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On the Simple Inductive Limits of Splitting Interval Algebras with Dimension Drops

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Abstract. A K-theoretic classification is given of the simple inductive limits of finite direct sums of the type I C^*-algebras known as splitting interval algebras with dimension drops. (These are the subhomogeneous C^*-algebras, each having spectrum a finite union of points and an open interval, and torsion K_1-group.)

Introduction

The Elliott invariant has been used with amazing success to classify simple unital C^*-algebras. By the great work of many people, a number of results concerning the classification of approximately homogeneous C^*-algebras (AH-algebras) were obtained (see for example [2, 3, 7–9, 11–13, 16–19, 23–25]). It is natural and important to consider the case of simple approximately subhomogeneous C^*-algebras (ASH-algebras). G. A. Elliott, K. Thomsen, X. Jiang, H. Su, and others have made a number of contributions in this direction (see for example [10, 20, 21, 32–35]). In [20, 21] the building blocks used are splitting interval algebras (without dimension drops) and dimension drop interval algebras (without splitting). This paper considers a mixed version of the two building blocks, namely, splitting interval algebras with dimension drops, by which is meant a C^*-subalgebra of $M_n(C[0,1])$ such that the endpoints 0 and 1 split, and each splitting point has a dimension drop. This is a special case of the Elliott–Thomsen building blocks introduced in [10] (see Definition 1.1). In this paper, the simple inductive limits built on this building block will be classified.

In the last few years, because of the important work of Kirchberg, Winter, and Lin (see for example [22, 26, 27, 37, 38]), it has become more and more important to investigate what is called tracial approximation by special C^*-algebras. G. A. Elliott and Z. Niu considered TAS-algebras in [15], where TAS means tracial approximation by splitting interval algebras. In fact, it is not enough to use only splitting interval algebras to make a tracial approximation. For example, the Jiang–Su algebra is not TAS.

So the building block considered here might be one starting point towards choosing a more general model for tracial approximation. In any case, classifying the inductive limits based on this special building block will be useful for classifying the inductive limits built on general Elliott–Thomsen building blocks. The K_1-group of a splitting interval algebra with dimension drops is finite, whereas the K_1-group of

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a general Elliott–Thomsen building block is an arbitrary finitely generated abelian group, but the case that the spectrum is a circle in which the K_1-group is torsion free of rank one is well understood (see [8, 29]).

The proof follows the lines of Elliott's intertwining argument used to verify the Elliott conjecture earlier for simpler kinds of ASH-algebras. The intertwining used here is a KK-intertwining, as used in [12], since the homotopy types of homomorphisms between the building blocks are not completely determined by the induced maps on K-theory but by the induced KK-elements. In this paper, the K-homology version of the universal coefficient theorem (UCT) will be used, as in [21]; this technique goes back to [6]. This is for the following reason: for this kind of building block, the usual UCT has a non trivial extension group (Ext) part, and on calculation, it is easily seen that the odd K-homology group of this building block vanishes, so for the UCT in terms of K-homology, the Ext part vanishes, and it has only the homomorphism group (Hom) part. This is very helpful in analyzing the homomorphisms between the building blocks.

The paper is organized as follows. In Section 1, the K-theory data, including the K-group, the K-homology group, and the KK-group of the building block are calculated. In Section 2, the trace data of the building block are given; because of the singularity of the endpoints 0 and 1, besides the usual trace data corresponding to [0,1], there are also some traces corresponding to the singular points. In Section 3, the local existence theorem will be proved. To establish this, there are two main ingredients: an approximation theorem due to [24, 33] and a reduction similar to that in [20]. Since the splitting points have dimension drops, the reduction process needs more work than [20], and becomes rather more complicated. In Section 4, the local uniqueness theorem will be established. As mentioned above, the crucial point is the role played by the KK-element in the K-homology version of the UCT. In Section 5, the two theorems above and Elliott's intertwining argument are combined to get the main classification theorem.

1 K-theoretical Data of Building Block

Definition 1.1 (Elliott–Thomsen Building Blocks) Let F_1 be a finite dimensional C^*-algebra, let φ_0 and φ_1 be two unital homomorphisms from F_1 to some matrix algebra $\mathrm{M}_m(\mathbb{C})$, and associate with this set of data the following subhomogeneous C^*-algebra:

$$\mathfrak{A}(\varphi_0,\varphi_1) = \big\{ (a,f) \in F_1 \oplus C\big([0,1], \mathrm{M}_m(\mathbb{C})\big) : f(0) = \varphi_0(a), f(1) = \varphi_1(a) \big\}.$$

Some important subhomogeneous algebras are included among these building blocks.

Example 1.2 The splitting interval algebra

$$\mathcal{S}(\bar{n}_0,\bar{n}_1) = \Big\{ f \in \mathrm{M}_m(C[0,1]) : f(x) \in \bigoplus_{i=1}^{r_x} \mathrm{M}_{n_{x_i}}(\mathbb{C}) \subseteq \mathrm{M}_m(\mathbb{C}), x = 0 \text{ or } 1 \Big\},$$

where $\bar{n}_{x_i} = (n_{x_1},\dots,n_{x_{r_x}})$, $x = 0$ or 1, is a partition of m.

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In this case, we take

$$F_1 = \bigoplus_{i=1}^{r_0} \mathrm{M}_{n_{0_i}}(\mathbb{C}) \oplus \bigoplus_{i=1}^{r_1} \mathrm{M}_{n_{1_i}}(\mathbb{C}), \quad \varphi_0(a \oplus b) = a, \quad \varphi_1(a \oplus b) = b,$$

where both $\bigoplus_{i=1}^{r_0} \mathrm{M}_{n_{0_i}}(\mathbb{C})$ and $\bigoplus_{i=1}^{r_1} \mathrm{M}_{n_{1_i}}(\mathbb{C})$ are regarded as unital subalgebras of $\mathrm{M}_m(\mathbb{C})$.

Example 1.3 The dimension drop interval algebra

$$\mathrm{I}[m_0, m, m_1] = \{f \in \mathrm{M}_m(C[0,1]) : f(0) = a_0 \otimes \mathrm{id}_{\frac{m}{m_0}}, f(1) = \mathrm{id}_{\frac{m}{m_1}} \otimes a_1\},$$

where a_0 and a_1 belong to $\mathrm{M}_{m_0}(\mathbb{C})$ and $\mathrm{M}_{m_1}(\mathbb{C})$ respectively.

In this case, we take

$$F_1 = \mathrm{M}_{m_0}(\mathbb{C}) \oplus \mathrm{M}_{m_1}(\mathbb{C}), \quad \varphi_0(a_0 \oplus a_1) = a_0 \otimes \mathrm{id}_{\frac{m}{m_0}}, \quad \varphi_1(a_0 \oplus a_1) = \mathrm{id}_{\frac{m}{m_1}} \otimes a_1.$$

In Definition 1.1, there are two homomorphisms, and it is well known that, up to unitary equivalence, each of them involves various repetitions on the diagonal of each direct summand of F_1. In this paper, an additional assumption will be put on each of the two homomorphisms; namely, with

$$F_1 = \bigoplus_{i=1}^{r_0} \mathrm{M}_{k_i}(\mathbb{C}) \oplus \bigoplus_{j=1}^{r_1} \mathrm{M}_{l_j}(\mathbb{C}),$$

let us assume that

$$\varphi_0\Big(\bigoplus_{j=1}^{r_1} \mathrm{M}_{l_i}(\mathbb{C})\Big) = 0, \quad \varphi_1\Big(\bigoplus_{i=1}^{r_0} \mathrm{M}_{k_i}(\mathbb{C})\Big) = 0.$$

Then the Elliott–Thomsen building blocks become splitting interval algebras with dimension drops, and their K_1-groups are finite (see Theorem 1.5).

One can see that

$$\mathrm{sp}(\mathfrak{A}(\varphi_0, \varphi_1)) = \{0_1, \dots, 0_{r_0}\} \cup \{1_1, \dots, 1_{r_1}\} \cup (0,1),$$

let us call the first two parts the fractional spectra and call the times of repetition the multiplicity of the corresponding fractional spectrum. Denote these multiplicities by $s_1, \dots, s_{r_0}, t_1, \dots, t_{r_1}$ and continue this notation throughout the paper. The non-Hausdorff topology on the $\mathrm{sp}(\mathfrak{A}(\varphi_0, \varphi_1))$ and the multiplicities (each of them is greater than or equal to 1) cause the main difficulties in this paper. The result and techniques developed in this paper will be useful when one considers the general case.

