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Numbers with Almost all Convergents in a Cantor Set

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Abstract. In 1984, K. Mahler asked how well elements in the Cantor middle third set can be approximated by rational numbers from that set and by rational numbers outside of that set. We consider more general missing digit sets C and construct numbers in C that are arbitrarily well approximable by rationals in C, but badly approximable by rationals outside of C. More precisely, we construct them so that all but finitely many of their convergents lie in C.

1 Introduction and Statement of the Main Result

Let $b \geq 3$ be an integer and let D be a proper subset of $\{0, 1, \ldots, b-1\}$ with at least two elements. We consider the Cantor set C, which consists of all real numbers in the interval $[0, 1]$ that admit a base b-expansion $\xi = (0.a_1a_2\cdots)_b = \sum_{k=1}^{\infty} a_k b^{-k}$ with digits a_k in D. This is a compact subset of $\mathbb{R}$ of measure zero. It is called the *middle third Cantor set* when $b = 3$ and $D = \{0, 2\}$. In 1984, K. Mahler [3] proposed a problem about this set, which also applies to any Cantor set. He asked how well irrational elements of C can be approximated by rational numbers from C, and how well they can be approximated by rational numbers outside of C. A construction of Y. Bugeaud [1] (see also [6, §2.2]) generalizing earlier work of J. Shallit and A. J. van der Poorten [4, 9] provides an interesting answer. For any monotone decreasing function $\psi: \mathbb{N} \to (0, \infty)$ on the set $\mathbb{N}$ of positive integers satisfying $\lim_{q\to\infty} q^2\psi(q) = 0$, it yields an irrational element ξ of C and a constant $c = c(b) > 0$ such that

$$|\xi - p/q| \leq \psi(q) \qquad \text{for infinitely many } p/q \in \mathbb{Q} \cap C,$$
$$|\xi - p/q| \geq c\psi(q) \qquad \text{for all } p/q \in \mathbb{Q},$$

with p/q in reduced form. However, because the construction is based on the folding lemma, such a number ξ possesses many good rational approximations p/q besides those for which $|\xi - p/q| \leq \psi(q)$. As we do not know whether they belong to C or not, we lack information about approximation to ξ by rational numbers outside of C. Our main result below is more precise in this aspect and, at the same time, answers a question of L. Fishman and D. Simmons in [2, §2.1] by providing irrational elements of C with all but finitely many convergents inside C (see [7, Chapter I] for the notion of convergents of a real number and the theory of continued fractions).

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Theorem 1.1 *Let C be as above. Then there is a constant c_1, depending only on b and D, with the following property. For any $\epsilon > 0$ and any function $\psi: \mathbb{N} \to (0,1]$, there exists $\xi \in C$ whose convergents $p/q \in \mathbb{Q}$ (in reduced form) with denominator $q \geq c_1$ all lie in C and satisfy*

$$\min\{\psi(q), q^{-q}\} > \left|\xi - \frac{p}{q}\right| > c_2 q^{-(1+\epsilon)q}\psi(q)$$

for a constant $c_2 = c_2(b,\epsilon) > 0$. If D contains $\{0,1\}$, we can take $c_1 = 1$, meaning that all convergents of ξ, starting with $0/1$, belong to C.

In particular, the numbers ξ of C that we construct are Liouville numbers that are ψ-approximable by rational numbers inside C and badly approximable by rational numbers outside of C. Indeed, if a fraction $p/q \in \mathbb{Q} \setminus C$, in reduced form, has denominator $q \geq c_1$, then p/q is not a convergent of ξ and so $|\xi - p/q| \geq 1/(2q^2)$. On the other hand, a result of L. Fishman and D. Simmons [2, Corollary 1.2] shows the existence of a constant $c_3 = c_3(b) > 0$ such that the inequality $|\xi - p/q| \leq c_3/q^2$ has infinitely many solutions $p/q \in \mathbb{Q} \setminus C$. Thus, the approximation to ξ by rational numbers outside of C is under control as well. As the proof will show, we even obtain explicit base b expansions for the convergents of ξ with large enough denominators.

Note that, for a general Cantor set C, there may exist no element of C with all its convergents in C. For example, if b is a large Fibonacci number and if $D = \{d, d+1\}$ where d is the preceding Fibonacci number, then $C \subseteq [d/(b-1), (d+1)/(b-1)]$ and all elements of C have the same initial convergents $0/1, 1/1, 1/2, 2/3, 3/5, \ldots$, none of which belong to C.

