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# REMARKS ON THE MIXED JOINT UNIVERSALITY FOR A CLASS OF ZETA FUNCTIONS

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**Abstract**

Two results related to the mixed joint universality for a polynomial Euler product $\varphi(s)$ and a periodic Hurwitz zeta function $\zeta(s, \alpha; \mathfrak{B})$, when $\alpha$ is a transcendental parameter, are given. One is the mixed joint functional independence and the other is a generalised universality, which includes several periodic Hurwitz zeta functions.

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## 1. Introduction

In our paper [5], we have shown a certain mixed joint universality theorem, which is valid for an Euler product of rather general form, and a periodic Hurwitz zeta function.

In the present note, we give two remarks related to the result in [5]. The first remark is on the mixed joint functional independence. It is well known that functional independence properties can be deduced from universality results. We will show that such a functional independence is also valid in the present mixed joint situation.

The second remark is on a generalisation of the result in [5]. It is an important problem to study how general the mixed joint universality property can be. We will prove a generalised limit theorem and a generalised universality theorem, which involve several periodic Hurwitz zeta functions, under a certain matrix condition.

## 2. Functional independence

The history of the problem on the functional independence of Dirichlet series goes back to the famous lecture of Hilbert [2] in 1900. He mentioned that the Riemann zeta function $\zeta(s)$ does not satisfy any nontrivial algebraic differential equation.

Recall the definition of $\zeta(s)$. Let $\mathbb{P}$ be the set of all prime numbers and $\mathbb{C}$ the set of all complex numbers. For $s = \sigma + it \in \mathbb{C}$, $\zeta(s)$ is given by

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s} = \prod_{p\in\mathbb{P}} \Big(1 - \frac{1}{p^s}\Big)^{-1}$$

for $\sigma > 1$, and can be analytically continued to the whole complex plane $\mathbb{C}$ except for a simple pole at the point $s = 1$ with residue 1.

In 1973, Voronin [16] proved the following functional independence result for $\zeta(s)$. Let $\mathbb{N}$ and $\mathbb{N}_0$ be the sets of positive integers and nonnegative integers, respectively.

Theorem 2.1 [16]. *Let $N \in \mathbb{N}$ and $n \in \mathbb{N}_0$. The function $\zeta(s)$ does not satisfy any differential equation*

$$\sum_{j=0}^{n} s^j F_j(\zeta(s), \zeta'(s), \ldots, \zeta^{(N-1)}(s)) \equiv 0$$

*for continuous functions $F_j$, $j = 0, \ldots, n$, not all identically zero.*

Later this result was generalised to other zeta and $L$-functions. For a survey, see the monographs by Laurinčikas [6] and Steuding [15].

Nowadays in analytic number theory the investigation of statistical properties (and also the functional independence) for a collection of various zeta functions, some of which have an Euler product expansion while others do not, is a very interesting problem since an important role is played by parameters in the definition of the functions.

The first result in this direction is due to Mishou. In 2007, he proved [12, Theorem 4] that the pair of zeta functions consisting of the Riemann zeta function $\zeta(s)$ and the Hurwitz zeta function $\zeta(s, \alpha)$ is functionally independent.

We recall that the Hurwitz zeta function $\zeta(s, \alpha)$ with a fixed parameter $\alpha$, $0 < \alpha \leq 1$, is defined by the Dirichlet series

$$\zeta(s, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m + \alpha)^s}$$

for $\sigma > 1$, and can be continued to the whole complex plane except for a simple pole at the point $s = 1$ with residue 1. In general, the function $\zeta(s, \alpha)$ has no Euler product over primes, except for the cases $\alpha = 1$ and $\alpha = \frac{1}{2}$, when $\zeta(s, \alpha)$ is essentially reduced to $\zeta(s)$. Then Mishou's result is the following statement.

Theorem 2.2 [12]. *Let $\alpha$ be transcendental. For $N \in \mathbb{N}$, $n \in \mathbb{N}_0$, let $F_j : \mathbb{C}^{2N} \to \mathbb{C}$ be a continuous function for each $j = 0, \ldots, n$. Suppose that*

$$\sum_{j=0}^{n} s^j \cdot F_j\,(\zeta(s), \zeta'(s), \ldots, \zeta^{(N-1)}(s), \zeta(s, \alpha), \zeta'(s, \alpha), \ldots, \zeta^{(N-1)}(s, \alpha)) \equiv 0.$$

*Then $F_j \equiv 0$ for $j = 0, \ldots, n$.*

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This is the first 'mixed joint' functional independence theorem. Later this result was generalised to the collection of a periodic zeta function and a periodic Hurwitz zeta function by the first author and Laurinčikas in [4].

In this paper, we will prove a rather general result on the mixed joint functional independence for a class of zeta functions, consisting of the so-called Matsumoto zeta functions and periodic Hurwitz zeta functions.

