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CONSTRUCTION OF UNITAL QUANTALES

SHENGWEI HAN AND JIAXIN LV

Abstract. Unital quantales constitute a significant subclass within quantale theory, which play a crucial role in the theoretical framework of quantale research. The main purpose of this article is to investigate the construction of unital quantales from a given quantale. Using Q-algebras, we prove that every quantale is embedded into a unital quantale, which generalizes the work of Paseka and Kruml for the construction of unital quantales. Based on which, we further show that every quantale can be transformed into a unitally non-distributive quantale, which expands the foundational work of Guriérrez García and Höhle for unitally non-distributive quantales. Finally, we provide a variety of methods for constructing unital quantales from some special quantales.

1. Introduction

Quantales were introduced to develop a framework for non-commutative spaces and quantum mechanics (see [20]). The literature on quantales often emphasizes that these structures are required to have units. Quantales characterized by unit, as a valued domain, have wide applications in theoretical computer science, logic, quantitative domain, enriched category, many-valued topological space and fuzzy algebra (see [1, 6, 11, 14, 21, 30, 31]). In [18], Paseka and Kruml investigated the construction of unital quantales from a given quantale, and they proved that every quantale Q can be extended to a unital quantale $Q[e]$. Based on such a result, Paseka and Kruml further showed that every quantale is embedded into a simple quantale (see [18]). Girard quantales introduced by Yetter are an important class of quantales, which provide an algebraic semantics for linear logic (see [30]). By unital quantales $Q[e]$, one can see that every quantale can also be embedded into a Girard quantale (see [9, 22]). Moreover, Han and Zhao used unital quantales $Q[e]$ to prove that the category of unital quantales is a reflective subcategory of the category of quantales (see [10]), and then Pan and Han used unital quantale $Q[e]$ to give the concrete form of free Q-algebras over sets (see [16]). Besides the method provided by Paseka and Kruml, are there any other methods to construct unital quantales from a given quantale? This question is the main motivation of this article.

In order to investigate the question posed by Hofmann and Clementino during the XIV Portuguese Category Seminar celebrated at Coimbra, Portugal in October 2023 (see [3]), Guriérrez García and Höhle presented a new class of quantales, called *unitally non-distributive quantales*. In [8], Guriérrez García and Höhle showed that

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under mild conditions, quantales on non-distributive lattices can be extended to unitally non-distributive quantales by addition of an isolated unit. Here, we have a natural question to ask whether every quantale can be extended to a unitally non-distributive quantale. In order to answer this question, we also need to consider a new method of constructing unital quantales. Therefore, investigating the construction of unital quantales will be the main task of the present article. Let 0 and 1 respectively denote the bottom element and the top element in a complete lattice. In the following, we shall review some basic concepts and results needed in this article.

Definition 1.1. (Rosenthal [22]) A *quantale* is a complete lattice Q with a semigroup multiplication & that distributes over arbitrary joins, that is,

$$a\&(\bigvee_{i\in I} b_i) = \bigvee_{i\in I}(a\&b_i) \quad \text{and} \quad (\bigvee_{i\in I} b_i)\&a = \bigvee_{i\in I}(b_i\&a)$$

for all $a \in Q, \{b_i\}_{i\in I} \subseteq Q$.

In fact, quantales can be seen as non-commutative complete residuated lattices. The study of such partially ordered algebraic structures goes back to a series of papers by Ward and Dilworth in the 1930s (see [4, 28, 29]). By the completeness of quantale Q, the semigroup multiplication & gives rise to a pair of binary operations $\rightsquigarrow$ and $\rightarrow$ satisfying

$$a \leq b \rightarrow c \Longleftrightarrow a\&b \leq c \Longleftrightarrow b \leq a \rightsquigarrow c \tag{1.1}$$

for all $a, b, c \in Q$. Note that every complete lattice Q can give rise to a quantale $(Q, \&_a)$ where $a \in Q$ and $\&_a$ is a semigroup multiplication determined by

$$\forall x, y \in Q,\ x\&_a y = \begin{cases} 0, & \text{if } x = 0 \text{ or } y = 0, \\ a, & \text{otherwise.} \end{cases} \tag{1.2}$$

A quantale Q is said to be *unital* provided that there exists an element $e \in Q$ such that $q\&e = q = e\&q$ for all $q \in Q$. Q is said to be *commutative* provided that $a\&b = b\&a$ for all $a, b \in Q$. An element x of Q is called *two-sided* (*non-zero*) provided that $x\&1 \leq x$ and $1\&x \leq x$ $(x \neq 0)$. Q is called *two-sided* if every element of Q is two-sided. A non-zero element x of Q is called a *zero-factor* provided that there exists a non-zero element y of Q such that $x\&y = 0$ or $y\&x = 0$. A subset S of Q is called a *subquantale* if S is closed under joins and &.

The following construction corresponds to the extension of a non-unital C^*-algebra to a unital C^*-algebra (see [19]).

