subsets S of $\mathcal{A}$ let $e_S = e_0 \wedge \cdots \wedge e_r \in \widehat{E}$ where $e_0 < e_1 < \cdots < e_r$ is the increasing enumeration of the elements of S in the ordering $<$. (The ordering $<$ may be taken to be $\prec$, but the role of these two orderings will be completely unrelated in the following.)

Lemma 2.1. *The free R-module $\widetilde{A}(\eta)$ is of finite rank, a basis is the set*

$$\{e_S \mid S \subset \mathcal{A},\ \textit{all } e \in S \textit{ are } (S, \widehat{\eta})\textit{-special with respect to } \prec\}.$$

Proof. This is [deS01, Theorem 2.5]: the 'broken circuit theorem' (there $R = K$, $\mathrm{char}(K) = 0$ and $\prec = <$, but this is irrelevant). ☐

Let $N_{\widehat{\eta}}$ be as in § 1. A subset $M_0 \subset N_{\widehat{\eta}}$ corresponds to a collection of lattices $(L_z)_{z \in M_0}$ with $L_0 \subsetneq L_z \subsetneq L_1$ for all $z \in M_0$. The following lemma is clear.

Lemma 2.2. *We have M_0 is stable with respect to $\widehat{\eta}$ if and only if for all $z_1, z_2 \in M_0$ the lattice $L_{z_1} \cap L_{z_2}$ represents an element of M_0.*

Suppose that $M_0 \subset N_{\widehat{\eta}}$ is stable with respect to $\widehat{\eta}$. In particular there is a $x_0 \in M_0$ with $L_{x_0} \subset L_z$ for all $z \in M_0$. We say that a collection $(\prec_z)_{z \in M_0}$, indexed by M_0, of linear orderings on $\mathcal{A}$ is *adapted to* $(M_0, \widehat{\eta})$ if the following conditions hold:

- for any $z \in M_0$ the ordering $\prec_z$ is adapted to the pointed k-simplex $(z, \widehat{\eta})$;
- for any $z_1, z_2 \in M_0$ with $L_{z_1} \subset L_{z_2}$ we have $[e \prec_{z_1} e' \Leftrightarrow e \prec_{z_2} e']$ for all pairs $e, e' \in \mathcal{A} \cap L_{z_1}$ and also for all pairs $e, e' \in \mathcal{A} \cap L_{z_2} - L_{z_1}$;
- for any $z \in M_0$ we have $[e \prec_{x_0} e' \Leftrightarrow e \prec_z e']$ for all pairs $e, e' \in \mathcal{A} \cap L_k - L_z$;
- we have $e \prec_{x_0} e'$ for all pairs $e, e' \in \mathcal{A}$ with $e \in \bigcup_{z \in M_0} L_z$ and $e' \notin \bigcup_{z \in M_0} L_z$.

Lemma 2.3. *Collections of linear orderings on $\mathcal{A}$ adapted to $(M_0,\widehat{\eta})$ exist.*

Proof. The referee suggested the following proof (our original was unnecessarily complicated). Let $U=\bigcup_{z\in M_0}L_z$. Fix a linear ordering $\prec$ on $\mathcal{A}$ adapted to $\widehat{\eta}$ which satisfies $e\prec e'$ for all pairs $e,e'\in\mathcal{A}$ with $e\in U$ and $e'\notin U$. For a $z\in M_0$ let $\prec_z$ be the ordering which satisfies firstly $e\prec_z e'\prec_z e''$ for all triples $e,e',e''\in\mathcal{A}$ with $e\in L_z$, with $e'\in U-L_z$ and with $e''\in L_k-U$, and secondly $[e\prec e'\Leftrightarrow e\prec_z e']$ for all pairs $e,e'\in\mathcal{A}\cap L_z$, for all pairs $e,e'\in\mathcal{A}\cap U-L_z$ and for all pairs $e,e'\in\mathcal{A}-(\mathcal{A}\cap U)$. It is straightforwardly checked that $(\prec_z)_{z\in M_0}$ is adapted to $(M_0,\widehat{\eta})$. □

Fix a collection $(\prec_z)_{z\in M_0}$ of linear orderings on $\mathcal{A}$ adapted to $(M_0,\widehat{\eta})$. Let

$$G(\widehat{\eta};M_0)=\left\{e_S\;\middle|\;\begin{array}{c}S\subset\mathcal{A},\ \text{for all } e\in S \text{ there is a}\\ z\in M_0 \text{ such that } e \text{ is } (S,(z,\widehat{\eta}))\text{-special}\end{array}\right\}$$

where $(S,(z,\widehat{\eta}))$-speciality is to be understood with respect to $\prec_z$. Let $J(\widehat{\eta};M_0)\subset\widetilde{E}$ be the ideal generated by the set

$$I(\eta)\bigcup\{g\in\widetilde{E}\mid g \text{ is a standard generator of } I(z) \text{ for all } z\in M_0\}.$$

Proposition 2.4. *We have $G(\widehat{\eta};M_0)$ is finite and generates $\widetilde{E}/J(\widehat{\eta};M_0)$ as an R-module.*

Proof. Clearly the set of *all* e_S with $S\subset\mathcal{A}$ is generating. Now suppose that $S\subset\mathcal{A}$ does not satisfy the condition defining $G(\widehat{\eta};M_0)$. That is, there exists an $e\in S$ such that for all $z\in M_0$ this e is *not* $(S,(z,\widehat{\eta}))$-special. Fix such an e.