Let $\mathfrak{B} = \{b_m : m \in \mathbb{N}_0\}$ be a periodic sequence of complex numbers (not all zero) with minimal period $k \in \mathbb{N}$, and let $0 < \alpha \le 1$. In 2006, Javtokas and Laurinčikas [3] introduced the periodic Hurwitz zeta function $\zeta(s, \alpha; \mathfrak{B})$. For $\sigma > 1$, it is given by the series

$$\zeta(s, \alpha; \mathfrak{B}) = \sum_{m=0}^{\infty} \frac{b_m}{(m+\alpha)^s}.$$

It is known that

$$\zeta(s, \alpha; \mathfrak{B}) = \frac{1}{k^s} \sum_{l=0}^{k-1} b_l \zeta\Big(s, \frac{l+\alpha}{k}\Big), \quad \sigma > 1.$$

Therefore, the function $\zeta(s, \alpha; \mathfrak{B})$ can be analytically continued to the whole complex plane except for a possible simple pole at the point $s = 1$ with residue

$$b := \frac{1}{k} \sum_{l=0}^{k-1} b_l.$$

If $b = 0$, the corresponding periodic Hurwitz zeta function is an entire function.

The functional independence of periodic Hurwitz zeta functions was proved by Laurinčikas in [7, Theorem 1].

Now we recall the definition of the polynomial Euler products $\widetilde{\varphi}(s)$ or so-called Matsumoto zeta functions.

For $m \in \mathbb{N}$, let $g(m) \in \mathbb{N}$ and, for $j \in \mathbb{N}, 1 \le j \le g(m)$, let $f(j, m) \in \mathbb{N}$. Denote by $p_m$ the $m$th prime number and let $a_m^{(j)} \in \mathbb{C}$. The zeta function $\widetilde{\varphi}$ was introduced by the second author in [11] and it is defined by the polynomial Euler product

$$\widetilde{\varphi}(s) = \prod_{m=1}^{\infty} \prod_{j=1}^{g(m)} \Big(1 - a_m^{(j)} p_m^{-sf(j,m)}\Big)^{-1}. \tag{2.1}$$

Suppose that

$$g(m) \le C_1 p_m^{\alpha} \quad \text{and} \quad |a_m^{(j)}| \le p_m^{\beta} \tag{2.2}$$

with a positive constant $C_1$ and nonnegative constants $\alpha$ and $\beta$. In view of (2.2), the function $\widetilde{\varphi}(s)$ converges absolutely for $\sigma > \alpha + \beta + 1$ (see the Appendix) and hence in this region it can be expressed as the Dirichlet series

$$\widetilde{\varphi}(s) = \sum_{k=1}^{\infty} \frac{\widetilde{c}_k}{k^s} \tag{2.3}$$

with coefficients $\widetilde{c}_k$. For brevity, denote the shifted version of the function $\widetilde{\varphi}(s)$ by

$$\varphi(s) = \widetilde{\varphi}(s+\alpha+\beta) = \sum_{k=1}^{\infty} \frac{\widetilde{c}_k}{k^{s+\alpha+\beta}} = \sum_{k=1}^{\infty} \frac{c_k}{k^s} \tag{2.4}$$

with $c_k = k^{-\alpha-\beta}\widetilde{c}_k$. Then, for $\sigma > 1$, the series on the right-hand side of (2.4) converges absolutely.

The aim of this paper is to obtain a mixed joint functional independence of the collection of zeta functions consisting of the Matsumoto zeta function $\varphi(s)$ belonging to the Steuding subclass $\widetilde{S}$, defined below, and periodic Hurwitz zeta functions $\zeta(s,\alpha;\mathfrak{B})$.

We say that the function $\varphi(s)$ belongs to the class $\widetilde{S}$ if the following conditions are satisfied:

(i) there exists a Dirichlet series expansion

$$\varphi(s) = \sum_{m=1}^{\infty} \frac{a(m)}{m^s}$$

with $a(m) = O(m^{\varepsilon})$ for every $\varepsilon > 0$;

(ii) there exists $\sigma_\varphi < 1$ such that $\varphi(s)$ can be meromorphically continued to the half-plane $\sigma > \sigma_\varphi$;

(iii) there exists a constant $c \geq 0$ such that

$$\varphi(\sigma+it) = O(|t|^{c+\varepsilon})$$

for every fixed $\sigma > \sigma_\varphi$ and $\varepsilon > 0$;

(iv) there exists an Euler product expansion over prime numbers, that is,

$$\varphi(s) = \prod_{p\in\mathbb{P}} \prod_{j=1}^{l} \Big(1 - \frac{a_j(p)}{p^s}\Big)^{-1};$$

(v) there exists a constant $\kappa > 0$ such that

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \sum_{p\leq x} |a(p)|^2 = \kappa,$$

where $\pi(x)$ denotes the number of primes $p$, $p \leq x$.

This class was introduced by Steuding in [15], and is a subclass of the class of Matsumoto zeta functions. For $\varphi \in \widetilde{S}$, let $\sigma^*$ be the infimum of all $\sigma_1$ for which

$$\frac{1}{2T}\int_{-T}^{T} |\varphi(\sigma+it)|^2\,dt \sim \sum_{m=1}^{\infty} \frac{|a(m)|^2}{m^{2\sigma}}$$

holds for any $\sigma \geq \sigma_1$. Then it is known that $\frac{1}{2} \leq \sigma^* < 1$ (see [15, Theorem 2.4]).