Theorem 1.2. (*Paseka and Kruml* [15, 18]) *Let Q be a quantale, $Q[e] = \{a \vee k : a \in Q, k \in \{0, e\}\}$, where e is an arbitrary element such that $e \notin Q$. We define the supremum on $Q[e]$ with $0 \vee e = e$,*

$$\bigvee_{i\in I}^{Q[e]}(a_i \vee k_i) = \begin{cases} (\bigvee_{i\in I}^{Q} a_i) \vee e, & \textit{if } \exists i \in I, k_i = e, \\ \bigvee_{i\in I}^{Q} a_i, & \textit{otherwise} \end{cases}$$

and the semigroup multiplication $\&'$ *on* $Q[e]$ *as follows*

$$(a \vee k')\&'(b \vee k'') = \begin{cases} a\&b, & \text{if } k' = 0, k'' = 0, \\ (a\&b) \vee a, & \text{if } k' = 0, k'' \neq 0, \\ (a\&b) \vee b, & \text{if } k' \neq 0, k'' = 0, \\ ((a\&b) \vee a \vee b) \vee e, & \text{if } k' \neq 0, k'' \neq 0. \end{cases}$$

Then $(Q[e], \&')$ *is a unital quantale with unit* e*, and the inclusion map* $i: Q \to Q[e]$ *is an embedding of quantales in which* $i(a) = a \vee 0$.

Note that each element $a \in Q$ is identified with the formal join $a \vee 0$. The research on unital quantales $Q[e]$ can be found in [10, 15, 18].

Definition 1.3. (Abramsky and Vickers [1], Paseka [17]) Let Q be a commutative quantale. A *right module* over Q (a *right* Q*-module* for short) is a pair $(M, \cdot)$, where M is a complete lattice and $\cdot: M \times Q \to M$ is a map (called a *module operator*) such that

(1) $(\bigvee S) \cdot q = \bigvee_{s \in S} s \cdot q$ for all $q \in Q, S \subseteq M$;
(2) $m \cdot (\bigvee T) = \bigvee_{t \in T} m \cdot t$ for all $m \in M, T \subseteq Q$;
(3) $m \cdot (p\&q) = (m \cdot p) \cdot q$ for all $p, q \in Q, m \in M$.

For the sake of shortness, from now on "Q-module" means "right Q-module". If Q is a unital quantale with unit e, then a Q-module $(M, \cdot)$ is called *unital* provided that $m \cdot e = m$ for all $m \in M$.

Definition 1.4. (Solovyov [26], Wang and Zhao [27]) Let Q be a commutative (unital) quantale. A (*unital*) Q*-algebra* is a triple $(M, \cdot, \otimes)$ such that

(1) $(M, \cdot)$ is a (unital) Q-module;
(2) $(M, \otimes)$ is a quantale;
(3) $(a \otimes b) \cdot q = (a \cdot q) \otimes b = a \otimes (b \cdot q)$ for all $q \in Q, a, b \in M$.

Note that every complete lattice M can give rise to a Q-module $(M, \cdot)$ with *trivial module operator* $\cdot$, that is, $m \cdot q = 0$ for all $m \in M, q \in Q$. Thus, every quantale $(M, \otimes)$ can give rise to a Q-algebra $(M, \cdot, \otimes)$.

For the notions and concepts, which are not explained, please refer to [2, 5, 7].

2. Constructing unital quantales from given quantales

As mentioned in the introduction, unital quantales play an important role in the study of quantales. To investigate the different constructions of unital quantales should be an interesting and meaningful work.

Let $(M, \otimes)$ and $(N, \otimes)$ be quantales, $M \times N$ denote the Cartesian product of M and N. Then $(M \times N, \otimes)$ is a quantale, where $(m, n) \otimes (m', n') = (m \otimes m', n \otimes n')$. It is easy to verify that $(M \times N, \otimes)$ is a unital quantale if and only if both M and N are unital quantales. Thus, we can not use this method to construct unital quantales from non-unital quantales. In this section, we shall provide some new methods of constructing unital quantales.

2.1. **Constructing unital quantales by Q-algebras.** Let Q be a commutative (unital) quantale and $(M, \cdot, \otimes)$ be a (unital) Q-algebra. We define a semigroup multiplication $\&.$ on $M \times Q$ and a map $\star: (M \times Q) \times Q \to M \times Q$ as follows

$$(m, q)\&.(n, p) = ((m \otimes n) \vee (m \cdot p) \vee (n \cdot q), q\&p) \tag{2.1}$$

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and

$$(m, q) \star p = (m \cdot p, q\&p).$$

Note that if $(M, \cdot, \otimes)$ be a Q-algebra with trivial module operator, then $(M \times Q, \otimes) = (M \times Q, \&.)$.

Lemma 2.1. (*Han and Zhao* [11]) *Let Q be a commutative* (*unital*) *quantale and $(M, \cdot, \otimes)$ be a* (*unital*) *Q-algebra. Then*

(1) *$(M \times Q, \star, \&.)$ is a* (*unital*) *Q-algebra.*
(2) *the inclusion map $M \hookrightarrow M \times Q$, assigning m to $(m, 0)$, is an embedding of Q-algebras.*

Corollary 2.2. Let Q be a commutative (unital) quantale and $(M, \cdot, \otimes)$ be a (unital) Q-algebra. Then $(M \times Q, \&.)$ is a (unital) quantale and the inclusion map $M \hookrightarrow M \times Q$ is an embedding of quantales.

Lemma 2.3. *Let Q be a commutative quantale without zero-factors. Then every complete lattice M can give rise to a Q-module $(M, *)$ with unital module operator $*$, that is,*

$$\forall m \in M, q \in Q, \quad m * q = \begin{cases} 0, & \text{if } q = 0, \\ m, & \text{otherwise.} \end{cases}$$

Thus, every quantale $(M, \otimes)$ can give rise to a Q-algebra $(M, *, \otimes)$.