We first claim that there is an $\widehat{e}\in S$ and a subset $\widehat{S}\subset S$ such that for all $z\in M_0$ the following statements (1) and (2) are satisfied:

(1) for all $e'\in\widehat{S}$ we have $e'\prec_z e\prec_z\widehat{e}$;

(2) The element $\delta(e_{\widehat{S}\cup\{e,\widehat{e}\}})$ belongs to $J(\widehat{\eta};M_0)$.

We distinguish three cases. First consider the case $e\in L_j-L_{j-1}$ for some $2\leqslant j\leqslant k$. Then we set $\widehat{S}=\{e'\in S\mid e'\prec_{x_0}e \text{ and } e'\notin L_{j-1}\}$ and

$$\widehat{e}=\max_{\prec_{x_0}}(\mathcal{A}\cap L((x_0,\widehat{\eta}),\prec_{x_0},S,e)).$$

The fact that e is not $(S,(x_0,\widehat{\eta}))$-special and the properties of the adapted collection $(\prec_z)_{z\in M_0}$ give statement (1). Moreover $\widehat{S}\cup\{e,\widehat{e}\}$ is linearly dependent modulo L_{j-1}, hence $\delta(e_{\widehat{S}\cup\{e,\widehat{e}\}})$ lies in $I(x_{j-1})\subset I(\eta)$ and we get statement (2).

Now consider the case $e\in L_1-\bigcup_{z\in M_0}L_z$. Then we set $\widehat{S}=\{e'\in S\mid e'\prec_{x_0}e\}$ and $\widehat{e}=\max_{\prec_{x_0}}(\mathcal{A}\cap L((x_0,\widehat{\eta}),\prec_{x_0},S,e))$. Again the fact that e is not $(S,(x_0,\widehat{\eta}))$-special and the properties of the adapted collection $(\prec_z)_{z\in M_0}$ give statement (1). For $z\in M_0$ let $\widehat{S}_z=\widehat{S}-(\widehat{S}\cap L_z)$. The fact that e is not $(S,(z,\widehat{\eta}))$-special tells us that $\widehat{S}_z\cup\{e,\widehat{e}\}$ is also linearly dependent modulo L_z. However then the subset of $\pi^{-1}L_z-L_z$ which defines the same elements in $\mathbb{P}(K^{d+1})$ as does $\widehat{S}\cup\{e,\widehat{e}\}$ is linearly dependent modulo L_z, because it contains $\widehat{S}_z\cup\{e,\widehat{e}\}$. Therefore, $\delta(e_{\widehat{S}\cup\{e,\widehat{e}\}})$ is a standard generator of $I(z)$. We have shown statement (2).

Finally the case $e\in\bigcup_{z\in M_0}L_z$. Let $z_e\in M_0$ be such that $e\in L_{z_e}$ and L_{z_e} is minimal with this property (this z_e is unique since M_0 is stable). We set $\widehat{S}=\{e'\in S\mid e'\prec_{z_e}e\}$ and $\widehat{e}=\max_{\prec_{z_e}}(\mathcal{A}\cap L((z_e,\widehat{\eta}),\prec_{z_e},S,e))$. The fact that e is not $(S,(z_e,\widehat{\eta}))$-special gives statements (1) and (2) in this case (here $\delta(e_{\widehat{S}\cup\{e,\widehat{e}\}})\in I(x_k)\subset I(\eta)$). Again this is straightforwardly checked using the properties of the adapted collection $(\prec_z)_{z\in M_0}$.

The claim established, statement (2) tells us that we may replace $e_{\widehat{S}\cup\{e\}}$ in $\widetilde{E}/J(\widehat{\eta};M_0)$ by a linear combination of elements $e_{S'}$ with each S' arising from $\widehat{S}\cup\{e,\widehat{e}\}$ by deleting an element of $\widehat{S}\cup\{e\}$.

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By statement (1) of our claim this means that we may replace e_S by a linear combination of elements $e_{S''}$ with each S'' satisfying $S \prec_z S''$ for all $z \in M_0$ (for the lexicographic ordering $\prec_z$ on the set of subsets of $\mathcal{A}$ of fixed cardinality derived from the ordering $\prec_z$ on $\mathcal{A}$). Repeating the process proves that $G(\widehat{\eta}; M_0)$ is generating. That it is finite follows from Lemma 2.1. We are done. □

We define the complex

$$K(\widehat{\eta}, M_0) = \left\{ \begin{array}{c} 0 \longrightarrow \dfrac{\widetilde{E}}{\bigcap_{z\in M_0} I(\eta\cup\{z\})} \longrightarrow \\ \prod\limits_{z\in M_0} \widetilde{A}(\eta\cup\{z\}) \longrightarrow \prod\limits_{\substack{z_1,z_2\in M_0\\ \{z_1,z_2\}\in X^1}} \widetilde{A}(\eta\cup\{z_1,z_2\}) \longrightarrow \cdots \end{array} \right\}.$$