Let

$$\mathfrak{I} = \big\{ (0, f) \in \mathfrak{A}(\varphi_0, \varphi_1) : f(0) = \varphi_0(0) = 0, f(1) = \varphi_1(0) = 0 \big\}.$$

Obviously, $\mathfrak{I}$ is an ideal of the basic building block, and it is isomorphic to $S\mathrm{M}_m(\mathbb{C})$, the suspension of $\mathrm{M}_m(\mathbb{C})$.

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Lemma 1.4 *There is a short exact sequence,* $0 \to \mathfrak{I} \to \mathfrak{A}(\varphi_0, \varphi_1) \to F_1 \to 0$.

Theorem 1.5

$$K_0(\mathfrak{A}(\varphi_0,\varphi_1)) = \left\{ (k_1,\dots,k_{r_0},l_1,\dots,l_{r_1}) \in \mathbb{Z}^{r_0} \times \mathbb{Z}^{r_1} \mid \varphi_{0_*}(k_1,\dots,k_{r_0}) = \varphi_{1_*}(l_1,\dots,l_{r_1}) \right\}.$$

$$K_1(\mathfrak{A}(\varphi_0,\varphi_1)) = \mathbb{Z}/\left\{ \varphi_{1_*}(l_1,\dots,l_{r_1}) - \varphi_{0_*}(k_1,\dots,k_{r_0}) \mid (k_1,\dots,k_{r_0},l_1,\dots,l_{r_1}) \in \mathbb{Z}^{r_0} \times \mathbb{Z}^{r_1} \right\}.$$

Proof The short exact sequence of Lemma 1.4 induces a six-term exact sequence of the K-groups:

$$\begin{array}{ccccc} \mathrm{K}_0(\mathfrak{I}) & \longrightarrow & \mathrm{K}_0(\mathfrak{A}(\varphi_0,\varphi_1)) & \longrightarrow & \mathrm{K}_0(F_1) \\ \partial \uparrow & & & & \downarrow \partial \\ \mathrm{K}_1(F_1) \oplus \mathrm{K}_1(F_1) & \longleftarrow & \mathrm{K}_1(\mathfrak{A}(\varphi_0,\varphi_1)) & \longleftarrow & \mathrm{K}_1(\mathfrak{I}). \end{array}$$

Note that

$$\mathrm{K}_0(\mathfrak{I}) = \mathrm{K}_0(S\mathrm{M}_m(\mathbb{C})) = \mathrm{K}_1(\mathrm{M}_m(\mathbb{C})) = 0, \text{ and}$$
$$\mathrm{K}_1(\mathfrak{I}) = \mathrm{K}_1(S\mathrm{M}_m(\mathbb{C})) = \mathrm{K}_0(\mathrm{M}_m(\mathbb{C})) = \mathbb{Z}.$$

From the diagram above, one obtains $\mathrm{K}_1(\mathfrak{A}(\varphi_0,\varphi_1)) = \mathrm{K}_1(\mathfrak{I})/\mathrm{Im}\partial = \mathbb{Z}/\mathrm{Im}\partial$, and $\mathrm{K}_0(\mathfrak{A}(\varphi_0,\varphi_1)) = \mathrm{Ker}\partial$. So it suffices to know the map ∂.

Compare the short exact sequence in Lemma 1.2 with the classical suspension short exact sequence:

$$0 \to S\mathrm{M}_m(\mathbb{C}) \to C[0,1] \otimes \mathrm{M}_m(\mathbb{C}) \to \mathrm{M}_m(\mathbb{C}) \oplus \mathrm{M}_m(\mathbb{C}) \to 0.$$

By the naturality of ∂, one has the following commutative diagram

$$\begin{array}{ccc} \mathrm{K}_0(F_1)\partial & \xrightarrow{\qquad\qquad} & \mathrm{K}_1(\mathfrak{I}) \\ \varphi_{0_*} \downarrow \varphi_{1_*} & & \downarrow \mathrm{id} \\ \mathrm{K}_0(\mathrm{M}_m(\mathbb{C})) \oplus \mathrm{K}_0(\mathrm{M}_m(\mathbb{C})) & \xrightarrow{\partial'} & \mathrm{K}_1(S\mathrm{M}_m(\mathbb{C})). \end{array}$$

It is well known that $\partial'(a,b) = b - a$, and so

$$\partial(k_1,\dots,k_{r_0},l_1,\dots,l_{r_1}) = \varphi_{1_*}(l_1,\dots,l_{r_1}) - \varphi_{0_*}(k_1,\dots,k_{r_0}).$$

The conclusion follows. ■

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Remark (1) Because φ_{0_*} and φ_{1_*} map (essentially) from $\mathbb{Z}^{r_0}$ and $\mathbb{Z}^{r_1}$ to $\mathbb{Z}$, using the multiplicities $s_1, \dots, s_{r_0}, t_1, \dots, t_{r_1}$, one has that

$$\varphi_{0_*}(k_1, \dots, k_{r_0}) = k_1 s_1 + \cdots + k_{r_0} s_{r_0}, \quad \varphi_{1_*}(l_1, \dots, l_{r_1}) = l_1 t_1 + \cdots + l_{r_1} t_{r_1}.$$

(2) $\mathrm{K}_1(\mathfrak{A}(\varphi_0, \varphi_1))$ is a torsion group.

Notice that this argument also works even we remove the restriction on the two homomorphisms.

In what follows, the K-homology and KK-groups of the building block will be calculated. The following notation will be used throughout the paper.

Define

$$V_{0_i} \colon \mathfrak{A}(\varphi_0, \varphi_1) \to \mathrm{M}_{k_i}(\mathbb{C}) \text{ by } V_{0_i}(a, f) = a_i,$$
$$V_{1_j} \colon \mathfrak{A}(\varphi_0, \varphi_1) \to \mathrm{M}_{l_j}(\mathbb{C}) \text{ by } V_{1_j}(a, f) = b_j,$$

where a_i is the direct sum decomposition of a in $\bigoplus_{i=1}^{r_0} \mathrm{M}_{k_i}(\mathbb{C})$, and b_j is the direct sum decomposition of a in $\bigoplus_{j=1}^{r_1} \mathrm{M}_{l_j}(\mathbb{C})$, and these are the irreducible decompositions of the building block.

Theorem 1.6 *$K^0(\mathfrak{A}(\varphi_0, \varphi_1))$ is generated by $[V_{0_i}]$ and $[V_{1_j}]$ for $i = 1, \dots, r_0$, $j = 1, \dots, r_1$, with the relation*

$$[\varphi_0 \circ V_{0_1}] + \cdots + [\varphi_0 \circ V_{0_{r_0}}] = [\varphi_1 \circ V_{1_1}] + \cdots + [\varphi_1 \circ V_{1_{r_1}}].$$

More explicitly, the relation is

$$s_1 [V_{0_1}] + \cdots + s_{r_0} [V_{0_{r_0}}] = t_1 [V_{1_1}] + \cdots + t_{r_1} [V_{1_{r_1}}].$$

Note that $\varphi_0 \circ V_{0_i}$ is still a representation of $\mathfrak{A}(\varphi_0, \varphi_1)$, and so $[\varphi_0 \circ V_{0_i}]$ is still in $K^0(\mathfrak{A}(\varphi_0, \varphi_1))$, and similarly for φ_1. $K^1(\mathfrak{A}(\varphi_0, \varphi_1)) = 0$.

Proof Denote by $\mathfrak{A}(\varphi_0, \varphi_1)_{[0,1/2]}$ the restriction of the building block to $[0, 1/2]$, by $\mathfrak{A}(\varphi_0, \varphi_1)_{[1/2,1]}$ the restriction to $[1/2, 1]$, and let $\mathfrak{A}(\varphi_0, \varphi_1)_{1/2}$ be the evaluation of the building block at $1/2$.

Then one has the pull-back diagram

$$\begin{tikzcd}
\mathfrak{A}(\varphi_0, \varphi_1) \arrow[r, "R_0"] \arrow[d, "R_1"] & \mathfrak{A}(\varphi_0, \varphi_1)_{[0, \frac{1}{2}]} \arrow[d, "V_{\frac{1}{2}}"] \\
\mathfrak{A}(\varphi_0, \varphi_1)_{[\frac{1}{2}, 1]} \arrow[r, "V_{\frac{1}{2}}"] & \mathfrak{A}(\varphi_0, \varphi_1)_{\frac{1}{2}} = \mathrm{M}_m(\mathbb{C}),
\end{tikzcd}$$

where R_0, R_1 are the corresponding restrictions, and $V_{1/2}$ is the evaluation at $1/2$.