Proof. Proof is straightforward. □

Theorem 2.4. *Let K be a commutative unital quantale without zero-factors and $(M, \otimes)$ be a quantale. Then $(M \times K, \&_*)$ is a unital quantale.*

Proof. By Lemma 2.3, we have that $(M, *, \otimes)$ is a unital K-algebra with unital module operator. It follows from Corollary 2.2 that $(M \times K, \&_*)$ is a unital quantale. Clearly, we see that $\forall (m, k), (n, k') \in M \times K$,

$$(m, k)\&_*(n, k') = ((m \otimes n) \vee (m * k') \vee (n * k), k\&k')$$

and $(0, e)$ is the unit of $(M \times K, \&_*)$, where e is the unit of K. □

Let L be a complete lattice and let $\triangleleft$ be the totally below relation on L, i.e. given $\alpha, \beta \in L$, we write $\beta \triangleleft \alpha$ if for any subset $A \subseteq L$ with $\alpha \leq \bigvee A$ there is an element $\gamma \in A$ such that $\beta \leq \gamma$. An element α of L is called $\triangleleft$-*approximable* if $\alpha = \bigvee\{\beta \in L : \beta \triangleleft \alpha\}$. A lattice L is called *strictly non-distributive* if there exist $\alpha, \beta, x \in L$ such that $x \wedge (\alpha \vee \beta) \nleq (x \wedge \alpha) \vee (x \wedge \beta)$ and $x \nleq \alpha \vee \beta$ (see [8]). Let L, M be complete lattices such that L or M is strictly non-distributive. Then $L \times M$ is also strictly non-distributive.

Example 2.5. (1) Let $\mathbf{3} = \{0, e, 1\}$ denote the three-element quantale with a semigroup multiplication & determined by the table below.

&	0	e	1
0	0	0	0
e	0	e	1
1	0	1	1

Then $\mathbf{3}$ is a commutative unital quantale without zero-factors.

(2) Let $\mathbf{2}$ denote the two-element unital quantale. Then $\mathbf{2}$ has no zero-factors.

(3) Let $\mathbf{K_6} = \{0, a, b, c, e, 1\}$ be a complete lattice determined by Figure 1 and & be a semigroup multiplication on $\mathbf{K_6}$ determined by the table below.

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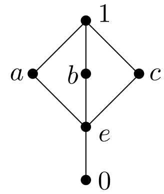

Figure 1. The partial order on $\mathbf{K_6}$

&	0	a	b	c	e	1
0	0	0	0	0	0	0
a	0	1	1	1	a	1
b	0	1	1	1	b	1
c	0	1	1	1	c	1
e	0	a	b	c	e	1
1	0	1	1	1	1	1

It is easy to check that $\mathbf{K_6}$ is a commutative unital quantale without zero-factors, and the underlying lattice of $\mathbf{K_6}$ is non-distributive, but not strictly non-distributive. Further, one can see that the unit e is $\lhd$-approximable.

(4) Let $\mathbf{K_8} = \{0, a, b, c, d, e, f, 1\}$ be a complete lattice determined by Figure 2 and & be a semigroup multiplication on $\mathbf{K_8}$ determined by the table below.

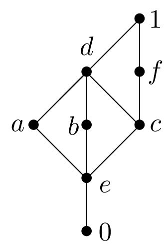

Figure 2. The partial order on $\mathbf{K_8}$

&	0	a	b	c	d	e	f	1
0	0	0	0	0	0	0	0	0
a	0	b	c	a	d	e	a	d
b	0	c	a	b	d	e	b	d
c	0	a	b	c	d	e	c	d
d	0	d	d	d	d	e	d	d
e	0	e	e	e	e	e	e	e
f	0	a	b	c	d	e	f	1
1	0	d	d	d	d	e	1	1

It is easy to check that $\mathbf{K_8}$ is a commutative unital quantale without zero-factors, and the underlying lattice of $\mathbf{K_8}$ is strictly non-distributive such that the unit f is $\lhd$-approximable.

The results in Corollary 2.6 generalize the work of Paseka and Kruml for unital quantales in [18].

Corollary 2.6. (1) Every quantale is embedded into a unital quantale.

(2) Let $(M, \otimes)$ be a quantale. Then $(M \times \mathbf{2}, \&_*)$ is isomorphic to the unital quantale $(M[e], \&')$.

Proof. The proof follows from Corollary 2.2, Theorem 2.4 and Example 2.5. □

In [8], Guriérrez García and Höhle showed that under mild conditions, quantales on non-distributive lattices can be extended to unitally non-distributive quantales by addition of an isolated unit. The following theorem indicates that every quantale can be extended to a unitally non-distributive quantale. First, we recall the concept of unitally non-distributive quantales.

A unital quantale Q is called *unitally non-distributive* if it satisfies the following properties:

(1) the unit e is $\lhd$-approximable.

(2) the unit e is non-distributive, that is, there exists a subset A of Q such that $e \wedge (\bigvee A) \nleq \bigvee_{a \in A}(e \wedge a)$.

Note that $\mathbf{K_8}$ is a unitally non-distributive quantale and the unit f is *strictly non-distributive*, that is, there exist $\alpha, \beta \in Q$ such that $f \wedge (\alpha \vee \beta) \nleq (f \wedge \alpha) \vee (f \wedge \beta)$ and $f \nleq \alpha \vee \beta$. In $\mathbf{K_6}$, the unit e is $\lhd$-approximable, but it is not non-distributive.