The original motivation for this paper was to determine whether or not Schmidt and Summerer's parametric geometry of numbers [5, 8] extends without qualitative change when restricting to points of the form $(1, \xi_1, \ldots, \xi_n)$ with $\xi_1, \ldots, \xi_n \in C$ instead of the full set of points with $\xi_1, \ldots, \xi_n \in \mathbb{R}$. For $n = 1$, the question amounts to determining whether or not for any irrational $\xi \in \mathbb{R}$ there exists $\xi' \in C$ and a constant $c > 1$ such that, for any convergent p/q of ξ (resp. p'/q' of ξ'), there exists a convergent p'/q' of ξ' (resp. p/q of ξ) with $q \leq cq'$ and $q' \leq cq$. We do not know the answer, but we observed that if the denominators of the convergents of ξ grow very fast, then ξ' must have essentially all its convergents in C, and the search for such numbers ξ' led us to the construction that we describe below.

2 Proof of the Theorem

We will assume, without loss of generality, that D consists of only two digits d_1, d_2 with $0 \leq d_1 < d_2 \leq b-1$. Let D^* denote the monoid of finite words on the alphabet D with the product given by concatenation, and let $|w|$ denote the length of a word $w \in D^*$. Then each rational number in C has an ultimately periodic base b expansion of the form

$$(0.v\,\overline{w})_b = \frac{(vw)_b - (v)_b}{b^m(b^N - 1)} \quad \text{with} \quad m = |v| \quad \text{and} \quad N = |w| > 0,$$

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where $v \in D^*$ is a possibly empty pre-period, and $w \in D^*$ is a non-empty period. The numerator in the right hand-side of the formula is the difference of two integers $(vw)_b$ and $(v)_b$, written in base b.

For each non-empty word $w \in D^*$, let w' be the word obtained from w by replacing its last letter or digit by the other element of the set D, so that w and w' differ only in their last digits. Our construction depends uniquely on the choice of a strictly increasing sequence of non-negative integers $(m_i)_{i\geq 1}$. We define a word v and a sequence of words $(w_i)_{i\geq 1}$ in D^* by

$$v = d_2^{m_1}, \quad w_1 = \begin{cases} d_1 d_2^{m_1} & \text{if } m_1 > 0, \\ d_2 & \text{if } m_1 = 0, \end{cases} \quad w_{i+1} = (w_i)^{m_{i+1}} w_i' \quad \text{for } i \geq 1.$$

Then the sequence of rational numbers, in reduced form,

$$\frac{p_i}{q_i} = (0.v\,\overline{w_i})_b \quad (i \geq 1)$$

is contained in C and converges to an element ξ of C. We claim that, for an appropriate choice of $(m_i)_{i\geq 1}$, they are consecutive convergents of ξ. The simplest case is when $D = \{0, 1\}$. As we will see, we can then choose $m_1 = 0$ so that v is the empty word and all fractions p_i/q_i have purely periodic base b expansion. The reader who wants to concentrate on this case can skip the technical Lemma 2.3. For the proof, define

$$u = d_2 - d_1 \quad \text{and} \quad N_i = |w_i| \quad \text{for each } i \geq 1,$$

so that $N_1 = m_1 + 1$ and $N_{i+1} = (m_{i+1} + 1)N_i$ for each $i \geq 1$. We start with a simple computation.

Lemma 2.1 *For each $i \geq 1$, we have*

$$\frac{p_{i+1}}{q_{i+1}} = \frac{p_i}{q_i} + \frac{(-1)^i u}{b^{m_1}(b^{N_{i+1}} - 1)}. \tag{2.1}$$

Proof Since w_i ends in d_2 for odd indices i and in d_1 for even ones, we find

$$(0.v\,\overline{w_{i+1}})_b = (0.v\,\overline{w_i})_b + (-1)^i u (0.0^{m_1}\overline{\epsilon})_b,$$

where ϵ consists of $N_{i+1} - 1$ zeros followed by a one. The result follows. ■

Lemma 2.2 *Suppose that the sequence $(q_i)_{i\geq 1}$ is strictly increasing. Then $(p_i/q_i)_{i\geq 1}$ consists of all convergents to ξ with denominator at least q_1 if and only if, for each $i \geq 1$, we have $b^{m_1}(b^{N_{i+1}} - 1) = u q_i q_{i+1}$.*