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Applying the Mayer–Vietoris sequence (cf. [1, Theorem 21.5.1]) to the diagram above, we obtain a six term exact sequence:

$$\begin{tikzcd}
\mathrm{K}^0(\mathrm{M}_m(\mathbb{C})) \arrow[r, "\left(-V_{\frac{1}{2}}^*, V_{\frac{1}{2}}^*\right)"] & \mathrm{K}^0(\mathfrak{A}(\varphi_0,\varphi_1)_{[0,\frac{1}{2}]}) \oplus \mathrm{K}^0(\mathfrak{A}(\varphi_0,\varphi_1)_{[\frac{1}{2},1]}) \arrow[r, "R_0^*+R_1^*"] & \mathrm{K}^0(\mathfrak{A}(\varphi_0,\varphi_1)) \arrow[d] \\
\mathrm{K}^1(\mathfrak{A}(\varphi_0,\varphi_1)) \arrow[u] & \mathrm{K}^1(\mathfrak{A}(\varphi_0,\varphi_1)_{[0,\frac{1}{2}]}) \oplus \mathrm{K}^1(\mathfrak{A}(\varphi_0,\varphi_1)_{[\frac{1}{2},1]}) \arrow[l] & \mathrm{K}^1(\mathrm{M}_m(\mathbb{C})) \arrow[l]
\end{tikzcd}$$

Define

$$\overline{V}_0\colon \mathfrak{A}(\varphi_0,\varphi_1)_{[0,\frac{1}{2}]} \to \bigoplus_{i=1}^{r_0} \mathrm{M}_{k_i}(\mathbb{C}) \text{ by } \overline{V}_0(a,f) = a',$$

$$\overline{\varphi}_0\colon \bigoplus_{i=1}^{r_0} \mathrm{M}_{k_i}(\mathbb{C}) \to \mathfrak{A}(\varphi_0,\varphi_1)_{[0,\frac{1}{2}]} \text{ by } \overline{\varphi}_0(a') = (a', f),$$

where

$$f(t) = \varphi_0(a'), \quad \forall t \in [0, \tfrac{1}{2}], \quad a = a' \oplus a''.$$

Then

$$\overline{V}_0 \circ \overline{\varphi}_0 = \mathrm{id}, \quad \overline{\varphi}_0 \circ \overline{V}_0(a,f) = (a', \varphi_0(a')).$$

Define

$$h_t\colon \mathfrak{A}(\varphi_0,\varphi_1)_{[0,\frac{1}{2}]} \to \mathfrak{A}(\varphi_0,\varphi_1)_{[0,\frac{1}{2}]} \text{ by } h_t(a,f) = (a', f(ts)),$$

then $\overline{\varphi}_0 \circ \overline{V}_0$ is homotopic to id. So $\mathfrak{A}(\varphi_0,\varphi_1)_{[0,1/2]}$ and $\bigoplus_{i=1}^{r_0} \mathrm{M}_{k_i}(\mathbb{C})$ are homotopy equivalent, similarly for $\mathfrak{A}(\varphi_0,\varphi_1)_{[1/2,1]}$ and $\bigoplus_{j=1}^{r_1} \mathrm{M}_{l_j}(\mathbb{C})$. Hence

$$\mathrm{K}^0(\mathfrak{A}(\varphi_0,\varphi_1)_{[0,\frac{1}{2}]}) = \mathbb{Z}^{r_0}, \quad \mathrm{K}^0(\mathfrak{A}(\varphi_0,\varphi_1)_{[\frac{1}{2},1]}) = \mathbb{Z}^{r_1},$$
$$\mathrm{K}^1(\mathfrak{A}(\varphi_0,\varphi_1)_{[0,\frac{1}{2}]}) = \mathrm{K}^1(\mathfrak{A}(\varphi_0,\varphi_1)_{[\frac{1}{2},1]}) = 0.$$

Therefore

$$\mathrm{K}^0(\mathfrak{A}(\varphi_0,\varphi_1)) = \mathbb{Z}^{r_0} \oplus \mathbb{Z}^{r_1}/\mathrm{Im}(-V_{\frac{1}{2}}^*, V_{\frac{1}{2}}^*),$$
$$\mathrm{K}^1(\mathfrak{A}(\varphi_0,\varphi_1)) = \mathrm{Ker}(-V_{\frac{1}{2}}^*, V_{\frac{1}{2}}^*).$$

Let us now make the necessary computations concerning $V_{1/2}^*$.

Define

$$V_{\frac{1}{2}}^*\colon \mathrm{K}^0(\mathrm{M}_m(\mathbb{C})) \to \mathrm{K}^0(\mathfrak{A}(\varphi_0,\varphi_1)_{[0,\frac{1}{2}]}) \text{ by } V_{\frac{1}{2}}^*([\rho]) = [\rho \circ V_{\frac{1}{2}}],$$

where ρ is a representation of $\mathrm{M}_m(\mathbb{C})$. Hence $V_{1/2}^*(1) = [\mathrm{id} \circ V_{1/2}]$, where $1 \in \mathrm{K}^0(\mathrm{M}_m(\mathbb{C}))$ is represented by $\mathrm{id}\colon \mathrm{M}_m(\mathbb{C}) \to \mathrm{M}_m(\mathbb{C})$. The generators of $\mathrm{K}^0(\mathfrak{A}(\varphi_0,\varphi_1)_{[0,1/2]})$ are given by $[V_{0_1}], \dots, [V_{0_{r_0}}]$; because $V_{1/2}$ is homotopic to V_0,

$$V_{\frac{1}{2}}^*(1) = [\varphi_0 \circ V_{0_1}] + \cdots + [\varphi_0 \circ V_{0_{r_0}}] = s_1[V_{0_1}] + \cdots + s_{r_0}[V_{0_{r_0}}].$$

Similarly for $V_{1/2}^*$ in the second component of $(-V_{1/2}^*, V_{1/2}^*)$, because $V_{1/2}$ is homotopic to V_1, and so

$$V_{\frac{1}{2}}^*(1) = [\varphi_1 \circ V_{1_1}] + \cdots + [\varphi_1 \circ V_{1_{r_1}}] = t_1[V_{1_1}] + \cdots + t_{r_1}[V_{1_{r_1}}].$$

The theorem follows. ■

Let $A_m = \mathfrak{A}(\varphi_0, \varphi_1), B_n = \mathfrak{B}(\psi_0, \psi_1)$; in the local uniqueness theorem later, one needs to investigate $\mathrm{KK}(\phi) \in \mathrm{KK}(A_m, B_n)$ for a homomorphism $\phi \colon A_m \to B_n$. Let us introduce some notation: let $\mathfrak{I}$ be the ideal as before of the algebra B_n, then

$$B_n/\mathfrak{I} = \bigoplus_{i=1}^{r_0^{B_n}} \mathrm{M}_{n_{0_i}}(\mathbb{C}) \oplus \bigoplus_{j=1}^{r_1^{B_n}} \mathrm{M}_{n_{1_j}}(\mathbb{C}),$$

$$\mathrm{KK}(A_m, B_n/\mathfrak{I}) = \bigoplus_{i=1}^{r_0^{B_n}} \mathrm{KK}\left(A_m, \mathrm{M}_{n_{0_i}}(\mathbb{C})\right) \oplus \bigoplus_{j=1}^{r_1^{B_n}} \mathrm{KK}\left(A_m, \mathrm{M}_{n_{1_j}}(\mathbb{C})\right).$$

For any $\rho \in \mathrm{KK}(A_m, B_n/\mathfrak{I})$, then

$$\rho = (\rho_{0_1^{B_n}}, \ldots, \rho_{0_{r_0}^{B_n}}, \rho_{1_1^{B_n}}, \ldots, \rho_{1_{r_1}^{B_n}}),$$

each component map belongs to $\mathrm{KK}(A_m, \mathrm{M}_{n_{0_i}}(\mathbb{C})) = \mathrm{K}^0(A_m)$. In the following theorem, all the fractional spectra and their multiplicities will be put with a superscript denoted the algebra, and any homomorphism $\phi \in \mathrm{Hom}(\mathbb{Z}, \mathrm{K}_0(B_n/\mathfrak{I}))$ can be written as

$$\phi = (\phi_{0_1^{B_n}}, \ldots, \phi_{0_{r_0}^{B_n}}, \phi_{1_1^{B_n}}, \ldots, \phi_{1_{r_1}^{B_n}}).$$