Theorem 2.7. *Every quantale can be embedded into a unitally non-distributive quantale.*

Proof. Let $(M, \otimes)$ be a quantale. Then by Theorem 2.4 and Example 2.5 we have that $(M \times \mathbf{K_8}, \&_*)$ is a unital quantale such that the unit $(0, f)$ of $(M \times \mathbf{K_8}, \&_*)$ is $\lhd$-approximable and the unit $(0, f)$ is non-distributive. Thus, $(M \times \mathbf{K_8}, \&_*)$ is unitally non-distributive. It follows from Corollary 2.2 that the inclusion map $M \hookrightarrow M \times \mathbf{K_8}$ is an embedding of quantales. □

Proposition 2.8. *Let Q be a commutative quantale and K be a commutative unital quantale without zero-factors. If $(M, \cdot, \otimes)$ is a Q-algebra, then $(M, \odot, \otimes)$ is a unital $(Q \times K)$-algebra.*

Proof. By Theorem 2.4, we have that $(Q \times K, \&_*)$ is a commutative unital quantale. Let $(M, \cdot, \otimes)$ be a Q-algebra. Then it follows from Lemma 2.3 that $(M, *, \otimes)$ is a unital K-algebra with unital module operator. We define a map $\odot\colon M \times (Q \times K) \to M$ as follows

$$\forall m \in M,\ (q, k) \in Q \times K,\ m \odot (q, k) := (m \cdot q) \vee (m * k).$$

Let $m, n, m_i \in M,\ (q, k), (p, k'), (q_i, k_i) \in Q \times K$. Then by Lemma 2.3 we have

$$\begin{aligned}
(\bigvee_i m_i) \odot (q, k) &= ((\bigvee_i m_i) \cdot q) \vee ((\bigvee_i m_i) * k) \\
&= (\bigvee_i m_i \cdot q) \vee (\bigvee_i m_i * k) \\
&= \bigvee_i (m_i \cdot q) \vee (m_i * k) \\
&= \bigvee_i m_i \odot (q, k),
\end{aligned}$$

$$\begin{aligned}
m \odot (\bigvee_i (q_i, k_i)) &= m \odot (\bigvee_i q_i, \bigvee_i k_i) \\
&= (m \cdot (\bigvee_i q_i)) \vee (m * (\bigvee_i k_i)) \\
&= (\bigvee_i (m \cdot q_i) \vee (m * k_i)) \\
&= \bigvee_i m \odot (q_i, k_i),
\end{aligned}$$

$$\begin{aligned}
(m \otimes n) \odot (q, k) &= ((m \otimes n) \cdot q) \vee ((m \otimes n) * k) \\
&= (m \otimes (n \cdot q)) \vee (m \otimes (n * k)) \\
&= m \otimes ((n \cdot q) \vee (n * k)) \\
&= m \otimes (n \odot (q, k)) \\
&= (m \otimes (n \cdot q)) \vee (m \otimes (n * k)) \\
&= ((m \cdot q) \otimes n) \vee ((m * k) \otimes n) \\
&= (m \odot (q, k)) \otimes n,
\end{aligned}$$

$$\begin{aligned}
m \odot ((q, k) \&_* (p, k')) &= m \odot ((q \& p) \vee (p * k) \vee (q * k'), k \& k') \\
&= (m \cdot ((q \& p) \vee (p * k) \vee (q * k'))) \vee (m * (k \& k')) \\
&= (m \cdot (q \& p)) \vee (m \cdot (p * k)) \vee (m \cdot (q * k')) \vee (m * (k \& k')) \\
&= ((m \cdot q) \cdot p) \vee ((m * k) \cdot p) \vee ((m \cdot q) * k') \vee ((m * k) * k') \\
&= (((m \cdot q) \vee (m * k)) \cdot p) \vee (((m \cdot q) \vee (m * k)) * k') \\
&= ((m \cdot q) \vee (m * k)) \odot (p, k') \\
&= (m \odot (q, k)) \odot (p, k'),
\end{aligned}$$

and $m \odot (0, e) = m$ where e is the unit of K.

By the above description, we have that $(M, \odot, \otimes)$ is a unital $(Q \times K)$-algebra. □

Corollary 2.9. (Pan and Han [16]) Let Q be a commutative quantale and $(M, \cdot, \otimes)$ be a Q-algebra. Then $(M, \odot, \otimes)$ is a unital $Q[e]$-algebra.

Proof. Taking $K = \mathbf{2}$, by Corollary 2.6 and Proposition 2.8 we have that $(M, \odot, \otimes)$ is a unital $Q[e]$-algebra. □

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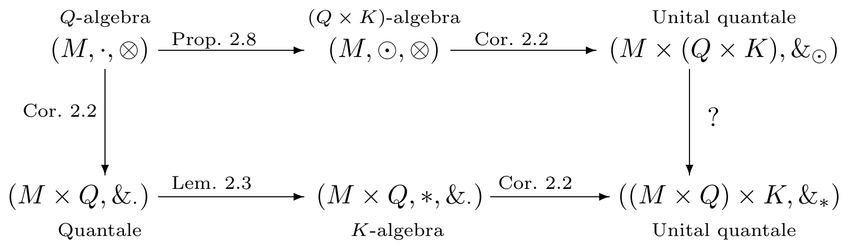
Let Q be a commutative quantale and K be a commutative unital quantale without zero-factors. Then for a given Q-algebra $(M,\cdot,\otimes)$, using the following methods we obtain two unital quantales $((M\times Q)\times K,\&_*)$ and $(M\times(Q\times K),\&_\odot)$.