Proof The formula (2.1) from Lemma 2.1 can be rewritten as

$$\det\begin{pmatrix} p_{i+1} & q_{i+1} \\ p_i & q_i \end{pmatrix} = (-1)^i \frac{u q_i q_{i+1}}{b^{m_1}(b^{N_{i+1}} - 1)} \quad (i \geq 1).$$

If p_i/q_i and p_{i+1}/q_{i+1} are consecutive convergents of ξ, then the above determinant is ± 1 and so $u q_i q_{i+1} = b^{m_1}(b^{N_{i+1}} - 1)$. Conversely, suppose that the latter equality holds for each $i \geq 1$. Then we have

$$\det\begin{pmatrix} p_{i+1} & q_{i+1} \\ p_i & q_i \end{pmatrix} = (-1)^i \quad (i \geq 1),$$

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and, since $(q_i)_{i\geq 1}$ is strictly increasing, we conclude that the sequence $(p_i/q_i)_{i\geq 1}$ consists of all convergents of its limit ξ, with denominator at least q_1. We leave the verification of this fact as an interesting exercise about continued fractions (we do not have a precise reference to propose). ■

The choice of m_1 is the most delicate part of the argument. It depends on the factorisation of $u = d_2 - d_1$ in the form

$$u = u_1 u_2,$$

where u_1, u_2 are positive integers with the prime factors of u_1 dividing b and those of u_2 not dividing b. Note that $u_1, u_2 \leq u \leq b-1$. In the statement below, φ denotes Euler's totient function.

Lemma 2.3 *Suppose that $m_1 = N-1$ where $N = \varphi(u_2^2(b-1)^2)$. Then we have $m_1 \geq 1$, $u_1 | b^{m_1}$, and the reduced fraction $p_1/q_1 = (0.v\,\overline{w_1})_b$ satisfies $u_2 q_1 = b^N - 1$.*

Proof Since $N \geq u_2(b-1) \geq b-1$, we have $m_1 \geq b-2 \geq u_1 - 1 \geq v_p(u_1)$ for any prime divisor p of u_1, where v_p denotes the valuation at p. Since any such prime p divides b, it follows that $u_1 | b^{m_1}$. Since $b \geq 3$, the inequality $m_1 \geq b-2$ also yields $m_1 \geq 1$.

For the last assertion, set $S = 1 + b + \cdots + b^{N-1}$ so that $b^N - 1 = (b-1)S$. Since $m_1 \geq 1$, we find

$$(0.v\,\overline{w_1})_b = (0.d_2^{m_1}\overline{d_1 d_2^{m_1}})_b = (0.\overline{d_2^{m_1} d_1})_b = \frac{(d_2^{m_1} d_1)_b}{b^N - 1} = \frac{d_2 S + u}{b^N - 1}.$$

Thus, we simply need to show that $\gcd(d_2 S + u, b^N - 1) = u_2$ or, equivalently, that

$$\min\left\{v_p(d_2 S + u), v_p(b^N - 1)\right\} = v_p(u_2)$$

for every prime factor p of $b^N - 1$. Fix such a prime number p.

Since b is coprime to $u_2(b-1)$, we have $b^N \equiv 1 \mod u_2^2(b-1)^2$ by the choice of N. Thus, $u_2^2(b-1)$ divides S. We also note that $p \nmid b$, and thus $p \nmid u_1$. If p divides $u_2(b-1)$, this implies that

$$v_p(S) \geq v_p\big(u_2^2(b-1)\big) > v_p(u_2) = v_p(u),$$

so $v_p(d_2 S + u) = v_p(u_2) < v_p(S) \leq v_p(b^N - 1)$, and we are done. Otherwise, p divides S but not u, so it does not divide $d_2 S + u$, and we are done again. ■

For the last lemma, recall the factorisation $u = u_1 u_2$, introduced just before Lemma 2.3 and the definition of the integers $N_i = |w_i|$, given just before Lemma 2.1.

Lemma 2.4 *Suppose that*

(i) $u_1 | b^{m_1}$ *and* $u_2 q_1 = b^{m_1+1} - 1$;
(ii) $q_0 p_1 (m_2 + 1) \equiv -1 \mod q_1$ *where* $q_0 = b^{m_1}/u_1$;
(iii) $q_i | m_{i+1}$ *for each* $i \geq 2$.