We state the first main result of this paper in the following theorem.

THEOREM 2.3. *Suppose that $\alpha$ is a transcendental number and the function $\varphi(s)$ belongs to the class $\widetilde{S}$. For $N \in \mathbb{N}$, $n \in \mathbb{N}_0$, let the functions $F_j : \mathbb{C}^{2N} \to \mathbb{C}$ be continuous for $j = 0, 1, \ldots, n$. If*

$$\sum_{j=0}^{n} s^j \cdot F_j(\varphi(s), \varphi'(s), \ldots, \varphi^{(N-1)}(s), \zeta(s, \alpha; \mathfrak{B}), \zeta'(s, \alpha; \mathfrak{B}), \ldots, \zeta^{(N-1)}(s, \alpha; \mathfrak{B}))$$

*is identically zero, then $F_j \equiv 0$ for $j = 0, 1, \ldots, n$.*

## 3. Proof of Theorem 2.3

For the proof of the mixed joint functional independence of the functions $\varphi(s)$ and $\zeta(s, \alpha; \mathfrak{B})$, we need two further propositions: the mixed joint universality theorem and the so-called denseness lemma.

**3.1. Mixed joint universality of the functions $\varphi(s)$ and $\zeta(s, \alpha; \mathfrak{B})$.** The proof of Theorem 2.3 is based on the mixed joint universality theorem in the Voronin sense for the functions $\varphi(s)$ and $\zeta(s, \alpha; \mathfrak{B})$. It was obtained by the authors in [5, Theorem 2.2]. We will give the statement of this universality theorem as a lemma. Let $\mathbb{R}$ be the set of real numbers and $D(a, b) = \{s \in \mathbb{C} : a < \sigma < b\}$ for any $a < b$. For any compact subset $K \subset \mathbb{C}$, denote by $H^c(K)$ the set of all $\mathbb{C}$-valued continuous functions defined on $K$, holomorphic in the interior of $K$. By $H_0^c(K)$ we mean the subset of $H^c(K)$ consisting of all elements which are nonvanishing on $K$.

LEMMA 3.1 [5]. *Suppose that $\varphi \in \widetilde{S}$ and $\alpha$ is a transcendental number. Let $K_1$ be a compact subset of $D(\sigma^*, 1)$ and $K_2$ be a compact subset of $D(\frac{1}{2}, 1)$, both with connected complements. Suppose that $f_1 \in H_0^c(K_1)$ and $f_2 \in H^c(K_2)$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\mu\Big\{\tau \in [0, T] : \sup_{s\in K_1} |\varphi(s + i\tau) - f_1(s)| < \varepsilon,$$
$$\sup_{s\in K_2} |\zeta(s + i\tau, \alpha; \mathfrak{B}) - f_2(s)| < \varepsilon\Big\} > 0,$$

*where $\mu\{A\}$ denotes the Lebesgue measure of the measurable set $A \subset \mathbb{R}$.*

Note that for the proof of Lemma 3.1 we use the joint mixed limit theorem in the sense of weakly convergent probability measures for the Matsumoto zeta functions $\varphi(s)$ and the periodic Hurwitz zeta function $\zeta(s, \alpha; \mathfrak{B})$. It was proved by the authors in [5, Theorem 2.1].

**3.2. A denseness lemma.** For the proof of Theorem 2.3, we also need a denseness lemma.

Define the map $u : \mathbb{R} \to \mathbb{C}^{2N}$ by the formula

$$u(t) = (\varphi(\sigma + it), \varphi'(\sigma + it), \ldots, \varphi^{(N-1)}(\sigma + it),$$
$$\zeta(\sigma + it, \alpha; \mathfrak{B}), \zeta'(\sigma + it, \alpha; \mathfrak{B}), \ldots, \zeta^{(N-1)}(\sigma + it, \alpha; \mathfrak{B}))$$

with $\sigma^* < \sigma < 1$.

Lemma 3.2. *Suppose that $\alpha$ is transcendental. Then the image of $\mathbb{R}$ by $u$ is dense in $\mathbb{C}^{2N}$.*

Proof. We will give a sketch, since the proof follows in the same way as Lemma 13 from [4] (see also [12, Theorem 3]).