Theorem 1.7

$$KK(A_m, B_n) = \Gamma_1 \oplus \Gamma_2,$$

$$\Gamma_1 = \left\{ \rho \in KK(A_m, B_n/\mathfrak{I}) \;\middle|\; \sum_{j=1}^{r_1} t_j^{B_n} \cdot \rho_{1_{j*}^{B_n}}(\,\cdot\,) - \sum_{i=1}^{r_0} s_i^{B_n} \cdot \rho_{0_{i*}^{B_n}}(\,\cdot\,) = 0, \forall(\,\cdot\,) \in K_0(A_m) \right\},$$

$$\Gamma_2 = \left\{ [\phi] \in \mathrm{Hom}(\mathbb{Z}, K_0(B_n/\mathfrak{I}))/\operatorname{Im}\alpha^* \;\middle|\; \sum_{j=1}^{r_1} t_j^{B_n} \cdot \phi_{1_j^{B_n}}(1) - \sum_{i=1}^{r_0} s_i^{B_n} \cdot \phi_{0_i^{B_n}}(1) \in p\mathbb{Z} \right\},$$

where $(\,\cdot\,)$ *is the abbreviation of* $(k_1^{A_m}, \ldots, k_{r_0}^{A_m}, l_1^{A_m}, \ldots, l_{r_1}^{A_m}) \in K_0(A_m)$.

Proof From the short exact sequence

$$0 \to \mathfrak{I} \to B_n \to B_n/\mathfrak{I} \to 0$$

one obtains a six-term exact sequence for KK-groups:

$$\begin{array}{ccccc}
\mathrm{KK}(A_m, \mathfrak{I}) & \longrightarrow & \mathrm{KK}(A_m, B_n) & \longrightarrow & \mathrm{KK}(A_m, B_n/\mathfrak{I}) \\
{\scriptstyle\delta}\uparrow & & & & \downarrow{\scriptstyle\delta} \\
\mathrm{KK}^1(A_m, B_n/\mathfrak{I}) & \longleftarrow & \mathrm{KK}^1(A_m, B_n) & \longleftarrow & \mathrm{KK}^1(A_m, \mathfrak{I}).
\end{array}$$

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We know that $\mathrm{KK}(A_m, \mathfrak{I}) = \mathrm{K}^1(A_m) = 0$. For our purpose we only need $\mathrm{KK}(A_m, B_n)$ ($\mathrm{KK}^1(A_m, B_n)$ can also be calculated), so it suffices to know the map δ. By the UCT, we have

$$\begin{array}{c} \mathrm{KK}(A_m, B_n/\mathfrak{I}) = \mathrm{Hom}(\mathrm{K}_0(A_m), \mathrm{K}_0(B_n/\mathfrak{I})) \oplus \mathrm{Ext}(\mathrm{K}_1(A_m), \mathrm{K}_0(B_n/\mathfrak{I})) \\ \Big\downarrow \delta \\ \mathrm{KK}^1(A_m, \mathfrak{I}) = \mathrm{Hom}(\mathrm{K}_0(A_m), \mathrm{K}_1(\mathfrak{I})) \oplus \mathrm{Ext}(\mathrm{K}_1(A_m), \mathrm{K}_1(\mathfrak{I})). \end{array}$$

So δ is decomposed into two parts δ_0 and δ_1, on Hom and Ext part respectively. This is because there are C^*-algebras B and C, such that

$$\mathrm{K}_0(B) = \mathrm{K}_0(A_m), \quad \mathrm{K}_1(B) = \mathrm{K}_0(C) = 0, \quad \mathrm{K}_1(C) = \mathrm{K}_1(A_m).$$

Then $B \oplus C$ is KK-equivalent to A_m. Therefore

$$\begin{aligned} \mathrm{KK}(A_m, B_n/\mathfrak{I}) &= \mathrm{KK}(B, B_n/\mathfrak{I}) \oplus \mathrm{KK}(C, B_n/\mathfrak{I}), \\ \mathrm{KK}^1(A_m, \mathfrak{I}) &= \mathrm{KK}^1(B, \mathfrak{I}) \oplus \mathrm{KK}^1(C, \mathfrak{I}), \end{aligned}$$

while we have

$$\begin{aligned} \mathrm{KK}(B, B_n/\mathfrak{I}) &= \mathrm{Hom}(\mathrm{K}_0(A_m), \mathrm{K}_0(B_n/\mathfrak{I})), \\ \mathrm{KK}(C, B_n/\mathfrak{I}) &= \mathrm{Ext}(\mathrm{K}_1(A_m), \mathrm{K}_0(B_n/\mathfrak{I})), \\ \mathrm{KK}^1(B, \mathfrak{I}) &= \mathrm{Hom}(\mathrm{K}_0(A_m), \mathrm{K}_1(\mathfrak{I})), \\ KK^1(C, \mathfrak{I}) &= \mathrm{Ext}(\mathrm{K}_1(A_m), \mathrm{K}_1(\mathfrak{I})). \end{aligned}$$

Recall Theorem 1.5, we have the index map $\partial\colon \mathrm{K}_0(B_n/\mathfrak{I}) \to \mathrm{K}_1(\mathfrak{I})$, both δ_0 and δ_1 are induced by ∂. We know that

$$B_n/\mathfrak{I} = \bigoplus_{i=1}^{r_0^{B_n}} \mathrm{M}_{n_{0_i}}(\mathbb{C}) \oplus \bigoplus_{j=1}^{r_1^{B_n}} \mathrm{M}_{n_{1_j}}(\mathbb{C}),$$

so

$$\mathrm{KK}(A_m, B_n/\mathfrak{I}) = \bigoplus_{i=1}^{r_0^{B_n}} \mathrm{KK}\left(A_m, \mathrm{M}_{n_{0_i}}(\mathbb{C})\right) \oplus \bigoplus_{j=1}^{r_1^{B_n}} \mathrm{KK}\left(A_m, \mathrm{M}_{n_{1_j}}(\mathbb{C})\right).$$

For any

$$\rho = (\rho_{0_1^{B_n}}, \dots, \rho_{0_{r_0}^{B_n}}, \rho_{1_1^{B_n}}, \dots, \rho_{1_{r_1}^{B_n}}) \in \mathrm{KK}(A_m, B_n/\mathfrak{I}),$$

it induces a $\rho_* \in \mathrm{Hom}(\mathrm{K}_0(A_m), \mathrm{K}_0(B_n/\mathfrak{I}))$ by

$$\rho_* = (\rho_{0_{1_*}^{B_n}}, \dots, \rho_{0_{r_{0_*}}^{B_n}}, \rho_{1_{1_*}^{B_n}}, \dots, \rho_{1_{r_{1_*}}^{B_n}}).$$

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Then

$$\delta_0(\rho_*) = \partial \circ \rho_* = \psi_{1_*}(\rho_{1^{B_n}_{1_*}}, \dots, \rho_{1^{B_n}_{r_{1_*}}}) - \psi_{0_*}(\rho_{0^{B_n}_{1_*}}, \dots, \rho_{0^{B_n}_{r_{0_*}}}),$$

$$\operatorname{Ker}(\delta_0) = \Big\{ \rho \in \mathrm{KK}\,(A_m, B_n/\mathcal{I}) \;\Big|\; \sum_{j=1}^{r_1} t_j^{B_n} \cdot \rho_{1^{B_n}_{j_*}}(\,\cdot\,) - \sum_{i=1}^{r_0} s_i^{B_n} \cdot \rho_{0^{B_n}_{i_*}}(\,\cdot\,) = 0 \Big\},$$

where $(\,\cdot\,)$ is the abbreviation of $(k_1^{A_m}, \dots, k_{r_0}^{A_m}, l_1^{A_m}, \dots, l_{r_1}^{A_m}) \in \mathrm{K}_0(A_m)$.