Then, for each $i \geq 1$, we have

$$b^{m_1}(b^{N_i} - 1) = u q_{i-1} q_i. \tag{2.2}$$

Proof We proceed by induction on i. If $i = 1$, we find $uq_0q_1 = b^{m_1}u_2q_1 = b^{m_1}(b^{N_1} - 1)$, since $N_1 = m_1 + 1$. Suppose now that the equality (2.2) holds for some integer $i \geq 1$. Since $N_{i+1} = (m_{i+1} + 1)N_i$, we have

$$b^{N_{i+1}} - 1 = (b^{N_i} - 1)S_{i+1}, \quad \text{where} \quad S_{i+1} = 1 + b^{N_i} + \cdots + b^{m_{i+1}N_i},$$

and so Lemma 2.1 yields

$$\frac{p_{i+1}}{q_{i+1}} = \frac{p_i}{q_i} + \frac{(-1)^i u}{uq_{i-1}q_iS_{i+1}} = \frac{R_{i+1}}{q_{i-1}q_iS_{i+1}}, \quad \text{where} \quad R_{i+1} = q_{i-1}p_iS_{i+1} + (-1)^i.$$

To complete the induction step, we simply need to show that q_i divides R_{i+1}, because, since $q_{i-1}S_{i+1}$ is coprime to R_{i+1}, this implies that $q_{i+1} = q_{i-1}S_{i+1}$, and so

$$b^{m_1}(b^{N_{i+1}} - 1) = b^{m_1}(b^{N_i} - 1)S_{i+1} = uq_{i-1}q_iS_{i+1} = uq_iq_{i+1}.$$

When $i = 1$, we use the fact that $b^{N_1} \equiv 1 \mod q_1$ by condition (i). This implies that $S_2 \equiv m_2 + 1 \mod q_1$, and thus, using condition (ii), we obtain

$$R_2 \equiv q_0p_1(m_2 + 1) + 1 \equiv 0 \mod q_1,$$

as needed. Now suppose that $i > 1$. Then (2.2) has the following two consequences. On the one hand, in combination with Lemma 2.1, it yields

$$p_iq_{i-1} - q_ip_{i-1} = q_{i-1}q_i\Big(\frac{p_i}{q_i} - \frac{p_{i-1}}{q_{i-1}}\Big) = (-1)^{i+1},$$

so $p_iq_{i-1} \equiv (-1)^{i+1} \mod q_i$, and thus $R_{i+1} \equiv (-1)^{i+1}(S_{i+1} - 1) \mod q_i$. On the other hand, it shows that q_i divides $b^{m_1}(b^{N_i} - 1)$, so q_i divides $b^{m_1}q_i^*$ where $q_i^* = \gcd(q_i, b^{N_i} - 1)$. Modulo q_i^*, we have $b^{N_i} \equiv 1$ and $m_{i+1} \equiv 0$ by condition (iii), thus $S_{i+1} \equiv m_{i+1} + 1 \equiv 1$. Since $u_1 | b^{m_1}$ and $N_i > N_1 > m_1$, we also have $S_{i+1} \equiv 1 \mod b^{m_1}$. As b^{m_1} and q_i^* are coprime, this implies that $b^{m_1}q_i^*$ divides $S_{i+1} - 1$ and so $R_{i+1} \equiv (-1)^{i+1}(S_{i+1} - 1) \equiv 0 \mod q_i$. ∎

Proof of Theorem 1.1 Fix a choice of $\epsilon > 0$ and of a function $\psi: \mathbb{N} \to (0, 1]$. If $d_2 \neq 1$, we take m_1 as in Lemma 2.3 so that Lemma 2.4(i) holds. Otherwise, we have $d_1 = 0$ and $u = d_2 = 1$, and we set $m_1 = 0$. This yields $p_1/q_1 = (0.\bar{1})_b = 1/(b - 1)$, and so Lemma 2.4(i) still holds. Moreover, in both cases, the product $q_0p_1 = p_1b^{m_1}/u_1$ is coprime to q_1. Thus, the integers m_2 satisfying Lemma 2.4(ii) form a congruence class modulo q_1. We choose m_2 to be the smallest positive element of that class with $m_2 \geq q_1$ for which the corresponding fraction $p_2/q_2 = (0.v\overline{w_2})_b$ satisfies $1/(q_1q_2) < \psi(q_1)$. More generally, once m_i and p_i/q_i are constructed for some index $i \geq 2$, we choose m_{i+1} to be the smallest positive multiple of q_i such that $p_{i+1}/q_{i+1} = (0.v\overline{w_{i+1}})_b$ satisfies