We can find a sequence $\{\tau_m : \tau_m \in \mathbb{R}\}$, $\lim_{m\to\infty} \tau_m = \infty$, such that the inequalities

$$|\varphi^{(j)}(\sigma + i\tau_m) - s_{1j}| < \frac{\varepsilon}{2N}$$

and

$$|\zeta^{(j)}(\sigma + i\tau_m, \alpha; \mathfrak{B}) - s_{2j}| < \frac{\varepsilon}{2N}$$

hold for every $\varepsilon > 0$ and arbitrary complex numbers $s_{lj}$, $l = 1, 2$, $j = 0, \ldots, N-1$. To show this, we consider the polynomial

$$p_{lN}(s) = \frac{s_{l,N-1} \cdot s^{N-1}}{(N-1)!} + \frac{s_{l,N-2} \cdot s^{N-2}}{(N-2)!} + \cdots + \frac{s_{l0}}{0!}, \quad l = 1, 2.$$

Then, for $j = 0, \ldots, N-1$ and $l = 1, 2$,

$$p_{lN}^{(j)}(0) = s_{lj}.$$

Now, in view of Lemma 3.1 and repeating analogous arguments as in the proof of [4, Lemma 13], we can prove the existence of the above sequence $\{\tau_m\}$ and obtain that the image of $\mathbb{R}$ by the map $u$ is dense in $\mathbb{C}^{2N}$. □

**3.3. Proof of Theorem 2.3.** Now we are ready to complete the proof of Theorem 2.3. The essential idea is due to Voronin (see, for example, [17]). We first prove that $F_n \equiv 0$.

Suppose that $F_n \not\equiv 0$. It follows that there exists a point

$$\underline{a} = (s_{10}, s_{11}, \ldots, s_{1,N-1}, s_{20}, s_{21}, \ldots, s_{2,N-1}) \in \mathbb{C}^{2N}$$

such that $F_n(\underline{a}) \neq 0$. From the continuity of the function $F_n$, we find a bounded domain $G \subset \mathbb{C}^{2N}$ such that $\underline{a} \in G$ and, for all $\underline{s} \in G$,

$$|F_n(\underline{s})| \geq c > 0. \tag{3.1}$$

By Lemma 3.2, there exists a sequence $\{\tau_m : \tau_m \in \mathbb{R}\}$, $\lim_{m\to\infty} \tau_m = \infty$, such that

$$\begin{aligned}&(\varphi(\sigma + it), \varphi'(\sigma + it), \ldots, \varphi^{(N-1)}(\sigma + it),\\ &\quad \zeta(\sigma + it, \alpha; \mathfrak{B}), \zeta'(\sigma + it, \alpha; \mathfrak{B}), \ldots, \zeta^{(N-1)}(\sigma + it, \alpha; \mathfrak{B})) \in G.\end{aligned}$$

But this together with (3.1) contradicts the hypothesis of the theorem if $\tau_m$ is sufficiently large. Hence, $F_n \equiv 0$.

Similarly, we can show that $F_{n-1} \equiv 0, \ldots, F_0 \equiv 0$, inductively. The proof is complete.

## 4. A generalisation

The mixed joint universality and the mixed joint functional independence theorem can be obtained in the following more general situation.

Suppose that $\alpha_j$ is a real number with $0 < \alpha_j < 1$, $l(j) \in \mathbb{N}$, $j = 1, \ldots, r$ and $\lambda = l(1) + \cdots + l(r)$. For each $j$ and $l$, $1 \le j \le r$, $1 \le l \le l(j)$, let $\mathfrak{B}_{jl} = \{b_{mjl} \in \mathbb{C} : m \in \mathbb{N}_0\}$ be a periodic sequence of complex numbers (not all zero) with the minimal period $k_{jl}$, and let $\zeta(s, \alpha_j; \mathfrak{B}_{jl})$ be the corresponding periodic Hurwitz zeta function. Denote by $k_j$ the least common multiple of periods $k_{j1}, \ldots, k_{jl(j)}$. Let $B_j$ be the matrix consisting of coefficients $b_{mjl}$ from the periodic sequences $\mathfrak{B}_{jl}$, $l = 1, \ldots, l(j)$, $m = 1, \ldots, k_j$, that is,

$$B_j = \begin{pmatrix} b_{1j1} & b_{1j2} & \ldots & b_{1jl(j)} \\ b_{2j1} & b_{2j2} & \ldots & b_{2jl(j)} \\ \ldots & \ldots & \ldots & \ldots \\ b_{k_j j1} & b_{k_j j2} & \ldots & b_{k_j jl(j)} \end{pmatrix}.$$

The functional independence for the above collection of periodic Hurwitz zeta functions was proved by Laurinčikas in [7, Theorem 3] under a certain matrix condition. The proof is based on the joint universality theorem among periodic Hurwitz zeta functions proved by Steuding in [14].

For the proof of mixed joint functional independence, we may adopt the method developed in a series of works by Laurinčikas and his colleagues (see, for example, [1, 8, 13]). Then it is possible to obtain the following generalisation of Theorem 2.3.