Proposition 2.5. *For non-empty M_0 as above the following statements hold:*

(a) *the complex $K(\widehat{\eta}, M_0)$ is exact;*

(b) $J(\widehat{\eta}; M_0) = \bigcap_{z\in M_0} I(\eta\cup\{z\})$;

(c) $\widetilde{E}/\bigcap_{z\in M_0} I(\eta\cup\{z\})$ *is a free R-module and $G(\widehat{\eta}; M_0)$ is a basis.*

Proof. For $n \geqslant 1$ let $(\mathrm{a})_n$, $(\mathrm{b})_n$ and $(\mathrm{c})_n$ be the corresponding statements for all M_0 with $1 \leqslant |M_0| \leqslant n$. We will prove these statements by simultaneous induction on n.

Statements $(\mathrm{a})_1$ and $(\mathrm{b})_1$ are clear, and $(\mathrm{c})_1$ is Lemma 2.1. Now we assume $n > 1$. First we prove $(\mathrm{a})_n$. Choose a $y \in M_0$ for which L_y is maximal, i.e. such that there is no $z \in M_0$ with $L_y \subsetneq L_z$. If we set

$$M_1 = M_0 - \{y\}, \quad M_1' = \{z \in M_1 \mid L_z \subset L_y\}$$

then M_1 and M_1' are stable with respect to $\widehat{\eta}$ respectively $(y, \widehat{\eta})$. We claim

$$\bigcap_{z'\in M_1'} I(\eta\cup\{y,z'\}) \subset I(\eta\cup\{y\}) + \bigcap_{z\in M_1} I(z). \tag{4}$$

For $x \in M_0$ and $S \subset \mathcal{A}$ let S^x be the uniquely determined subset of $L_x - \pi L_x$ which determines the same set of lines in K^{d+1} (or the same subset of $\mathbb{P}(K^{d+1})$) as does S (recall that by convention we view S as a subset of $L_k - L_0$). By induction hypothesis $(\mathrm{b})_{n-1}$, applied to the pointed k-cell $(y, \widehat{\eta})$ and $M_1' \subset N_{(y,\widehat{\eta})}$, a typical generator g of the left-hand side of (4) satisfies at least one of the following conditions:

(i) $g \in I(\eta\cup\{y\})$;

(ii) $g = \delta(e_0 \wedge \cdots \wedge e_r)$ for some $S = \{e_0, \ldots, e_r\} \subset \mathcal{A}$ such that $S^{z'}$ is linearly dependent modulo $\pi L_{z'}$ for all $z' \in M_1'$.

To show that g lies in the right-hand side of (4) is difficult only if (i) does not hold. In that case we claim $g \in \bigcap_{z\in M_1} I(z)$. So let S be as in (ii) and let $z \in M_1$ be given. Since M_0 is stable with respect to $\widehat{\eta}$ there exists a $z' \in M_1'$ such that $L_y \cap L_z = L_{z'}$. Now $S^{z'}$ is linearly dependent modulo $\pi L_{z'}$. If S^y was linearly dependent modulo πL_y we would have $g \in I(y)$, but then (i) would hold. Hence S^y is linearly independent modulo πL_y, hence so is $S^{z'} \cap S^y$, and hence $\widetilde{S} = S^{z'} - (S^{z'} \cap S^y)$ is linearly dependent modulo πL_z (since $S^{z'}$ is). On the other hand $\widetilde{S} \subset \pi L_y$ and this implies $\widetilde{S} \cap \pi L_z = \emptyset$ (since $\pi L_y \cap \pi L_z = \pi L_{z'}$). Thus, $\widetilde{S}$ is a subset of $L_z - \pi L_z$ which is linearly dependent modulo πL_z. In particular, S^z is linearly dependent modulo πL_z (since $\widetilde{S} \subset S^z$), hence $g \in I(z)$.

We have established the claim (4). Next we claim that

$$\widetilde{E} \longrightarrow \prod_{z\in M_0} \widetilde{A}(\eta\cup\{z\}) \longrightarrow \prod_{\substack{z_1,z_2\in M_0\\ \{z_1,z_2\}\in X^1}} \widetilde{A}(\eta\cup\{z_1,z_2\}) \tag{5}$$

is exact; this is equivalent with exactness of $K(\widehat{\eta},M_0)$ at the first non trivial degree. Let $(s_z)_{z\in M_0}$ be an element of the kernel of the second arrow in (5). By induction hypothesis $(\mathrm{a})_{n-1}$ for M_1 we may assume, after modifying $(s_z)_{z\in M_0}$ by the image of an element of $\widetilde{E}$, that $s_z=0$ for all $z\in M_1$. Then it follows that

$$s_y \in \operatorname{Ker}\biggl[\widetilde{A}(\eta\cup\{y\}) \longrightarrow \prod_{z'\in M_1'} \widetilde{A}(\eta\cup\{z',y\})\biggr].$$