For δ_1 on Ext part, assume $\mathrm{K}_1(A_m) = \mathbb{Z}_p$. Then we find a projective resolution of $\mathrm{K}_1(A_m)$:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z} \xrightarrow{\pi} \mathrm{K}_1(A_m) \longrightarrow 0,$$

where α is the map of multiplication by p. Then by definition,

$$\operatorname{Ext}(\mathrm{K}_1(A_m), \mathrm{K}_0(B_n/\mathcal{I})) = \operatorname{Hom}(\mathbb{Z}, \mathrm{K}_0(B_n/\mathcal{I}))/\mathrm{Im}\alpha^*,$$

$$\operatorname{Ext}(\mathrm{K}_1(A_m), \mathrm{K}_1(\mathcal{I})) = \operatorname{Hom}(\mathbb{Z}, \mathrm{K}_1(\mathcal{I}))/\mathrm{Im}\alpha^* = \mathbb{Z}_p.$$

For any $\phi \in \operatorname{Hom}(\mathbb{Z}, \mathrm{K}_0(B_n/\mathcal{I}))$, we know that

$$\partial(\alpha^*\phi)(\gamma) = \partial\big(\phi(\alpha(\gamma))\big) = \alpha^*(\partial \circ \phi)(\gamma), \forall \gamma \in \mathbb{Z}.$$

So ∂ induces a map from $\operatorname{Ext}(\mathrm{K}_1(A_m), \mathrm{K}_0(B_n/\mathcal{I}))$ to $\operatorname{Ext}(\mathrm{K}_1(A_m), \mathrm{K}_1(\mathcal{I}))$. Recalling the isomorphism between $\operatorname{Ext}(\mathrm{K}_1(A_m), \mathrm{K}_1(\mathcal{I}))$ and $\mathbb{Z}_p$, we know that

$$[\phi'] \in \operatorname{Ext}(\mathrm{K}_1(A_m), \mathrm{K}_1(\mathcal{I})) = 0$$

if and only if $\phi'(1)$ is a multiple of p. For a homomorphism $\phi \in \operatorname{Hom}(\mathbb{Z}, \mathrm{K}_0(B_n/\mathcal{I}))$, it can be written as

$$\phi = (\phi_{0^{B_n}_1}, \dots, \phi_{0^{B_n}_{r_0}}, \phi_{1^{B_n}_1}, \dots, \phi_{1^{B_n}_{r_1}}),$$

then

$$\partial \circ \phi(1) = \sum_{j=1}^{r_1} t_j^{B_n} \cdot \phi_{1^{B_n}_j}(1) - \sum_{i=1}^{r_0} s_i^{B_n} \cdot \phi_{0^{B_n}_i}(1), \text{ and}$$

$$\operatorname{Ker}(\delta_1) = \Big\{ [\phi] \in \operatorname{Hom}(\mathbb{Z}, \mathrm{K}_0(B_n/\mathcal{I}))/\operatorname{Im}\alpha^* \;\Big|\; \sum_{j=1}^{r_1} t_j^{B_n} \cdot \phi_{1^{B_n}_j}(1) - \sum_{i=1}^{r_0} s_i^{B_n} \cdot \phi_{0^{B_n}_i}(1) \in p\mathbb{Z} \Big\}.$$

The theorem follows. ∎

Remark From the calculation above, one can see that for any homomorphism $\phi\colon A_m \to B_n$, the induced element $\mathrm{KK}\,(\phi)$ is determined by several K-homology classes $[V_{0_i} \circ \phi]$, $[V_{1_j} \circ \phi]$, $i = 1, \dots, r_0$, $j = 1, \dots, r_1$. Note that the odd K-homology of the building block vanishes. From the K-homology version of the UCT, one obtains an isomorphism $\eta\colon \mathrm{KK}\,(A_m, B_n) \to \operatorname{Hom}(\mathrm{K}^0(B_n), \mathrm{K}^0(A_m))$. This will be used later in the local uniqueness part.

Lemma 1.8 *Suppose that $\rho\colon \mathfrak{A}(\varphi_0,\varphi_1) \to M_n(\mathbb{C})$ is a unital homomorphism. Then there exists a unitary $\mathfrak{U}$, an increasing sequence $(\lambda_1,\dots,\lambda_\mu)$ of numbers in $[0,1]$, and two sequences of nonnegative integers $(\mu_{0_i})_{i=1}^{r_0}$ and $(\mu_{1_j})_{j=1}^{r_1}$, where at least one μ_{0_i} is strictly less than s_i, and at least one μ_{1_j} is strictly less than t_j such that*

$$\rho(a,f) = \mathfrak{U}^* \begin{pmatrix} \operatorname{diag}(a_i \otimes id_{\mu_{0_i}}) & 0 & \dots & 0 & 0 \\ 0 & f(\lambda_1) & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & f(\lambda_\mu) & 0 \\ 0 & 0 & \dots & 0 & \operatorname{diag}(b_j \otimes id_{\mu_{1_j}}) \end{pmatrix}.$$

Let us call this the *compressed diagonalization form* of ρ. Then $(\lambda_1,\dots,\lambda_\mu)$ is uniquely determined by ρ up to a permutation. We assume $\lambda_1 \le \lambda_2 \le \dots \le \lambda_\mu$, and the sequence

$$L(\rho) = (\mu_{0_1},\dots,\mu_{0_{r_0}},\lambda_1,\dots,\lambda_\mu,\mu_{1_1},\dots,\mu_{1_{r_1}})$$

is called the *eigenvalue* list of ρ.

Proof Every representation is a finite direct sum of irreducible representations up to unitary equivalence, the only thing we need to do is, whenever all the irreducible representations at 0 appear enough times (namely, reach all the corresponding multiplicities), replace them by an $f(0)$. Whenever all the irreducible representations at 1 appear enough times, we replace them by an $f(1)$. Then the lemma follows. ∎

Following the lemma above, define

$$\zeta(\rho) = (\mu_{0_1},\dots,\mu_{0_{r_0}},\mu,\mu_{1_1},\dots,\mu_{1_{r_1}}).$$

This form above is frequently used in this paper; it is stable in the following sense.

Lemma 1.9 *Let $\rho_i\colon \mathfrak{A}(\varphi_0,\varphi_1) \to M_n(\mathbb{C})$, $i = 0,1$, be two unital homomorphisms. Then the following statements are equivalent:*

(i) *ρ_0 and ρ_1 are homotopic;*
(ii) *$[\rho_0] = [\rho_1] \in K^0(\mathfrak{A}(\varphi_0,\varphi_1))$;*
(iii) *$\zeta(\rho_0) = \zeta(\rho_1)$.*

Proof (i) $\Rightarrow$ (ii) is obvious. By Theorem 1.6, (ii) $\Rightarrow$ (iii) is also obvious. So we only need to prove (iii) $\Rightarrow$ (i). Suppose

$$\zeta(\rho_0) = \zeta(\rho_1) = (\mu_{0_1},\dots,\mu_{0_{r_0}},\mu,\mu_{1_1},\dots,\mu_{1_{r_1}}),$$

then for $k = 0,1$, there are unitaries $u_k \in \mathrm{M}_n(\mathbb{C})$, and numbers $\lambda_1(k),\dots,\lambda_\mu(k)$ in $[0,1]$ such that

$$\rho_k(a,f) = u_k^* \begin{pmatrix} \operatorname{diag}(a_i \otimes \mathrm{id}_{\mu_{0_i}}) & 0 & \dots & 0 & 0 \\ 0 & f(\lambda_1(k)) & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & f(\lambda_\mu(k)) & 0 \\ 0 & 0 & \dots & 0 & \operatorname{diag}(b_j \otimes \mathrm{id}_{\mu_{1_j}}) \end{pmatrix} u_k.$$

Let $u(s)$ be a continuous unitary path in $\mathrm{M}_n(\mathbb{C})$ connecting u_0 and u_1, and let $\lambda_j(s)$ be a continuous path connecting $\lambda_j(0)$ and $\lambda_j(1)$. Define a homotopy by

$$\rho_s(a,f) =$$
$$u(s)^* \begin{pmatrix} \mathrm{diag}(a_i \otimes \mathrm{id}_{\mu_{0_i}}) & 0 & \dots & 0 & 0 \\ 0 & f(\lambda_1(s)) & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & f(\lambda_\mu(s)) & 0 \\ 0 & 0 & \dots & 0 & \mathrm{diag}(b_j \otimes \mathrm{id}_{\mu_{1_j}}) \end{pmatrix} u(s).$$

This completes the proof. ∎

Corollary 1.10 *Let $\rho_x \colon \mathfrak{A}(\varphi_0,\varphi_1) \to M_n(\mathbb{C})$ $(x \in [0,1])$ be a continuous family of morphisms. For each x, let*

$$L(\rho_x) = (\mu_{0_1}(x), \dots, \mu_{0_{r_0}}(x), \lambda_1(x), \dots, \lambda_{\mu_x}(x), \mu_{1_1}(x), \dots, \mu_{1_{r_1}}(x))$$

be the eigenvalue list of ρ_x. Then the following statements hold:

(i) *Each of $\mu_{0_i}(x)$, $\mu_{1_j}(x)$, μ_x is a constant function.*
(ii) *For each i, $\lambda_i(x)$ is continuous.*

2 Trace Data and Compatible Pairs

Lemma 2.1 *There are two kinds of tracial states on $\mathfrak{A}(\varphi_0,\varphi_1)$:*

(i) *Any probability measure μ on $[0,1]$ defines a tracial state on the building block*

$$\mu(a,f) = \int Tr(f)d\mu,$$

where Tr is the normalized trace on $M_m(\mathbb{C})$.