Proposition 2.10. *Let Q be a commutative quantale and K be a commutative unital quantale without zero-factors. If $(M,\cdot,\otimes)$ is a Q-algebra, then $((M\times Q)\times K,\&_*)\cong(M\times(Q\times K),\&_\odot)$.*

Proof. We define a map $f\colon (M\times Q)\times K\to M\times(Q\times K)$ as follows

$$\forall((m,q),k)\in(M\times Q)\times K,\quad f((m,q),k)=(m,(q,k)).$$

It is easy to show that f is an isomorphism of quantales, that is, $((M\times Q)\times K,\&_*)\cong(M\times(Q\times K),\&_\odot)$. □

2.2. **Constructing unital quantales by upper sets.** In order to provide a unified semantics for a wide class of substructural logics, Rump and Yang introduced the concept of quantum B-algebras (see [23, 24]). Quantum B-algebras can be regarded as implicational subreducts of quantales. A *quantum B-algebra* is a poset X with two binary operations $\to$ and $\rightsquigarrow$ satisfying the following three conditions

$$y\le z\Longrightarrow x\to y\le x\to z$$
$$x\le y\to z\Longleftrightarrow y\le x\rightsquigarrow z$$
$$x\rightsquigarrow(y\to z)=y\to(x\rightsquigarrow z)$$

for all $x,y,z\in X$. A quantum B-algebra X is called *unital* if there exists an element $u\in X$ such that $u\to x=x=u\rightsquigarrow x$ for all $x\in X$.

Note that by (1.1) we see that a quantale is a quantum B-algebra. Rump and Yang used upper sets to build a relation between quantum B-algebras and quantales. Let X be a quantum B-algebra, $U(X)$ denote the set of all upper sets of X. Then $U(X)$ is a complete lattice with respect to set-theoretic union. For $A,B\in U(X)$, we define a semigroup multiplication $\odot$ on $U(X)$ as follows

$$A\odot B:=\{x\in X:\exists b\in B,b\to x\in A\}. \tag{2.2}$$

Note that $A\odot B=\{x\in X:\exists a\in A,a\rightsquigarrow x\in B\}=\{x\in X:\exists a\in A,b\in B,s.t.\ a\le b\to x\}=\{x\in X:\exists a\in A,b\in B,s.t.\ b\le a\rightsquigarrow x\}$.

Proposition 2.11. (*Rump and Yang* [24]) *Let $(X,\to,\rightsquigarrow,\le)$ be a quantum B-algebra. Then $(U(X),\odot)$ is a quantale, called an upper-set quantale.*

By upper-set quantales, Rump and Yang proved that the category of quantum B-algebras and the category of logical quantales are dually equivalent (see [24]). In [12], Han, Xu and Qin gave a sufficient and necessary condition for the upper-set quantale $U(X)$ to be a unital quantale. A subset E of a quantum B-algebra X is called *positive* provided that E is an upper set of X, and $x\rightsquigarrow y\in E\Longleftrightarrow x\le y\Longleftrightarrow x\to y\in E$ for all $x,y\in X$. Note that if a quantum B-algebra has a positive subset, then it is unique (see Proposition 3.6 in [13]). A quantum B-algebra is called *positive* if it has a positive subset. Clearly, a unital quantum B-algebra is positive, but a positive quantum B-algebra is in general not unital (see [12]).

Proposition 2.12. (*Han et al.* [12]) *Let* $(X, \rightarrow, \rightsquigarrow, \leq)$ *be a quantum* B*-algebra. Then* $(U(X), \odot)$ *is a unital quantale if and only if* X *is positive.*

Corollary 2.13. If a quantale Q as a quantum B-algebra is positive, then $(U(Q), \odot)$ is a unital quantale.

Remark 2.14. (1) If a quantale $(Q, \&)$ is considered as a posemigroup, then $A \odot B = \{x \in Q : \exists a \in A, b \in B,\ a\&b \leq x\}$ for $A, B \in U(Q)$.
(2) A quantale Q is unital if and only if Q as a quantum B-algebra is unital.
(3) One may ask such a question whenever a quantale as quantum B-algebra is positive, it must be unital? In Example 2.15, we shall give a negative answer.

Example 2.15. Let $Q = \{0, a, b, 1\}$ be a complete lattice determined by Figure 3 and & be a semigroup multiplication on Q determined by the table below.

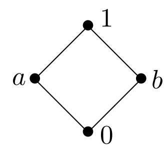


Figure 3. The partial order on Q

&	0	a	b	1
0	0	0	0	0
a	0	a	1	1
b	0	1	b	1
1	0	1	1	1

It is easy to verify that $(Q, \&)$ is a non-unital quantale, and $\{a, b, 1\}$ is a positive subset of Q. Thus, by Corollary 2.13 we have that $(U(Q), \odot)$ is a unital quantale.

When quantales are considered as quantum B-algebras, we can use upper sets to construct upper-set quantales. Further, if quantales are positive, then the corresponding upper-set quantales are unital. When quantales are considered as partially ordered semigroups (posemigroups for short), we can use lower sets to construct lower-set quantales (see [11, 15]). A natural question is under what condition the lower-set quantale $L(Q)$ of a quantale Q is unital.

Let $(S, \cdot, \leq)$ be a posemigroup, $L(S)$ denote the set of all lower sets of S. Then $L(S)$ is a complete lattice with respect to set-theoretic union. We now define a semigroup multiplication $\otimes$ on $L(S)$ as follows

$$A \otimes B = \{x \in S : \exists a \in A, b \in B,\ x \leq a \cdot b\}. \tag{2.3}$$

Proposition 2.16. (*Krum and Paseka* [15]) *Let* $(S, \cdot, \leq)$ *be a posemigroup. Then* $(L(S), \otimes)$ *is a quantale, called a lower-set quantale.*

A subset K of a posemigroup $(S, \cdot, \leq)$ is called *normal* provided that K satisfies the following three conditions:
(1) K is a lower set of S.
(2) $k \cdot q \leq q$ and $q \cdot k \leq q$ for all $q \in S, k \in K$.
(3) For any $q \in S$, there exist $k, k' \in K$ such that $q = k \cdot q$ and $q = q \cdot k'$.