This means that s_y can be lifted to an element $\tilde{s}_y$ of the left-hand side of (4). By (4) there exist $\tilde{b}\in I(\eta\cup\{y\})$ and $\tilde{c}\in\bigcap_{z\in M_1} I(\eta\cup\{z\})$ with $\tilde{s}_y=\tilde{b}+\tilde{c}$. Thus $\tilde{c}$ is a preimage of $(s_z)_{z\in M_0}$ and the exactness of (5) is proven. If we define the complex

$$K^y(\widehat{\eta},M_0)=\left\{\begin{array}{c} 0\longrightarrow \dfrac{\widetilde{E}}{\bigcap_{z\in M_1} I(\eta\cup\{z\})}\longrightarrow \\ \dfrac{\widetilde{E}}{\bigcap_{z'\in M_1'} I(\eta\cup\{y,z'\})}\times\prod_{z\in M_1}\widetilde{A}(\eta\cup\{z\})\longrightarrow \\ \prod_{\substack{z_1,z_2\in M_0\\ \{z_1,z_2\}\in X^1}}\widetilde{A}(\eta\cup\{z_1,z_2\})\longrightarrow \prod_{\substack{z_1,z_2,z_3\in M_0\\ \{z_1,z_2,z_3\}\in X^2}}\widetilde{A}(\eta\cup\{z_1,z_2,z_3\})\longrightarrow\cdots \end{array}\right\}$$

then we have a short exact sequence

$$0\longrightarrow K((y,\widehat{\eta}),M_1')[-1]\longrightarrow K^y(\widehat{\eta},M_0)\longrightarrow K(\widehat{\eta},M_1)\longrightarrow 0.$$

By induction hypothesis the complexes $K((y,\widehat{\eta}),M_1')$ and $K(\widehat{\eta},M_1)$ are exact; hence the complex $K^y(\widehat{\eta},M_0)$ is exact. The exactness of $K^y(\widehat{\eta},M_0)$ shows the exactness of $K(\widehat{\eta},M_0)$ except at the first non-trivial degree, but at the first non-trivial degree we have already seen exactness. Hence $K(\widehat{\eta},M_0)$ is exact and $(\mathrm{a})_n$ is proven. $\square$

If we define the complex

$$N^y(\widehat{\eta},M_0)=\left\{0\longrightarrow\frac{\bigcap_{z\in M_1} I(\eta\cup\{z\})}{\bigcap_{z\in M_0} I(\eta\cup\{z\})}\longrightarrow\frac{\bigcap_{z\in M_1'} I(\eta\cup\{y,z\})}{I(\eta\cup\{y\})}\longrightarrow 0\longrightarrow\cdots\right\}$$

then we have a short exact sequence

$$0\longrightarrow N^y(\widehat{\eta},M_0)\longrightarrow K(\widehat{\eta},M_0)\longrightarrow K^y(\widehat{\eta},M_0)\longrightarrow 0.$$

Since we have seen exactness of $K(\widehat{\eta},M_0)$ and of $K^y(\widehat{\eta},M_0)$ we get exactness of $N^y(\widehat{\eta},M_0)$. Now to prove $(\mathrm{b})_n$ and $(\mathrm{c})_n$ we first suppose $R=\mathbb{Z}$. By exactness of $N^y(\widehat{\eta},M_0)$ we get

$$\begin{aligned}\operatorname{rank}_{\mathbb{Z}}\biggl(\frac{\widetilde{E}}{\bigcap_{z\in M_0} I(\eta\cup\{z\})}\biggr)-\operatorname{rank}_{\mathbb{Z}}\biggl(\frac{\widetilde{E}}{\bigcap_{z\in M_1} I(\eta\cup\{z\})}\biggr)\\ =\operatorname{rank}_{\mathbb{Z}}(\widetilde{A}(\eta\cup\{y\}))-\operatorname{rank}_{\mathbb{Z}}\biggl(\frac{\widetilde{E}}{\bigcap_{z\in M_1'} I(\eta\cup\{y,z\})}\biggr).\end{aligned} \tag{6}$$

Observe that the collection $(\prec_z)_{z\in M_1}$, respectively $(\prec_{z'})_{z'\in M_1'}$, respectively $\prec_y$ is adapted to $(M_1,\widehat{\eta})$, respectively to $(M_1',(y,\widehat{\eta}))$, respectively $(\{y\},\widehat{\eta})$. We associate the sets $G(\widehat{\eta};M_1)$,

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respectively $G((y,\widehat{\eta});M_1')$, respectively $G(\widehat{\eta};\{y\})$ as before and claim

$$G(\widehat{\eta};M_0)=G(\widehat{\eta};M_1)\cup G(\widehat{\eta};\{y\}) \tag{7}$$
$$G((y,\widehat{\eta});M_1')=G(\widehat{\eta};M_1)\cap G(\widehat{\eta};\{y\}). \tag{8}$$