(ii) *At endpoints 0 and 1, there are fractional tracial states*

$$\delta_{0_i}(a,f) = Tr(a_i), i = 1,\dots,r_0, \quad \delta_{1_j}(a,f) = Tr(b_j), j = 1,\dots,r_1,$$

where Tr is the normalized trace on the corresponding matrix. Moreover,

$$\delta_0 = \sum_{i=1}^{r_0} \frac{m_{0_i}}{m}\delta_{0_i}, \quad \delta_1 = \sum_{j=1}^{r_1} \frac{m_{1_j}}{m}\delta_{1_j},$$

where m_{0_i} is the size of $\varphi_0(a_i)$, and m_{1_j} is the size of $\varphi_1(b_j)$.

The following result is less trivial but still standard.

Lemma 2.2 *Any $f \in Aff T\mathfrak{A}(\varphi_0,\varphi_1)$ determines a real-valued function $\delta^*(f)$ on* $\mathrm{sp}(\mathfrak{A}(\varphi_0,\varphi_1))$ *by $\delta^*(f)(x) = f(\delta_x)$.*

So this map identifies $Aff T\mathfrak{A}(\varphi_0,\varphi_1)$ with the space of all real-valued functions f on $\mathrm{sp}(\mathfrak{A}(\varphi_0,\varphi_1))$ *satisfying*

(i) *f is continuous on* $(0, 1)$,

(ii) *for* $x = 0, 1$, $\lim_{[\bar{x}]\to x} f(\bar{x}) = \sum_i \frac{m_{x_i}}{m} f(x_i)$.

Recall that $[\,\cdot\,]\colon \operatorname{sp}(\,\cdot\,) \to [0, 1]$ *is the canonical quotient map.*

Remark Let $\mathfrak{B}(\psi_0, \psi_1)$ be another building block. Then, by Lemma 2.1, to define a morphism $\theta\colon T\mathfrak{B}(\psi_0, \psi_1) \to T\mathfrak{A}(\varphi_0, \varphi_1)$, it suffices to define $\theta(\delta_y)$ for $y \in \operatorname{sp}(\mathfrak{B}(\psi_0, \psi_1))$ in a way that is continuous on $(0, 1)$ and satisfies the right boundary condition as in the lemma above. This will be used later.

In what follows, we are going to investigate compatible pairs for the building blocks.

Let A, B be two C^*-algebras; recall that two homomorphisms $\kappa\colon \mathrm{K}_0(A) \to \mathrm{K}_0(B)$ and $\theta\colon T(B) \to T(A)$ are compatible if $\langle e, \theta(t)\rangle = \langle \kappa(e), t\rangle$ for any $e \in \mathrm{K}_0(A)$ and any $t \in TB$.

By choosing proper sub-projections of B and standard a "cut-down" argument, we can focus on compatible pairs for the basic building blocks (without worrying about direct sums).

Definition 2.3 Let $A_m = \mathfrak{A}(\varphi_0, \varphi_1), B_n = \mathfrak{B}(\psi_0, \psi_1)$ be two building blocks (m and n refer to the size of generic fibre), $\alpha \in \mathrm{KK}\,(A_m, B_n)$, and let $\theta\colon TB_n \to TA_m$ be a continuous affine map. Then (α, θ) is compatible if (α_0, θ) is compatible, where α_0 is the induced map of α on K_0 -group.

In the proof of the local existence theorem, one of the crucial techniques is the decomposition of a compatible pair.

Definition 2.4 Let A, B be C^*-algebras, $\alpha \in \mathrm{KK}\,(A, B)$, and $\theta\colon TB \to TA$ be a continuous affine map. Let (α, θ) be a compatible pair for A, B. A decomposition of (α, θ), denoted by $(\alpha, \theta) = \sum_j (\alpha_j, \theta_j)$, consists of

(i) mutually orthogonal C^*-subalgebras $B_1, \dots, B_p$ of B such that $1_B \in B_1 + \cdots + B_p$;

(ii) a compatible pair (α_j, θ_j) for each (A, B_j), $\alpha_j \in \mathrm{KK}\,(A, B_j)$, such that $\alpha = \sum_j ((\alpha_j) \otimes_{B_j} [\iota_j])$, where $[\iota_j]$ is the KK-element induced by the inclusion $\iota_j\colon B_j \to B$, and $\theta(t) = \sum_j \theta_j(t|B_j)$, $t \in TB_j$, where θ_j is naturally extended.

For the building block, there are three kinds of basic compatible pairs.

Definition 2.5 Let

$$\begin{aligned} A_m &= \mathfrak{A}(\varphi_0, \varphi_1) \\ &= \bigl\{ (a, f) \in F_1 \oplus C\bigl([0, 1], \mathrm{M}_m(\mathbb{C})\bigr) : f(0) = \varphi_0(a), f(1) = \varphi_1(a) \bigr\}, \end{aligned}$$

recall that

$$\mathrm{K}_0(A_m) = \bigl\{ (k_1, \dots, k_{r_0}, l_1, \dots, l_{r_1}) \in \mathbb{Z}^{r_0} \times \mathbb{Z}^{r_1} \mid \varphi_{0_*}(k_1, \dots, k_{r_0}) = \varphi_{1_*}(l_1, \dots, l_{r_1}) \bigr\}.$$

(i) Let

$$B = \mathrm{M}_m(\mathbb{C}), \quad \alpha \in \mathrm{KK}\,(A_m, B), \quad \alpha_0(k_1, \dots, k_{r_0}, l_1, \dots, l_{r_1}) = \varphi_{0_*}(k_1, \dots, k_{r_0}),$$

and let $\theta(Tr) = \mu$, then (α, θ) is compatible for (A_m, B) and it is called a generic pair.

(ii) Let

$$B = A_m(0) \subseteq \underbrace{\mathrm{M}_{n_{0_1}}(\mathbb{C}) \oplus \cdots \oplus \mathrm{M}_{n_{0_1}}(\mathbb{C})}_{s_1} \oplus \underbrace{\mathrm{M}_{n_{0_2}}(\mathbb{C}) \oplus \cdots \oplus \mathrm{M}_{n_{0_2}}(\mathbb{C})}_{s_2} \oplus \dots$$
$$\cdots \oplus \underbrace{\mathrm{M}_{n_{0_{r_0}}}(\mathbb{C}) \oplus \cdots \oplus \mathrm{M}_{n_{0_{r_0}}}(\mathbb{C})}_{s_{r_0}}.$$

Let

$$\alpha \in \mathrm{KK}\,(A_m, B), \quad \alpha_0(k_1, \dots, k_{r_0}, l_1, \dots, l_{r_1}) = (\underbrace{k_1, \dots, k_1}_{s_1}, \dots, \underbrace{k_{r_0}, \dots, k_{r_0}}_{s_{r_0}}),$$

and $\theta(Tr_i) = \delta_{0_i}$ (Tr_i is the normalized trace on $\mathrm{M}_{n_{0_i}}(\mathbb{C})$), then (α, θ) is compatible and it is called a broken pair at 0. Similarly there are also broken pairs at 1.

(iii) Let

$$B = \mathrm{M}_{n_{0_i}}(\mathbb{C}), \quad \alpha \in \mathrm{KK}\,(A_m, B), \quad \alpha_0(k_1, \dots, k_{r_0}, l_1, \dots, l_{r_1}) = k_i,$$

and $\theta(Tr) = \delta_{0_i}$, then (α, θ) is compatible and it is called a fractional pair at 0_i, $i = 1, \dots, r_0$. Similarly we have fractional pair at 1_j, $j = 1, \dots, r_1$.