Note that if a posemigroup has a normal subset, then it is unique. To see this, we let K and K' be two normal subsets. Then for $k \in K$ there exists $k' \in K'$ such that $k = k' \cdot k \leq k' \in K'$, which implies that $k \in K'$, that is, $K \subseteq K'$. Similarly, we have $K' \subseteq K$. Thus, $K = K'$. A posemigroup is called *normal* if it has a normal subset. A unital posemigroup is normal, but the converse is in general not true (see Example 2.18).

Proposition 2.17. *Let $(S, \cdot, \leq)$ be a posemigroup. Then $(L(S), \otimes)$ is a unital quantale if and only if S is normal.*

Proof. Let K be the unit of quantale $(L(S), \otimes)$. Then $(\downarrow q) \otimes K = \downarrow q = K \otimes (\downarrow q)$ for all $q \in S$. By (2.3), we see that K is a normal subset of S, that is, S is normal.

Conversely, let K be the normal subset of S. Then it is easy to check that K is the unit of quantale $(L(S), \otimes)$. □

Example 2.18. Let $S = \{0, a, b, c, d\}$ be a poset determined by Figure 4 and & be a semigroup multiplication on S determined by the table below.

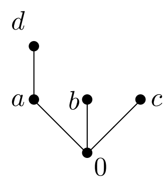


Figure 4. The partial order on S

&	0	a	b	c	d
0	0	0	0	0	0
a	0	a	0	0	a
b	0	0	b	0	d
c	0	0	0	c	0
d	0	a	d	0	a

It is easy to verify that $(S, \cdot, \leq)$ is a non-unital posemigroup, and $\{0, a, b, c\}$ is a normal subset of S. Thus, by Proposition 2.17 we have that $L(S), \otimes)$ is a unital quantale.

If quantales are considered as posemigroups, then we have the following interesting result.

Proposition 2.19. *Let $(Q, \&)$ be a quantale. Then $(L(Q), \otimes)$ is a unital quantale if and only if Q is a unital quantale.*

Proof. Let Q be a unital quantale with unit e. Then $\downarrow e$ is the unit of $(L(Q), \otimes)$.

Conversely, let $(L(Q), \otimes)$ be a unital quantale with unit K. By Proposition 2.17, we see that K is a normal subset of Q. We let $e = \bigvee K$. It is easy to show that e is the unit of Q, that is, Q is unital. □

2.3. **Constructing unital quantales by adding elements.** For some special quantales, Guriérrez García and Höhle investigated the constructions of unital quantales by addition of an isolated element (see [8]). Next, based on the work of Guriérrez García and Höhle, we shall continue the research on the construction of unital quantales.

Adding two elements to construct unital quantales. Let L be a complete lattice, $\alpha \in L \backslash \{0, 1\}$. Then α is called *isolated* (see [8]) in L if there exist two elements $\alpha^- \leq \alpha$ and $\alpha^+ \geq \alpha$ of L such that the following properties hold:

$$(\downarrow \alpha) \backslash \{\alpha\} = \downarrow \alpha^- \text{ and } (\uparrow \alpha) \backslash \{\alpha\} = \uparrow \alpha^+.$$

Lemma 2.20. (*Guriérrez García and Höhle* [8]) *Let L be a complete lattice, and $\top$ and e be two different elements satisfying the condition $L \cap \{\top, e\} = \emptyset$. Then for every $\gamma \in L$ there exists a unique complete lattice-structure on $L^\gamma = L \cup \{\top, e\}$ satisfying the following conditions:*

(1) *L^γ is an extension of L.*
(2) *The element $\top$ is the universal upper bound of L^γ.*
(3) *The element e is isolated and satisfies the conditions $e^- = \gamma$ and $e^+ = \top$.*

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Theorem 2.21. (*Guriérrez García and Höhle* [8]) *Let $(Q,\&)$ be a quantale, $\gamma \in Q\backslash\{1\}$ and let $Q^\gamma = Q \cup \{\top, e\}$ be the extension of the underlying lattice by an isolated element e in the sense of Lemma* 2.20. *Then there exists a unique quantale structure on Q^γ with unit e and subquantale Q if and only if Q satisfies the following conditions for all $\alpha \in Q$ and $\beta \in Q$ with $\beta \not\leq \gamma$:*

$$(\gamma\&\alpha) \vee (\alpha\&\gamma) \leq \alpha, \tag{2.4}$$

$$1\&\alpha \leq (\beta\&\alpha) \vee \alpha \text{ and } \alpha\&1 \leq (\alpha\&\beta) \vee \alpha. \tag{2.5}$$

Remark 2.22. (1) If a quantale $(Q,\&)$ has an element γ satisfying (2.4) and (2.5), then Q can be extended to a unital quantale Q^γ with unit e where the semigroup multiplication $\&^\gamma$ on Q^γ is determined by

$$\forall x, y \in Q^\gamma,\ x\&^\gamma y = \begin{cases} x\&y, & \text{if } x, y \in Q, \\ y, & \text{if } x = e, \\ x, & \text{if } y = e, \\ (1\&y) \vee y, & \text{if } x = \top, y \in Q, \\ x \vee (x\&1), & \text{if } y = \top, x \in Q, \\ \top, & \text{if } x = y = \top. \end{cases}$$

(2) Since every non-top element γ of a two-sided quantale satisfies the properties (2.4) and (2.5), every two-sided quantale Q can be extended to a unital quantale Q^γ.