Theorem 4.1. *Suppose $\alpha_1, \ldots, \alpha_r$ are algebraically independent over $\mathbb{Q}$, $\mathrm{rank} B_j = l(j)$, $1 \le j \le r$ and $\varphi(s)$ belongs to the class $\widetilde{S}$. Let the function $F_j : \mathbb{C}^{N(\lambda+1)} \to \mathbb{C}$ be a continuous function for each $j = 0, \ldots, n$. Suppose that the function*

$$\begin{aligned} \sum_{j=0}^{n} s^j \cdot F_j(&\varphi(s), \varphi'(s), \ldots, \varphi^{(N-1)}(s), \\ &\zeta(s, \alpha_1; \mathfrak{B}_{11}), \zeta'(s, \alpha_1; \mathfrak{B}_{11}), \ldots, \zeta^{(N-1)}(s, \alpha_1; \mathfrak{B}_{11}), \ldots, \\ &\zeta(s, \alpha_1; \mathfrak{B}_{1l(1)}), \zeta'(s, \alpha_1; \mathfrak{B}_{1l(1)}), \ldots, \zeta^{(N-1)}(s, \alpha_1; \mathfrak{B}_{1l(1)}), \ldots, \\ &\zeta(s, \alpha_r; \mathfrak{B}_{r1}), \zeta'(s, \alpha_r; \mathfrak{B}_{r1}), \ldots, \zeta^{(N-1)}(s, \alpha_r; \mathfrak{B}_{r1}), \ldots, \\ &\zeta(s, \alpha_r; \mathfrak{B}_{rl(r)}), \zeta'(s, \alpha_r; \mathfrak{B}_{rl(r)}), \ldots, \zeta^{(N-1)}(s, \alpha_r; \mathfrak{B}_{rl(r)})) \end{aligned}$$

*is identically zero. Then $F_j \equiv 0$ for $j = 0, \ldots, n$.*

This theorem is a consequence of the following mixed joint universality theorem, which is a generalisation of Lemma 3.1. This theorem is also an analogue of a result of Genys *et al.* [1, Theorem 3], which treats the case that $\varphi(s)$ is replaced by $\zeta(s)$.

Theorem 4.2. *Suppose $\alpha_1, \ldots, \alpha_r$ are algebraically independent over $\mathbb{Q}$, $\mathrm{rank} B_j = l(j)$, $1 \le j \le r$ and $\varphi(s)$ belongs to the class $\widetilde{S}$. Let $K_1$ be a compact subset of $D(\sigma^*, 1)$*

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*and $K_{2jl}$ be compact subsets of $D(\frac{1}{2},1)$, all with connected complements. Suppose that $f_1 \in H_0^c(K_1)$ and $f_{2jl} \in H^c(K_{2jl})$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T}\mu\Big\{\tau \in [0,T] : \sup_{s\in K_1} |\varphi(s+i\tau) - f_1(s)| < \varepsilon,$$
$$\max_{1\le j\le r}\max_{1\le l\le l(j)}\sup_{s\in K_{2jl}} |\zeta(s+i\tau,\alpha_j;\mathfrak{B}_{jl}) - f_{2jl}(s)| < \varepsilon\Big\} > 0.$$

Remark 4.3. Consider the case when all $l(j) = 1$, $1 \le j \le r$. Write $\mathfrak{B}_{j1} = \mathfrak{B}_j$. In this case, the condition $\mathrm{rank} B_j = l(j)$ trivially holds. Therefore, the joint universality and the functional independence among the functions $\varphi(s), \zeta(s,\alpha_1,\mathfrak{B}_1),\ldots,\zeta(s,\alpha_r,\mathfrak{B}_r)$ are valid without any matrix condition, only under the assumption that $\alpha_1,\ldots,\alpha_r$ are algebraically independent over $\mathbb{Q}$. When Laurinčikas started his study of the universality of periodic Hurwitz zeta functions, he assumed various matrix-type conditions, but, finally, the joint universality among periodic Hurwitz zeta functions without any matrix condition was established by Laurinčikas and Skerstonaitė in [9, Theorem 3]. Our Theorems 4.1 and 4.2 include the 'mixed' generalisation of this theorem.

## 5. Proof of Theorems 4.1 and 4.2

In the proof of Theorems 4.1 and 4.2, the crucial role is played by a mixed joint limit theorem in the sense of weakly convergent probability measures in the space of analytic functions.

**5.1. A generalised mixed joint limit theorem.** Let $D_1$ be an open subset of $D(\sigma^*,1)$ and $D_2$ be an open subset of $D(\frac{1}{2},1)$. For any set $S$, by $\mathcal{B}(S)$ we denote the set of all Borel subsets of $S$. For any region $D$, denote by $H(D)$ the set of all holomorphic functions on $D$. Let $\underline{H}$ be the Cartesian product of $\lambda+1$ such spaces, that is,

$$\underline{H} = H(D_1) \times \underbrace{H(D_2)\times\cdots\times H(D_2)}_{\lambda}.$$

Moreover, let

$$\Omega_1 = \prod_{p\in\mathbb{P}} \gamma_p \quad \text{and} \quad \Omega_2 = \prod_{m=0}^{\infty} \gamma_m,$$

where $\gamma_p = \gamma$ for all $p \in \mathbb{P}$, $\gamma_m = \gamma$ for all $m \in \mathbb{N}_0$ and $\gamma = \{s \in \mathbb{C} : |s| = 1\}$, and define

$$\underline{\Omega} = \Omega_1 \times \Omega_{21} \times \cdots \times \Omega_{2r},$$

where $\Omega_{2j} = \Omega_2$ for all $j = 1,\ldots,r$. Then, by the Tikhonov theorem, $\underline{\Omega}$ is a compact topological Abelian group also. Then we have the probability space $(\underline{\Omega},\mathcal{B}(\underline{\Omega}),\underline{m}_H)$. Here $\underline{m}_H$ is the product of Haar measures $m_{H1}, m_{H21},\ldots,m_{H2r}$, where $m_{H1}$ is the probability Haar measure on $(\Omega_1,\mathcal{B}(\Omega_1))$ and $m_{H2j}$ is the probability Haar measure