Lemma 2.6 *Let $A_m = \mathfrak{A}(\varphi_0, \varphi_1)$ be a building block, $B = M_n(\mathbb{C})$, and (κ, θ) is compatible for A_m and B. Then $\theta(Tr) \in T\mathfrak{A}(\varphi_0, \varphi_1)$ uniquely determines a vector $\lambda(\theta(Tr)) = (\bar{\lambda}_0, \bar{\lambda}_1, \lambda) \in \mathbb{R}_+^{r_0} \times \mathbb{R}_+^{r_1} \times \mathbb{R}_+$ such that*

$$\theta(Tr) = \sum_{x \in \{0,1\}} \sum_{i=1}^{r_x} \lambda_{x_i} \cdot \delta_{x_i} + \lambda \cdot \mu,$$

where $\lambda \cdot \mu$ is the maximum of all possible values that can be realized by a homomorphism from A_m to B, $\bar{\lambda}_x = (\lambda_{x_1}, \dots, \lambda_{x_{r_x}})$ satisfies that at least one λ_{0_i} is strictly less than $\frac{n_{0_i}}{n} s_i$ and at least one λ_{1_j} is strictly less than $\frac{n_{1_j}}{n} t_j$, μ is a Radon probability measure on $[0, 1]$. This is called the standard form of $\theta(Tr)$, $\lambda \cdot \mu$ is called the principal part, and the rest is called the residual part.

Proof Any $t \in TA_m$ has a decomposition. For the uniqueness of the coefficient, we make the following simplification: recall that $\delta_0 = \sum_i \frac{n_{0_i} \cdot s_i}{m} \delta_{0_i}$, $\delta_1 = \sum_j \frac{n_{1_j} \cdot t_j}{m} \delta_{1_j}$; then

$$\frac{m}{n} \delta_0 = \sum_i \frac{n_{0_i} \cdot s_i}{n} \delta_{0_i}, \quad \frac{m}{n} \delta_1 = \sum_j \frac{n_{1_j} \cdot t_j}{n} \delta_{0_i}.$$

If every fractional spectrum appears enough times, namely, reaches its multiplicity, then we can replace the fractional trace by a $\frac{m}{n}\delta_0$ or a $\frac{m}{n}\delta_1$. Otherwise we just leave it there, which means that at least one of the fractional spectra does not reach its multiplicity, then the lemma follows. ∎

The following lemma is basic for the proof of the local existence theorem.

Lemma 2.7 *Let $A_m = \mathfrak{A}(\varphi_0, \varphi_1)$ be a building block, let*

$$B = \left\{ \operatorname{diag}(\underbrace{a_1, \ldots, a_1}_{k_1}, \ldots, \underbrace{a_s, \ldots, a_s}_{k_s}, \ldots) \mid a_s \in M_{n_s}(\mathbb{C}) \right\}, \quad n = \sum_s k_s \cdot n_s,$$

and $\alpha \in KK(A_m, B)$, and let $\theta \colon TB \to TA_m$ be a continuous affine map. Then $(\alpha, \theta) = \sum_j (\alpha_j, \theta_j)$ such that

(i) *each (α_j, θ_j) is either generic, broken or fractional;*

(ii) *if $\theta(Tr) = \sum_{x \in \{0,1\}} \sum_{i=1}^{r_x} \lambda_{x_i} \cdot \delta_{x_i} + \lambda \cdot \mu$ is the standard form of $\theta(Tr)$, then the number of fractional pairs at x_i is $n \cdot \lambda_{x_i} / n_{x_i}$, where Tr is the normalized trace on $M_n(\mathbb{C})$.*

Proof First we deal with the special case where $B = \mathrm{M}_n(\mathbb{C})$. In this case, because B has only one block, there is no broken pair in the decomposition. By choosing an orthogonal projection with rank equal to m_{x_i} (the size of corresponding fractional part), $x = 0$ or 1, then cutting B by this projection, we get a subalgebra of B and a fractional pair at x_i. Similarly we also have generic pairs. To calculate the number of fractional pairs at x_i we use a counting argument. Assume this number is k. From the definition of decomposition of a compatible pair, we know that $\theta(Tr) = \sum_j \theta_j(Tr|B_j)$, where $\{B_j\}_j$ are all the subalgebras. From the definition, there is no contribution to the residual part from generic pairs. Suppose that B_j corresponds to a fractional pair at x_i, then

$$\theta(Tr|B_j) = \theta\left(\frac{n_{x_i}}{n} Tr_{x_i}\right) = \frac{n_{x_i}}{n} \delta_{x_i},$$

where Tr_{x_i} is the normalized trace on $\mathrm{M}_{n_{x_i}}(\mathbb{C})$. So

$$k \cdot \frac{n_{x_i}}{n} = \lambda_{x_i}, \quad k = n \cdot \frac{\lambda_{x_i}}{n_{x_i}}.$$

The general case follows by applying the special case to each block of B. But in the general case, some fractional pairs arising from different blocks may add up to a broken pair; namely, if all the fractional pairs reach their multiplicities, then we will take direct sums of them and replace them by a single broken pair. Again only fractional pairs contribute to the residual part of $\theta(Tr)$, so the same argument works in the general case. ∎

Remark The fractions $n \cdot \lambda_{x_i} / n_{x_i}$ above are integers under the assumption of compatibility. This is not true without this assumption.

3 Local Existence Theorem

In this section, a local existence theorem will be established, and this theorem will be used in the main classification theorem of Section 5. The proof has two crucial ingredients: a modified version of an approximation theorem by [33], as improved

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in [24], and a reduction process of compatible pairs for the building blocks based on Lemma 2.7.

Let $A_m = \mathfrak{A}(\varphi_0, \varphi_1), B_n = \mathfrak{B}(\psi_0, \psi_1)$ be the building blocks. For the local existence theorem, one needs to realize (approximately) the homomorphisms between the Elliott's invariants by a homomorphism between the C^*-algebras. Let us look at the homomorphisms between the building blocks. As proved in [32, Theorem 3.1], a similar theorem holds:

Theorem 3.1 *Let A_m, B_n be the building blocks above. For any homomorphism $\phi\colon A_m \to B_n$, any $\epsilon > 0$, and any finite subset $F \subseteq A_m$, there is a unital homomorphism $\phi'\colon A_m \to B_n$ such that*

(i) *$\|\phi(a, f) - \phi'(a, f)\| < \epsilon$ for all $(a, f) \in F$;*
(ii) *there exist continuous maps $\{\lambda_j(s)\}_{j=1}^{p}\colon [0,1] \to \mathrm{sp}(A_m)$, and a unitary $W \in M_n(C[0,1])$ such that*

$$\phi'(a, f)(s) = W^*(s) \begin{pmatrix} f(\lambda_1(s)) & 0 & \dots & 0 & 0 \\ 0 & f(\lambda_2(s)) & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & f(\lambda_{p-1}(s)) & 0 \\ 0 & 0 & \dots & 0 & f(\lambda_p(s)) \end{pmatrix} W(s)$$

for all $(a, f) \in A_m$ and all $s \in [0,1]$.

Any homomorphism of the form above is called a *standard homomorphism*, and $\{\lambda_j(s)\}_{j=1}^{p}$ are called the *eigenvalue maps*.

The following result appears in [20], and it is essentially due to [24, 33].