Using a method similar to adding isolated elements, we shall give a new method to construct unital quantales.

Lemma 2.23. *Let L be a complete lattice, and $\perp$ and e be two different elements satisfying the condition $L \cap \{\perp, e\} = \emptyset$. Then for every $\gamma \in L$ there exists a unique complete lattice-structure on $L_\gamma = L \cup \{\perp, e\}$ satisfying the following conditions:*

(1) *L_γ is an extension of L.*
(2) *The element $\perp$ is the universal lower bound of L_γ.*
(3) *The element e is isolated and satisfies the conditions $e^+ = \gamma$ and $e^- = \perp$.*

Let $(Q,\&)$ be a quantale and γ be a non-zero element of Q. Define a semigroup multiplication $\&_\gamma$ as follows

$$\forall x, y \in Q_\gamma,\ x\&_\gamma y = \begin{cases} x\&y, & \text{if } x, y \in Q, \\ y, & \text{if } x = e, \\ x, & \text{if } y = e, \\ \perp, & \text{if } x = \perp \text{ or } y = \perp. \end{cases}$$

Theorem 2.24. *Let γ be a non-zero element of a quantale Q and let $Q_\gamma = Q \cup \{\perp, e\}$ be the extension of the underlying lattice by an isolated element e in the sense of Lemma* 2.23. *Then $(Q_\gamma, \&_\gamma)$ is a unital quantale with unit e if and only if γ is the unit of Q.*

Proof. The proof is similar to that of Theorem 2.21. □

Adding one element to construct unital quantales. Inspired by the work of Guriérrez García and Höhle, we will consider the method of adding one element to construct unital quantales. Let r be a non-zero element of quantale Q and $P_r(Q) \triangleq \{y \in Q \mid (r\&y) \vee (x\&y) = y \vee (x\&y)$ and $(y\&r) \vee (y\&x) = y \vee (y\&x)$ for all $x \in Q\backslash\{0\}\}$.

Proposition 2.25. *Let Q be a quantale, $r \in Q\backslash\{0\}$. Then we have*

(1) *$P_r(Q)$ is a subquantale of Q.*
(2) $1 \in P_r(Q) \Longleftrightarrow r\&1 = 1 = 1\&r$.
(3) *if Q is a unital quantale with unit u, then $u \in P_r(Q) \Longleftrightarrow u = r$.*
(4) *if Q is a unital quantale with unit u, then $P_u(Q) = Q$.*

Proof. Proof is straightforward. □

Definition 2.26. A non-zero element r of quantale Q is called a *weak unit* if $P_r(Q) = Q$.

Remark 2.27. (1) If Q is a quantale with $\& = \&_1$ defined by (1.2), then every non-zero element of Q is a weak unit.
(2) If Q is a quantale with $\& = \&_a$ and $a \neq 1$ defined by (1.2), then no element of Q is a weak unit.
(3) If Q is a unital quantale, then by Proposition 2.25(3) we see that the unit is the unique weak unit in Q. Conversely, if a quantale Q has a unique weak unit, then Q is not necessarily a unital quantale (see Example 2.28).

Example 2.28. Let $Q = \{0, a, b, 1\}$ be a complete lattice with $0 < a < b < 1$ and $\&$ be a semigroup multiplication on Q determined by the table below.

$\&$	0	a	b	1
0	0	0	0	0
a	0	b	b	b
b	0	b	b	b
1	0	b	b	1

It is easy to verify that Q is a non-unital quantale and there is a unique weak unit 1 in Q.

Let r be a non-zero element of quantale Q and $e \notin Q$. We denote by $\overline{Q_r}$ the set $Q \cup \{e\}$ and define a partial order $\leq_r$ on $\overline{Q_r}$ and a semigroup multiplication $\&_e$ on $\overline{Q_r}$ as follows

$$\leq_r = \leq \cup \{(e, x) \mid r \leq x, x \in Q\} \cup \{(e, e), (0, e)\}$$

and

$$\forall x, y \in \overline{Q_r}, \quad x\&_e y = \begin{cases} x\&y, & \text{if } x, y \in Q, \\ y, & \text{if } x = e, \\ x, & \text{if } y = e. \end{cases} \tag{2.6}$$

Proposition 2.29. *Let Q be a quantale and r be a non-zero element of Q. Then we have*

(1) *$(\overline{Q_r}, \leq_r)$ is a complete lattice and $e \vee x = r \vee x$ for all $x \in Q\backslash\{0\}$.*
(2) *e is an isolated element in $\overline{Q_r}$ and $e^+ = r, e^- = 0$.*
(3) *$(\overline{Q_r}, \leq_r, \&_e)$ is a unital posemigroup if and only if $x \leq (x\&r) \wedge (r\&x)$ for all $x \in Q$.*

Proof. Proof is straightforward. □

Proposition 2.30. *Let Q be a quantale and r be a non-zero element of Q. Then $(\overline{Q_r}, \leq_r, \&_e)$ is a unital quantale if and only if r is a weak unit of Q.*

Proof. Let $(\overline{Q_r}, \leq_r, \&_e)$ be a unital quantale with unit e. Then by Proposition 2.29 we have that $(r\&y)\vee(x\&y) = (r\vee x)\&y = (e\vee x)\&_e y = (e\&_e y)\vee(x\&_e y) = y\vee(x\&y)$ for all $y \in Q, x \in Q\backslash\{0\}$. Similarly, we have $(y\&r) \vee (y\&x) = y \vee (y\&x)$. Thus, r is a weak unit of Q.