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on $(\Omega_{2j}, \mathcal{B}(\Omega_{2j}))$, $j = 1, \dots, r$. Let $\omega_1(p)$ be the projection of $\omega_1 \in \Omega_1$ to $\gamma_p$, and, for every $m \in \mathbb{N}$, define

$$\omega_1(m) = \prod_{p^\alpha \| m} \omega_1(p)^\alpha$$

with respect to the factorisation of $m$ to primes. Denote by $\omega_{2j}(m)$ the projection of $\omega_{2j} \in \Omega_{2j}$ to $\gamma_m$, $m \in \mathbb{N}_0$, $j = 1, \dots, r$.

For brevity, we write $\underline{\alpha} = (\alpha_1, \dots, \alpha_r)$, $\underline{\mathfrak{B}} = (\mathfrak{B}_{11}, \dots, \mathfrak{B}_{1l(1)}, \dots, \mathfrak{B}_{r1}, \dots, \mathfrak{B}_{rl(r)})$, $\underline{s} = (s_1, s_{211}, \dots, s_{21l(1)}, \dots, s_{2r1}, \dots, s_{2rl(r)}) \in \mathbb{C}^{\lambda+1}$ and

$$\begin{aligned} \underline{Z}(\underline{s}, \underline{\alpha}; \underline{\mathfrak{B}}) = (\varphi(s_1), \zeta(s_{211}, \alpha_1; \mathfrak{B}_{11}), \dots, \zeta(s_{21l(1)}, \alpha_1; \mathfrak{B}_{1l(1)}), \\ \dots, \zeta(s_{2r1}, \alpha_r; \mathfrak{B}_{r1}), \dots, \zeta(s_{2rl(r)}, \alpha_r; \mathfrak{B}_{rl(r)})). \end{aligned}$$

Let $\underline{\omega} = (\omega_1, \omega_{21}, \dots, \omega_{2r}) \in \underline{\Omega}$. Define the $\underline{H}$-valued random element $\underline{Z}(\underline{s}, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{B}})$ on the probability space $(\underline{\Omega}, \mathcal{B}(\underline{\Omega}), \underline{m}_H)$ by the formula

$$\begin{aligned} \underline{Z}(\underline{s}, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{B}}) = (\varphi(s_1, \omega_1), \zeta(s_{211}, \alpha_1, \omega_{21}; \mathfrak{B}_{11}), \dots, \zeta(s_{21l(1)}, \alpha_1, \omega_{21}; \mathfrak{B}_{1l(1)}), \\ \dots, \zeta(s_{2r1}, \alpha_r, \omega_{2r}; \mathfrak{B}_{r1}), \dots, \zeta(s_{2rl(r)}, \alpha_r, \omega_{2r}; \mathfrak{B}_{rl(r)})), \end{aligned}$$

where

$$\varphi(s, \omega_1) = \sum_{m=1}^{\infty} \frac{c_m \omega_1(m)}{m^s}, \quad s \in D(\sigma^*, 1),$$

and

$$\zeta(s, \alpha_j, \omega_{2j}; \mathfrak{B}_{jl}) = \sum_{m=0}^{\infty} \frac{b_{mjl} \omega_{2j}(m)}{(m + \alpha_j)^s}, \quad s \in D\Big(\frac{1}{2}, 1\Big), j = 1, \dots, r, l = 1, \dots, l(r),$$

respectively. These series are convergent for almost all $\omega_1 \in \Omega_1$ and $\omega_{2j} \in \Omega_{2j}$, $j = 1, \dots, r$. Denote by $P_{\underline{Z}}$ the distribution of the random element $\underline{Z}(\underline{s}, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{B}})$, that is,

$$P_{\underline{Z}}(A) = \underline{m}_H(\underline{\omega} \in \underline{\Omega} : \underline{Z}(\underline{s}, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{B}}) \in A), \quad A \in \mathcal{B}(\underline{H}).$$

Now we are ready to state our mixed joint limit theorem for the functions $\varphi(s)$, $\zeta(s, \alpha_1; \mathfrak{B}_{11}), \dots, \zeta(s, \alpha_r; \mathfrak{B}_{rl(r)})$ as a lemma.