Here (7) and $\subset$ in (8) are very easy. To prove $\supset$ in (8), let $e_S\in G(\widehat{\eta};M_1)\bigcap G(\widehat{\eta};\{y\})$ and let $e\in S$. Then e is $(S,(y,\widehat{\eta}))$-special and $(S,(z,\widehat{\eta}))$-special for some $z\in M_1$. Let $z'\in M_1'$ be the element with $L_y\cap L_z=L_{z'}$. We will show that e is $(S,(z,y,\widehat{\eta}))$-special. If $e\in L_y$, then

$$e=\max_{\prec_z}(\mathcal{A}\cap L((z,\widehat{\eta}),\prec_z,S,e))=\max_{\prec_{z'}}(\mathcal{A}\cap L((z',y,\widehat{\eta}),\prec_{z'},S,e))$$

where the first equality follows from the $(S,(z,\widehat{\eta}))$-speciality of e and the second one from $L_y\cap L_z=L_{z'}$. If, however, $e\notin L_y$, then

$$e=\max_{\prec_y}(\mathcal{A}\cap L((y,\widehat{\eta}),\prec_y,S,e))=\max_{\prec_{z'}}(\mathcal{A}\cap L((z',y,\widehat{\eta}),\prec_{z'},S,e))$$

where the first equality follows from the $(S,(y,\widehat{\eta}))$-speciality of e and the second one is clear.

From (7) and (8) we deduce

$$|G(\widehat{\eta};M_0)|=|G(\widehat{\eta};M_1)|+|G(\widehat{\eta};\{y\})|-|G((y,\widehat{\eta});M_1')|. \tag{9}$$

By induction hypothesis $(\mathrm{c})_{n-1}$ we know

$$|G(\widehat{\eta};M_1)|=\mathrm{rank}_{\mathbb{Z}}\left(\frac{\widetilde{E}}{\bigcap_{z\in M_1}I(\eta\cup\{z\})}\right)$$
$$|G(\widehat{\eta};\{y\})|=\mathrm{rank}_{\mathbb{Z}}(\widetilde{A}(\eta\cup\{y\}))$$
$$|G((y,\widehat{\eta});M_1')|=\mathrm{rank}_{\mathbb{Z}}\left(\frac{\widetilde{E}}{\bigcap_{z\in M_1'}I(\eta\cup\{y,z\})}\right).$$

Thus we may compare (6) with (9) to obtain

$$\mathrm{rank}_{\mathbb{Z}}\left(\frac{\widetilde{E}}{\bigcap_{z\in M_0}I(\eta\cup\{z\})}\right)=|G(\widehat{\eta};M_0)|.$$

Comparing with Proposition 2.4 we see that source and target of the canonical surjection

$$\frac{\widetilde{E}}{J(\widehat{\eta};M_0)}\longrightarrow\frac{\widetilde{E}}{\bigcap_{z\in M_0}I(\eta\cup\{z\})}$$

have the same finite $\mathbb{Z}$-rank. However, the source is free over $\mathbb{Z}$ (because the $\mathbb{Z}$-submodule $J(\widehat{\eta};M_0)$ of $\widetilde{E}$ has a set of generators each of which is a linear combination, *with coefficients in* $\{-1,1\}$, of elements of the obvious (countable) $\mathbb{Z}$-basis of $\widetilde{E}$). Hence this surjection is bijective, so $(\mathrm{b})_n$ and $(\mathrm{c})_n$ follow if $R=\mathbb{Z}$, and then for arbitrary R by base change.

Theorem 2.6. *For any $\mathcal{A}\subset\mathbb{P}(K^{d+1})$, possibly infinite, the CCSs $\widetilde{A}$ and A satisfy $\mathcal{S}(k)$ for any $1\leqslant k\leqslant d$.*

Proof. The condition $\mathcal{S}(k)$ for $\widetilde{A}$ requires that for any pointed $(k-1)$-simplex $\widehat{\eta}$ and any subset $M_0\subset N_{\widehat{\eta}}$ which is stable with respect to $\widehat{\eta}$ the sequence

$$\widetilde{A}(\eta)\longrightarrow\prod_{z\in M_0}\widetilde{A}(\eta\cup\{z\})\longrightarrow\prod_{\substack{z_1,z_2\in M_0\\ \{z_1,z_2\}\in X^1}}\widetilde{A}(\eta\cup\{z_1,z_2\})$$

is exact. For fixed $\widehat{\eta}$ we may pass to a suitable finite subset of $\mathcal{A}$ without changing any of the involved groups. Hence we are in the situation considered above and what we need to show is precisely the exactness of $K(\widehat{\eta}, M_0)$ at its first nontrivial degree. This we did in Proposition 2.5. Hence $\widetilde{A}$ satisfies $\mathcal{S}(k)$. However, then A also satisfies $\mathcal{S}(k)$ because of the split exact sequence (3). ☐

Corollary 2.7. *The CCS $\widetilde{A}$ and A on X are acyclic in positive degrees: for any $k \geqslant 1$ we have $H^k(X, A) = 0$ and $H^k(X, \widetilde{A}) = 0$.*

Proof. See Theorems 1.2 and 2.6. ☐

For the rest of this section we assume $\mathrm{char}(K) = 0$ and take $\mathcal{A} = \mathbb{P}(K^{d+1})$. We write A_R instead of A in order to specify the chosen base ring R. Let $\Omega_K^{(d+1)}$ be Drinfel'd's symmetric space of dimension d over K. This is the K-rigid space obtained by removing all K-rational hyperplanes from projective d-space $\mathbb{P}_K^d$. There is a natural $\mathrm{GL}_{d+1}(K)$-equivariant *reduction map*

$$r : \Omega_K^{(d+1)} \longrightarrow X$$

(see, e.g., [deS01] for the precise meaning of r). For a simplex σ of X let $]\sigma[= r^{-1}(\mathrm{Star}(\sigma))$, the preimage in $\Omega_K^{(d+1)}$ of the star of σ: the star of σ is the union of the open simplices whose closure contains σ. This $]\sigma[$ is an admissible open subset of $\Omega_K^{(d+1)}$, and the collection of all the $]\sigma[$ forms an admissible open covering of $\Omega_K^{(d+1)}$.