$$\frac{1}{q_iq_{i+1}} < \psi(q_i). \tag{2.3}$$

This is possible at each step $i \geq 1$, because $N_{i+1} = |w_{i+1}| = (m_{i+1}+1)N_i$ tends to infinity with m_{i+1}, and so, according to Formula (2.2) in Lemma 2.4, the ratio

$$\frac{1}{q_iq_{i+1}} = \frac{u}{b^{m_1}(b^{N_{i+1}} - 1)} \tag{2.4}$$

tends to 0.

We claim that, upon putting $N_0 = 1$, we have

$$b^{m_i N_{i-1}} \le q_i < b^{(m_i+1)N_{i-1}} = b^{N_i} \tag{2.5}$$

for each $i \ge 1$. For $i = 1$, this follows from

$$b^{m_1} \le \frac{(b-1)b^{m_1}}{u_2} \le \frac{b^{m_1+1}-1}{u_2} = q_1 < b^{m_1+1}.$$

If $i > 1$ and if we assume that (2.5) holds for all smaller values of i, then we have $q_{i-1} \ge q_0$, and, since $u \le b^{m_1} \le uq_0$, we find

$$b^{N_i - N_{i-1}} \le \frac{b^{N_i}}{q_{i-1}+1} \le \frac{b^{m_1}(b^{N_i}-1)}{uq_{i-1}} = q_i < \frac{b^{m_1+N_i}}{uq_0} \le b^{N_i}.$$

So, by induction, (2.5) holds for all $i \ge 1$.

In particular, the sequence $(q_i)_{i\ge1}$ is strictly increasing, and thanks to (2.2), Lemma 2.2 shows that $(p_i/q_i)_{i\ge1}$ is a sequence of consecutive convergents to its limit $\xi \in C$. Fix an index $i \ge 1$. By the theory of continued fractions, we have

$$\frac{1}{2q_iq_{i+1}} < \left|\xi - \frac{p_i}{q_i}\right| < \frac{1}{q_iq_{i+1}}.$$

According to (2.3), this implies that $|\xi - p_i/q_i| < \psi(q_i)$. By (2.5), we also have $q_{i+1} \ge b^{m_{i+1}N_i} \ge q_i^{q_i}$, because $m_{i+1} \ge q_i$, and this further yields $|\xi - p_i/q_i| < 1/q_{i+1} \le q_i^{-q_i}$.

To provide a lower bound for $|\xi - p_i/q_i|$ when $i \ge 2$, we note that if $m_{i+1} > q_i$, then, by virtue of (2.4) and of the choice of m_{i+1}, we have

$$\psi(q_i) \le \frac{u}{b^{m_1}(b^{N_{i+1}-q_iN_i}-1)} \le \frac{2}{b^{N_{i+1}-q_iN_i}} \le \frac{2}{b^{N_{i+1}-(q_i+1)N_i}}.$$

This also holds if $m_{i+1} = q_i$, because, by hypothesis, $\psi(q_i) \le 1$. Thus, we obtain

$$\begin{aligned}\left|\xi - \frac{p_i}{q_i}\right| > \frac{1}{2q_iq_{i+1}} &= \frac{u}{2b^{m_1}(b^{N_{i+1}}-1)}\\ &\ge \frac{1}{2b^{m_1+N_{i+1}}} \ge \frac{\psi(q_i)}{4b^{m_1+(q_i+1)N_i}} \ge \frac{\psi(q_i)}{4b^{m_1}q_i^{(q_i+1)(1+m_i)/m_i}},\end{aligned}$$

where the last inequality uses (2.5). As m_i tends to infinity with i (because $m_i \ge q_{i-1}$ for $i \ge 2$), this means that $|\xi - p_i/q_i| > \psi(q_i)q_i^{-(1+\epsilon)q_i}$ for each sufficiently large index i.

This proves the theorem with $c_1 = q_1$, as q_1 depends only on b and u. If $d_2 = 1$, we have $D = \{0,1\}$, and we can even take $c_1 = 1$, because $0 = 0/1$ is the only missing convergent of ξ among $(p_i/q_i)_{i\ge1}$, and $0 = (0.\overline{0})_b$ belongs to C. ■

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