Lemma 5.1. *Suppose that the numbers $\alpha_1, \dots, \alpha_r$ are algebraically independent over $\mathbb{Q}$, and $\varphi \in \widetilde{S}$. Then the probability measure $P_T$ defined by*

$$P_T(A) = \frac{1}{T} \mathrm{meas}\{\tau \in [0, T] : \underline{Z}(\underline{s} + i\tau, \underline{\alpha}; \underline{\mathfrak{B}}) \in A\}, \quad A \in \mathcal{B}(\underline{H}), \tag{5.1}$$

*converges weakly to $P_{\underline{Z}}$ as $T \to \infty$.*

When $r = 1$ and $l(1) = 1$, this lemma is [5, Theorem 2.1]. On the other hand, the same type of limit theorem with $\varphi$ replaced by the Riemann zeta function is given in [1, Theorem 4]. The proof of the above lemma is quite similar to that of those results, so here we give a very brief outline.

Let $\varphi_n(s)$, $\varphi_n(s,\widehat{\omega}_1)$, $\zeta_n(s,\alpha;\mathfrak{B})$, $\zeta_n(s,\alpha,\widehat{\omega}_2;\mathfrak{B})$ be the same as in [5, Section 3] and define

$$\underline{Z}_n(\underline{s},\underline{\alpha};\underline{\mathfrak{B}})=(\varphi_n(s_1),\zeta_n(s_{211},\alpha_1;\mathfrak{B}_{11}),\ldots,\zeta_n(s_{21l(1)},\alpha_1;\mathfrak{B}_{1l(1)}),\\ \ldots,\zeta_n(s_{2r1},\alpha_r;\mathfrak{B}_{r1}),\ldots,\zeta_n(s_{2rl(r)},\alpha_r;\mathfrak{B}_{rl(r)}))$$

and

$$\underline{Z}_n(\underline{s},\underline{\alpha},\underline{\widehat{\omega}};\underline{\mathfrak{B}})=(\varphi_n(s_1,\widehat{\omega}_1),\zeta_n(s_{211},\alpha_1,\widehat{\omega}_{21};\mathfrak{B}_{11}),\ldots,\zeta_n(s_{21l(1)},\alpha_1,\widehat{\omega}_{21};\mathfrak{B}_{1l(1)}),\\ \ldots,\zeta_n(s_{2r1},\alpha_r,\widehat{\omega}_{2r};\mathfrak{B}_{r1}),\ldots,\zeta_n(s_{2rl(r)},\alpha_r,\widehat{\omega}_{2r};\mathfrak{B}_{rl(r)})),$$

where $\underline{\widehat{\omega}}=(\widehat{\omega}_1,\widehat{\omega}_{21},\ldots,\widehat{\omega}_{2r})\in\underline{\Omega}$. The probability measures $P_{T,n}$, $\widehat{P}_{T,n}$ and $\widehat{P}_T$ on $\underline{H}$ are defined similarly to $P_T$ in (5.1), replacing $\underline{Z}(\underline{s}+i\tau,\underline{\alpha};\underline{\mathfrak{B}})$ respectively by $\underline{Z}_n(\underline{s}+i\tau,\underline{\alpha};\underline{\mathfrak{B}})$, $\underline{Z}_n(\underline{s}+i\tau,\underline{\alpha},\underline{\widehat{\omega}};\underline{\mathfrak{B}})$ and $\underline{Z}(\underline{s}+i\tau,\underline{\alpha},\underline{\widehat{\omega}};\underline{\mathfrak{B}})$. As an analogue of [5, Lemma 3.2] or [1, Lemma 2], we can show that both measures $P_{T,n}$ and $\widehat{P}_{T,n}$ converge weakly to a certain probability measure $P_n$ as $T\to\infty$. (In the course of the proof, a key is [1, Lemma 1], which is based on the assumption that $\alpha_1,\ldots,\alpha_r$ are algebraically independent over $\mathbb{Q}$.)

Next we need an approximation lemma in the mean value sense, similar to [5, Lemma 3.3]. In the proof of [5, Lemma 3.3], we used the result of [3]. The desired approximation can be shown by using, instead of [3], mean value results given in [10, (2.3) and (2.5)].

Then we can prove that both $P_T$ and $\widehat{P}_T$ converge weakly to a certain probability measure $P$. This is an analogue of [5, Lemma 3.4] or [1, Lemma 5]. Finally, we can show that $P=P_{\underline{Z}}$, by the usual ergodic argument. This completes the proof of Lemma 5.1.

**5.2. Completion of the proof.** Now we complete the proof of Theorems 4.1 and 4.2. Hereafter we assume that $\alpha_1,\ldots,\alpha_r$ are algebraically independent over $\mathbb{Q}$, $\operatorname{rank} B_j=l(j)$, $1\le j\le r$ and $\varphi(s)$ belongs to the class $\widetilde{S}$.

Let $S_\varphi$ be the set of all $f\in H(D_1)$ which are nonvanishing on $D_1$ or identically $\equiv 0$ on $D_1$.

LEMMA 5.2. *The support of the measure $P_{\underline{Z}}$ is $S_\varphi\times H(D_2)^\lambda$.*

This can be shown analogously to [1, Theorem 5]. Then Theorem 4.2 follows from Lemmas 5.1 and 5.2 by the standard argument; see again [1].