Proposition 2.8 (de Shalit) [deS01]. *For a simplex σ of X denote by $H^k_{dR}(]\sigma[)$ the kth de Rham cohomology group of the K-rigid space $]\sigma[$. There is a natural isomorphism*

$$H^k_{dR}(]\sigma[) \cong A^k_K(\sigma).$$

Corollary 2.9.

(1) *(Local acyclicity.) Let σ be a simplex. For any $k \geqslant 0$ the sequence*

$$0 \longrightarrow H^k_{dR}\Big(\bigcup_{x\in\sigma}]x[\Big) \longrightarrow \prod_{x\in\sigma} H^k_{dR}(]x[) \longrightarrow \prod_{\substack{\tau\in X^1\\ \tau\subset\sigma}} H^k_{dR}(]\tau[) \longrightarrow \cdots$$

is exact.

(2) *(Global acyclicity, de Shalit.) The sequence*

$$0 \longrightarrow H^k_{dR}(\Omega_K^{(d+1)}) \longrightarrow \prod_{x\in X^0} H^k_{dR}(]x[) \longrightarrow \prod_{\sigma\in X^1} H^k_{dR}(]\sigma[) \longrightarrow \cdots$$

is exact.

Proof. (1) Choose a vertex $x \in \sigma$. Then $M_0 = \sigma - \{x\}$ is (as a set of vertices) stable with respect to x. Since the CCS A_K satisfies $\mathcal{S}(k)$ for any k, we derive just as in the proof of Proposition 2.5 that the sequence

$$\prod_{z\in\sigma} A_K(z) \longrightarrow \prod_{\substack{\tau\in X^1\\ \tau\subset\sigma}} A_K(\tau) \longrightarrow \cdots$$

is exact. Inserting Proposition 2.8 it becomes the exact sequence

$$\prod_{x\in\sigma} H^k_{dR}(]x[) \longrightarrow \prod_{\substack{\tau\in X^1\\ \tau\subset\sigma}} H^k_{dR}(]\tau[) \longrightarrow \cdots .$$

On the other hand, we have the spectral sequence

$$E_1^{rs} = \prod_{\substack{\tau\in X^r\\ \tau\subset\sigma}} H^s_{dR}(]\tau[) \Longrightarrow H^{s+r}_{dR}\Big(\bigcup_{x\in\sigma}]x[\Big).$$

Together (1) follows. The proof of (2) works the same way, using Corollary 2.7 instead of Theorem 2.6. □

Corollary 2.9 gives a precise expression of $H^k_{dR}(\Omega^{(d+1)}_K)$ through all the $H^k_{dR}(]\sigma[)$. The natural $\mathbb{Z}$-structures $A^k_{\mathbb{Z}}(\sigma)$ in the $A^k_K(\sigma)$ provide natural $\mathbb{Z}$-structures $H^k_{\mathbb{Z}}(]\sigma[)$ in the $H^k_{dR}(]\sigma[)$; hence the $\mathrm{GL}_{d+1}(K)$-stable subgroup

$$H^k_{\mathbb{Z}}(\Omega^{(d+1)}_K) = \mathrm{Ker}\Big[\prod_{x\in X^0} H^k_{\mathbb{Z}}(]x[) \longrightarrow \prod_{\sigma\in X^1} H^k_{\mathbb{Z}}(]\sigma[)\Big]$$

of $H^k_{dR}(\Omega^{(d+1)}_K)$. Now our Corollary 2.7 expresses $H^k_{\mathbb{Z}}(\Omega^{(d+1)}_K)$ *precisely* through the local terms $A^k_{\mathbb{Z}}(\sigma)$: it tells us that $H^k_{\mathbb{Z}}(\Omega^{(d+1)}_K)$ is quasiisomorphic with the complex

$$\prod_{x\in X^0} H^k_{\mathbb{Z}}(]x[) \longrightarrow \prod_{\sigma\in X^1} H^k_{\mathbb{Z}}(]\sigma[) \longrightarrow \prod_{\sigma\in X^2} H^k_{\mathbb{Z}}(]\sigma[) \longrightarrow \cdots . \tag{10}$$