Conversely, we let r be a weak unit of Q. By Proposition 2.29, it is easy to check that the semigroup multiplication $\&_e$ distributes over arbitrary joins, that is, $(\overline{Q_r}, \leq_r, \&_e)$ is a unital quantale. □

Corollary 2.31. If quantale Q has a weak unit r, then Q can be embedded into a unital quantale $\overline{Q_r}$.

Example 2.32. (Guriérrez García and Höhle [8]) Let $(G, \cdot, e)$ be a group with $\mid G \mid\geq 2$. We provide G with the discrete order " $=$ " and subsequently we apply the MacNeille completion. This construction leads to a complete lattice $G_\infty = G \cup \{\bot, \top\}$ by adding the universal bounds to the discretely ordered set G. The binary operation $*$ on G_∞ is determined by

$$\forall x, y \in Q_\infty, \quad x * y = \begin{cases} x \cdot y, & \text{if } \ x, y \in G, \\ \bot, & \text{if } \ x = \bot \text{ or } y = \bot, \\ \top, & \text{otherwise.} \end{cases}$$

It is easy to check that $(G_\infty, *)$ is a unital quantale and the unit e is an isolated element of G_∞ with $e^+ = \top, e^- = \bot$.

Proposition 2.33. *Let Q be a unital quantale such that the unit e is an isolated element and $e^- = 0$. Then $Q \cong \overline{K_r}$ for some quantale K with weak unit r if and only if $Q\backslash\{e\}$ is a subquantale of Q.*

Proof. Let K be a quantale with weak unit r. Then by Proposition 2.30 we have that $\overline{K_r}$ is a unital quantale with unit e and K is a subquantale of $\overline{K_r}$. Since $Q \cong \overline{K_r}$, we have that $Q\backslash\{e\} \cong K$, which implies that $Q\backslash\{e\}$ is a subquantale of Q.

Conversely, we let $r = e^+$ and $K = Q\backslash\{e\}$. Then by Proposition 2.29 we have that r is a weak unit of K. Further, we see that $Q = \overline{K_r}$. □

Corollary 2.34. Let Q be a unital quantale such that the unit e is an isolated element and $e^- = 0$. Then $Q \cong \overline{K_r}$ for some quantale K with weak unit r if and only if $x\&y = e$ implies that $x = e = y$ for all $x, y \in Q$.

Remark 2.35. (1) Let $(G_\infty, *)$ be the unital quantale in Example 2.32. Then $(G_\infty, *)$ is not of form $\overline{K_r}$, that is, $G_\infty \ncong \overline{K_r}$.
(2) Let $Q = \{0, a, 1\}$ be a quantale with $\& = \&_1$. Then 1 is a weak unit of Q. By Proposition 2.30, we see that $\overline{Q_1}$ is a unital quantale determined by Figure 5 and the following table.

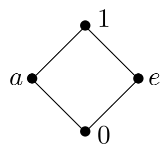

Figure 5. The partial order on $\overline{Q_1}$

$\&$	0	a	e	1
0	0	0	0	0
a	0	1	a	1
e	0	a	e	1
1	0	1	1	1

Similarly, we can give another method of adding one element to construct unital quantales. Let Q be a quantale with $e \notin Q$. For a non-top element r of Q, we

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denote by $\overline{Q^r}$ the set $Q \cup \{e\}$ and define a partial order $\leq_r$ on $\overline{Q^r}$ and a binary operation $\&_e$ on $\overline{Q^r}$ as (2.6)

$$\leq_r = \leq \cup \{(x, e) \mid x \leq r, x \in Q\} \cup \{(e, e), (e, 1)\}.$$

Lemma 2.36. *Let Q be a quantale and r be a non-top element of Q. Then we have*

(1) *$(\overline{Q^r}, \leq_r)$ is a complete lattice and $e \vee x = 1$ for all $x \in Q$ with $x \not\leq r$.*
(2) *e is an isolated element in $\overline{Q^r}$ and $e^+ = 1, e^- = r$.*
(3) *$(\overline{Q^r}, \leq_r, \&_e)$ is a posemigroup with unit e if and only if $(x\&r) \vee (r\&x) \leq x \leq (x\&1) \wedge (1\&x)$ for all $x \in Q$.*

Theorem 2.37. *Let $(Q, \&)$ be a quantale with non-top element r and let $\overline{Q^r}$ be the extension of the underlying lattice in the sense of Lemma 2.36. Then there exists a unique quantale structure on $\overline{Q^r}$ with unit e and subquantale Q if and only if Q satisfies the following conditions for all $x \in Q$ and $y \in Q$ with $y \not\leq r$:*

$$(r\&x) \vee (x\&r) \leq x,$$

$$1\&x = (y\&x) \vee x \text{ and } x\&1 = (x\&y) \vee x.$$

Proof. The proof is similar to that of Theorem 2.21. □

Let Q be a quantale with $\top \notin Q$ and $\widehat{Q}_\top$ denote the set $Q \cup \{\top\}$. We define a partial order $\leq_\top$ on $\widehat{Q}_\top$ and a semigroup multiplication $\&_\top$ on $\widehat{Q}_\top$ as follows

$$\leq_\top = \leq \cup \{(x, \top) : x \in \widehat{Q}_\top\}$$

and

$$\forall x, y \in \widehat{Q}_\top, \quad x \&_\top y = \begin{cases} x\&y, & \text{if } x, y \in Q, \\ y, & \text{if } x = \top, \\ x, & \text{if } y = \top. \end{cases}$$

Proposition 2.38. *Let $(Q, \&)$ be a quantale. Then $(\widehat{Q}_\top, \&_\top)$ is a unital quantale if and only if Q is a two-sided quantale.*

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