Next, as a generalisation of Lemma 3.2, we can show that the image of the map $u:\mathbb{R}\to\mathbb{C}^{(\lambda+1)N}$ by the formula

$$\begin{aligned}u(t)=(&\varphi(\sigma+it),\varphi'(\sigma+it),\ldots,\varphi^{(N-1)}(\sigma+it),\\ &\zeta(\sigma+it,\alpha_1;\mathfrak{B}_{11}),\zeta'(\sigma+it,\alpha_1;\mathfrak{B}_{11}),\ldots,\zeta^{(N-1)}(\sigma+it,\alpha_1;\mathfrak{B}_{11}),\ldots,\\ &\zeta(\sigma+it,\alpha_1;\mathfrak{B}_{1l(1)}),\zeta'(\sigma+it,\alpha_1;\mathfrak{B}_{1l(1)}),\ldots,\zeta^{(N-1)}(\sigma+it,\alpha_1;\mathfrak{B}_{1l(1)}),\ldots,\\ &\zeta(\sigma+it,\alpha_r;\mathfrak{B}_{r1}),\zeta'(\sigma+it,\alpha_r;\mathfrak{B}_{r1}),\ldots,\zeta^{(N-1)}(\sigma+it,\alpha_r;\mathfrak{B}_{r1}),\ldots,\\ &\zeta(\sigma+it,\alpha_r;\mathfrak{B}_{rl(r)}),\zeta'(\sigma+it,\alpha_r;\mathfrak{B}_{rl(r)}),\ldots,\zeta^{(N-1)}(\sigma+it,\alpha_r;\mathfrak{B}_{rl(r)}))\end{aligned}$$

is dense in $\mathbb{C}^{(\lambda+1)N}$. Using this denseness result, similar to the proof of Theorem 2.3, we obtain the assertion of Theorem 4.1.

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## Appendix

Here we give a comment on [11]. On [11, page 179], it is mentioned that the Euler product (2.1) is convergent absolutely, under the condition (2.2), in the region $\sigma > \alpha + \beta + 1$. This can be seen by the estimate

$$\sum_{m=1}^{\infty}\sum_{j=1}^{g(m)} |a_m^{(j)}| p_m^{-\sigma f(j,m)} \le \sum_{m=1}^{\infty}\sum_{j=1}^{g(m)} p_m^{\beta-\sigma} \ll \sum_{m=1}^{\infty} p_m^{\alpha+\beta-\sigma}$$

and hence the Dirichlet series expansion (2.3) is also valid in that region.

On the same page of [11], the estimate $\widetilde{c}_k = O(k^{\alpha+\beta})$ is also stated. However, this is to be amended as follows. From (2.1),

$$\widetilde{\varphi}(s) = \prod_{m=1}^{\infty}\sum_{l=0}^{\infty}\Big(\sum\nolimits^{*} (a_m^{(1)})^{h_1}\cdots(a_m^{(g(m))})^{h_{g(m)}}\Big) p_m^{-ls},$$

where $\sum^*$ means the summation over all tuples $(h_1, \ldots, h_{g(m)})$ of nonnegative integers satisfying

$$h_1 f(1,m) + \cdots + h_{g(m)} f(g(m), m) = l.$$

Denote by $C(m,l)$ the number of such tuples. Using (2.2),

$$\sum\nolimits^{*} (a_m^{(1)})^{h_1}\cdots(a_m^{(g(m))})^{h_{g(m)}} \le \sum\nolimits^{*} p_m^{(h_1+\cdots+h_{g(m)})\beta} \le p_m^{l\beta} C(m,l).$$

To estimate $C(m,l)$, it suffices to consider the case when $f(1,m) = \cdots = f(g(m),m) = 1$ and in this case

$$C(m,l) = \binom{g(m)+l-1}{l} \le g(m)^l \le (C_1 p_m^{\alpha})^l.$$

Therefore, $\sum^* \le C_1^l p_m^{(\alpha+\beta)l}$, which yields, if $k = p_1^{l_1}\ldots p_r^{l_r}$,

$$\widetilde{c}_k \le C_1^{l_1+\cdots+l_r}(p_1^{l_1}\ldots p_r^{l_r})^{\alpha+\beta} = C_1^{\Omega(k)} k^{\alpha+\beta},$$

where $\Omega(k)$ denotes the total number of prime divisors of $k$.

For any $\varepsilon > 0$, we see that $C_1 p_m^{\alpha} \le p_m^{\alpha+\varepsilon}$ if $m$ is sufficiently large, and then we have $\sum^* \le (p_m^{\alpha+\beta+\varepsilon})^l$. This implies that $\widetilde{c}_k = O(k^{\alpha+\beta+\varepsilon})$ if all prime factors of $k$ are large.

## Note added in proof

Lemma 5.1 can be shown, more generally, for any Matsumoto zeta-function $\varphi$ (as in [5, Theorem 2.1]). The statement is valid as it is. The only point to change is that in the definition of $\underline{H}$ in Section 5.1, $D_1$ is to be chosen as a subset of $D_\varphi \cap \{\sigma < 1\}$, where $D_\varphi$ is defined in [5].

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