Let us explain why this should have an application to a challenging problem on p-adic Abel–Jacobi mappings raised by Raskind and Xarles [RX03]. Let $\Gamma \subset \mathrm{PGL}_{d+1}(K)$ be a cocompact discrete subgroup such that the quotient $Y = \Gamma\backslash\Omega^{(d+1)}_K$, a smooth projective K-scheme, has strictly semistable reduction. For $1 \leqslant k \leqslant d$, Raskind and Xarles associate to Y a certain rigid analytic torus $J^k(Y)$, a 'p-adic intermediate Jacobian' ($J^1(Y)$ is the Picard variety of Y and $J^d(Y)$ is the Albanese variety of Y). The device for the construction of $J^k(Y)$ is a canonical $\mathbb{Z}$-structure in the graded pieces $\mathrm{Gr}^M_* H^k(Y)$ of the monodromy filtration on the cohomology $H^k(Y)$ of Y—both ℓ-adic ($\ell \neq p$) and log crystalline ($\cong$ de Rham) cohomology. This $\mathbb{Z}$-structure results from the fact that for any component intersection Z of the reduction of Y (so each Z is a smooth projective k-scheme), the cycle map

$$CH^k(Z \times_k \overline{k}) \otimes W(-k)/\mathrm{tors} \longrightarrow H^{2k}_{\mathrm{crys}}(Z \times_k \overline{k}/W)/\mathrm{tors}$$

is bijective (here $W(-k)$ is the ring of Witt vectors W with the action of Frobenius multiplied by p^k); similarly for ℓ-adic cohomology, as was recently proved by Ito [Ito03]. That is, the $\mathbb{Z}$-structure is essentially given by the collection of Chow groups for all Z. Then they define an Abel–Jacobi mapping

$$CH^k(Y)_{\mathrm{hom}} \longrightarrow J^k(Y)(K)$$

with $CH^k(Y)_{\mathrm{hom}}$ the group of cycles that are homologically equivalent to zero, using the ℓ-adic ($\ell \neq p$ and $\ell = p$) Abel–Jabobi mapping which involves the Galois cohomology groups $H^1_g(K, \cdot)$ defined by Bloch and Kato. As they point out, it would be helpful to define the Abel–Jabobi mapping by analytic means.

We expect that such a definition involves p-adic integration of cycles on the (contractible!) K-rigid space $\Omega^{(d+1)}_K$, similar to Besser's p-adic integration on K-varieties with good reduction. The link would be the covering spectral sequence

$$E^{rs}_2 = H^r(\Gamma, H^s(\Omega^{(d+1)}_K)) \Longrightarrow H^{r+s}(Y)$$

(which exists for both ℓ-adic ($\ell \neq p$) and de Rham cohomology). Indeed, we know that the associated filtration on $H^k(Y)$ *is* the monodromy filtration [deS05, Gro, Ito03]), therefore the $\mathbb{Z}$-structure $H^k_{\mathbb{Z}}(\Omega^{(d+1)}_K)$ in $H^k(\Omega^{(d+1)}_K)$ gives a $\mathbb{Z}$-structure in $\mathrm{Gr}^M_* H^k(Y)$. A comparison with that of Raskind and Xarles probably needs the resolution (10): the component intersections Z considered by them correspond precisely to the simplices of the quotient simplicial complex $\Gamma\backslash X$.

3. Local systems arising from representations

Let K, $\mathcal{O}_K$, π, k, X and its orientation be as in § 2. We fix a natural number $n \geqslant 1$ and let

$$U = U^{(n)} = \{g \in \mathrm{GL}_{d+1}(\mathcal{O}_K) \mid g \equiv 1 \bmod \pi^n\}$$

denote the principal congruence subgroup of level n in G. For a vertex $x \in X^0$ we let

$$U_x = U_x^{(n)} = gUg^{-1} \quad \text{if } x = g([\mathcal{O}_K^{d+1}]) \text{ for some } g \in G$$

and for a simplex $\tau = \{x_1, \ldots, x_k\}$ we let

$$U_\tau = \text{the subgroup generated by } U_{x_1} \cup \cdots \cup U_{x_k}.$$

This is a pro-p-group, $p = \mathrm{char}(k)$.

Lemma 3.1. *Suppose the lattices L_z, L_{x_1} and L_{x_2} represent vertices z, x_1 and x_2 in X^0 such that both x_1 and x_2 are incident to z and such that $L_z = L_{x_1} \cap L_{x_2}$. Then $U_z \subset U_{x_1}U_{x_2}$.*

Proof. Applying a suitable $g \in G$ we may assume that $L_z = \mathcal{O}_K^{d+1}$ and $L_{x_s} = t^s\mathcal{O}_K^{d+1}$ for diagonal matrices $\mathrm{id} \neq t^s = (t_0^s, \ldots, t_d^s)$ satisfying $\{1\} \subset \{t_j^1, t_j^2\} \subset \{1, \pi^{-1}\}$ for all $0 \leqslant j \leqslant d$. However, then $U_z = U$ as defined above, and $U_{x_s} = t^sU(t^s)^{-1}$ and an easy matrix argument gives the claim. □

Proposition I.3.1 in [SS97] significantly strengthens Lemma 3.1. It is this interpolation property of the groups U_x which also underlies the acyclicity proof in [SS93] and the much more general theory in [SS97].

Let V be a smooth representation of G on a (not necessarily free) $\mathbb{Z}[1/p]$-module V which is generated, as a G-representation, by its U-fixed vectors. Because of $U_\sigma \subset U_\tau$ if $\sigma \subset \tau$ we can form the HCS $\underline{V} = (V^{U_\tau})$ of subspaces of fixed vectors

$$V^{U_\tau} = \{v \in V \mid gv = g \text{ for all } g \in U_\tau\}$$

with the obvious inclusions as transition maps. In the special case where our V is a G representation on a $\mathbb{C}$-vector space (not just on a $\mathbb{Z}[1/p]$-module), the following theorem (and its version for $n = 1$) was proved in [SS93].

Theorem 3.2. *Suppose $n > 1$. Then the chain complex $C_\bullet(X, \underline{V})$ is a resolution of V.*

Proof. To see $H_k(X, \underline{V}) = 0$ for $k \geqslant 1$ it suffices, by Theorem 1.7, to prove $\mathcal{S}^*(k)$, i.e. to prove that for any pointed $(k-1)$-simplex $\widehat{\eta}$ with underlying $(k-1)$-simplex η and for any subset M_0 of $N_{\widehat{\eta}}$ which is stable with respect to $\widehat{\eta}$, the sequence

$$\bigoplus_{\substack{z,z' \in M_0 \\ \{z,z'\} \in X^1}} V^{U_{\{z,z'\} \cup \eta}} \xrightarrow{\partial_k} \bigoplus_{z \in M_0} V^{U_{\{z\} \cup \eta}} \xrightarrow{\partial_{k-1}} V^{U_\eta}$$

is exact. We use induction on $|M_0|$. If M_0 is non-empty choose a $y \in M_0$ for which L_y is maximal, i.e. there is no $z \in M_0$ with $L_y \subsetneq L_z$. Then $M_1 = M_0 - \{y\}$ is stable with respect to $\widehat{\eta}$. Letting

$$M_1' = \{z' \in M_1 \mid \{y, z'\} \in X^1\}$$

we first claim that

$$\bigoplus_{z' \in M_1'} V^{U_{\{y,z'\} \cup \eta}} \longrightarrow V^{U_y} \bigcap \sum_{z \in M_1} V^{U_{\{z\} \cup \eta}} \tag{11}$$

is surjective. Let $v = \sum_{z \in M_1} v_z$ be an element of the right-hand side with $v_z \in V^{U_{\{z\} \cup \eta}}$ for all $z \in M_1$. Since V is smooth we can find a (finitely generated) submodule V' of V containing v which

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is stable under U_η and U_z for all $z \in M_0$. The action of U_y on V' factors through a finite quotient $\overline{U}_y$ of U_y. Since v is fixed by $\overline{U}_y$ it follows that

$$v = \frac{1}{|\overline{U}_y|} \sum_{g \in \overline{U}_y} g \cdot v = \sum_{z \in M_1} \frac{1}{|\overline{U}_y|} \sum_{g \in \overline{U}_y} g \cdot v_z.$$

Since M_0 is stable, there exists for any $z \in M_1$ a $z' \in M_1'$ such that $L_{z'} = L_z \cap L_y$. It will be enough to show $\sum_{g \in \overline{U}_y} g \cdot v_z \in V^{U_{\{y,z'\}\cup\eta}}$. The stability under U_y is clear. Now let $h \in U_{z'}$. By Lemma 3.1 we may factor h as $h = h_y h_z$ with $h_y \in U_y$ and $h_z \in U_z$. Since $n > 1$ and since there is a vertex incident to both z and y we have $g^{-1} U_z g = U_z$ for any $g \in U_y$, hence $h_z g = g h_z^g$ with $h_z^g \in U_z$. Thus

$$h \sum_{g \in \overline{U}_y} g \cdot v_z = h_y \sum_{g \in \overline{U}_y} g h_z^g \cdot v_z = \sum_{g \in \overline{U}_y} g \cdot v_z,$$

i.e. $\sum_{g \in \overline{U}_y} g.v_z$ is stable under $U_{z'}$. Finally let $h \in U_x$ for some $x \in \eta$. Since x and y are incident we have $g^{-1} U_x g = U_x$, hence there is for any $g \in U_y$ a $h^g \in U_x$ with $hg = gh^g$. Then

$$h \sum_{g \in \overline{U}_y} g \cdot v_z = \sum_{g \in \overline{U}_y} g h^g \cdot v_z = \sum_{g \in \overline{U}_y} g \cdot v_z$$

so we have shown stability under U_η. The surjectivity of (11) is proven. Now let $c = (c_z)_{z \in M_0}$ be an element of $\mathrm{Ker}(\partial_{k-1})$. Then necessarily

$$c_y \in V^{U_y} \cap \sum_{z \in M_1} V^{U_{\{z\}\cup\eta}}.$$

By the surjectivity of (11) we may therefore modify c by an element of $\mathrm{Im}(\partial_k)$ such that for the new $c = (c_z)_{z \in M_0} \in \mathrm{Ker}(\partial_{k-1})$ we have $c_y = 0$. However, then the induction hypothesis, applied to M_1, tells us that after another such modification we can achieve $c = 0$. We have shown that $C_\bullet(X, \underline{V})$ is exact in positive degrees. It remains to observe that the hypothesis that V is generated by V^U is equivalent with the surjectivity of

$$C_0(X, \underline{V}) = \bigoplus_{x \in X^0} V^{U_x} \longrightarrow V. \qquad \square$$

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