Theorem 3.2 *For any finite set $F \subseteq C[0,1]$ and any $\epsilon > 0$, there is a constant N such that for any homomorphism $\theta\colon T(C[0,1]) \to T(C[0,1])$ and any $q \geq N$, there are exactly q endomorphisms ϕ_k of $C[0,1]$ satisfying*

$$\Big\| \theta(t)(f) - \frac{1}{q} \sum_k \phi_k^*(t)(f) \Big\| < \epsilon$$

for all $f \in F$ and all $t \in T(C[0,1])$. Moreover, if $\theta(\delta_0) = \frac{1}{n} \sum_{j=1}^{n} \delta_{x(j)}$ for some $x(j) \in [0,1]$ and $\theta(\delta_1) = \frac{1}{n} \sum_{j=1}^{n} \delta_{y(j)}$ for some $y(j) \in [0,1]$, and $n|q$, then those q endomorphisms can be chosen such that

$$\theta(\delta_0) = \frac{1}{q} \sum_{k=1}^{q} \phi_k^*(\delta_0), \quad \theta(\delta_1) = \frac{1}{q} \sum_{k=1}^{q} \phi_k^*(\delta_1).$$

Theorem 3.3 (Local Existence) *Let $A_m = \mathfrak{A}(\varphi_0, \varphi_1)$ be a basic building block. Then for any $\epsilon > 0$ and any finite subset $F \subseteq TA_m$, there is a $N \in \mathbb{N}$, such that if*

(i) *$B_n = \mathfrak{B}(\psi_0, \psi_1)$, where the size of each direct summand of F_1 (of B_n) is greater than or equal to N,*
(ii) *$\theta\colon TB_n \to TA_m$ is a continuous affine map,*

(iii) *$\alpha \in KK(A_m, B_n)$ and (α, θ) is compatible,*

then it follows that there is a unital homomorphism $\phi\colon A_m \to B_n$ such that $KK(\phi) = \alpha$ and

$$\|\phi^*(\tau)(a, f) - \theta(\tau)(a, f)\| < \epsilon + \frac{s_1 + \cdots + s_{r_0} + t_1 + \cdots + t_{r_1} + 2}{n} N\|f\|$$

for all $(a, f) \in F$ and all $\tau \in TB_n$.

Proof The proof is inspired by the local existence theorem in [20], but it is more complicated, and at some crucial steps new techniques are needed. For any $\epsilon > 0$, any finite subset F of A_m, set

$$\widetilde{F} = \{Tr(f) \in C[0, 1] \mid (a, f) \in A_m\},$$

where $Tr(f)$ is given by taking trace at each point. So by Theorem 3.2, there is a natural number N such that for any morphism $\theta\colon TC[0, 1] \to TC[0, 1]$, any $q \geq N$, there are exactly q endomorphisms ϕ_k of $C[0, 1]$ (corresponding to q continuous paths $\beta_k\colon [0, 1] \to [0, 1]$) such that

$$\Big\|\theta(t)(f) - \frac{1}{q}\sum_k \phi_k^*(t)(f)\Big\| < \epsilon$$

for all $f \in \widetilde{F}$ and all $t \in TC[0, 1]$.

In what follows, it will be shown that the compatible pair (α, θ) has a decomposition $\sum_j (\alpha_j, \theta_j)$, and after doing some grouping of the pairs (α_j, θ_j) we can then apply Theorem 3.2 to the main part of (α, θ) (discarding the fractional pairs (α_j, θ_j)). First, one can decompose the compatible pair at the endpoints 0 and 1. Define R_0 and R_1 to be the restrictions of B_n onto $B_n(0)$ and $B_n(1)$. Because

$$B_n(0) \subseteq \Big\{ \operatorname{diag}(\underbrace{a_1, \ldots, a_1}_{s_1}, \ldots, \underbrace{a_k, \ldots, a_k}_{s_k}, \ldots) \mid a_k \in \mathrm{M}_{n_k}(\mathbb{C})\Big\},$$

$$B_n(1) \subseteq \Big\{ \operatorname{diag}(\underbrace{b_1, \ldots, b_1}_{t_1}, \ldots, \underbrace{b_l, \ldots, b_l}_{t_l}, \ldots) \mid b_l \in \mathrm{M}_{n_l}(\mathbb{C})\Big\},$$

we can apply Lemma 2.7 to get the decompositions $(\alpha^0, \theta^0) = \sum_j (\alpha_j^0, \theta_j^0)$ at $x = 0$, and $(\alpha^1, \theta^1) = \sum_j (\alpha_j^1, \theta_j^1)$ at $x = 1$. While $\alpha^0 = \alpha \otimes_{B_n} [R_0]$, $\alpha^1 = \alpha \otimes_{B_n} [R_1]$, where $[R_0], [R_1]$ are the KK-elements induced by $[R_0], [R_1]$, $\theta^0 = \theta \circ R_0^*$, $\theta^1 = \theta \circ R_1^*$, and α^0 and α^1 are homotopic, so by compatibility, one obtains that $\theta^0(Tr)$ and $\theta^1(Tr)$ have the same standard form (Tr is the normalized trace on $\mathrm{M}_n(\mathbb{C})$). By Lemma 2.7, the number of fractional pairs at x_i $(x = 0, 1)$ are the same for (α^0, θ^0) and (α^1, θ^1), and by compatibility, the total number of summand pairs are the same. Let us assume that (α_j^0, θ_j^0) is fractional at x_i if and only if (α_j^1, θ_j^1) is fractional at the same x_i.

Second, the interior should be filled in such a way that one can get a decomposition of (α, θ). Recall that

$$F_1 = \bigoplus_{i=1}^{r_0} \mathrm{M}_{n_{0_i}}(\mathbb{C}) \oplus \bigoplus_{j=1}^{r_1} \mathrm{M}_{n_{1_j}}(\mathbb{C})$$

(this F_1 is of B_n).

(i) If (α_j^0, θ_j^0) is fractional at x_i, then so is (α_j^1, θ_j^1). We shall treat all the fractional pairs at x_i together, and this is one of the main differences from [20]. As in the proof of Lemma 2.7, at $x = 0$, $\mathrm{M}_{m_{x_i}}(\mathbb{C})$ may come from several direct summands of F_1 (of B_n). Assume that these are the p_1-th, p_2-th, …, p_u-th direct summands of F_1 (of B_n), where $1 \leq p_1, p_2, \dots, p_u \leq r_0$, and assume that at $x = 1$, $\mathrm{M}_{m_{x_i}}(\mathbb{C})$ comes from the q_1-th, q_2-th, …, q_v-th direct summands of F_1 (of B_n), where $1 \leq q_1, q_2, \dots, q_v \leq r_1$. In this case, define B_j in the following way. Let $\mathrm{I}_{m_{x_i}}$ denote the identity of $\mathrm{M}_{m_{x_i}}(\mathbb{C})$, and put

$$P_j^0 = \psi_0\Big(0 \oplus \cdots \oplus 0 \oplus \left(\begin{smallmatrix} \mathrm{I}_{m_{x_i}} & 0 \\ 0 & 0 \end{smallmatrix}\right)_{n_{0_{p_1}} \times n_{0_{p_1}}} \oplus 0 \oplus \cdots \oplus 0 \oplus \left(\begin{smallmatrix} \mathrm{I}_{m_{x_i}} & 0 \\ 0 & 0 \end{smallmatrix}\right)_{n_{0_{p_u}} \times n_{0_{p_u}}} \oplus 0 \oplus \cdots \oplus 0\Big),$$

$$P_j^1 = \psi_1\Big(0 \oplus \cdots \oplus 0 \oplus \left(\begin{smallmatrix} \mathrm{I}_{m_{x_i}} & 0 \\ 0 & 0 \end{smallmatrix}\right)_{n_{0_{q_1}} \times n_{0_{q_1}}} \oplus 0 \oplus \cdots \oplus 0 \oplus \left(\begin{smallmatrix} \mathrm{I}_{m_{x_i}} & 0 \\ 0 & 0 \end{smallmatrix}\right)_{n_{0_{q_v}} \times n_{0_{q_v}}} \oplus 0 \oplus \cdots \oplus 0\Big).$$

Because the number of fractional pairs at x_i are the same for $x = 0$ and 1, one gets that the ranks of P_j^0 and P_j^1 are the same, and so there is a continuous path $P_j(t)$ of projections in $\mathrm{M}_n(\mathbb{C})$ such that $P_j(0) = P_j^0$, $P_j(1) = P_j^1$. Define $B_j = P_j B_n P_j$, and $(\alpha_j)_0 \colon \mathrm{K}_0(A_m) \to \mathrm{K}_0(B_j)$ is given by

$$(\alpha_j)_0(k_1, \dots, k_{r_0}, l_1, \dots, l_{r_1}) = (0, \dots, 0, k_i, 0, \dots, 0, k_i, 0, \dots, 0),$$

where all the k_i are in the corresponding position.

(ii) If (α_j^0, θ_j^0) is not fractional, then (α_j^0, θ_j^0) is either broken or generic. In this case, if B_j^0 is generic and comes from the p_1-th direct summand of F_1 (of B_n), then

$$B_j(0) = \psi_0\big(\underbrace{0 \oplus 0 \oplus \cdots \oplus 0}_{p_1 - 1} \oplus B_j^0 \oplus 0 \cdots \oplus 0 \oplus 0 \oplus 0 \oplus \cdots \oplus 0\big).$$

If B_j^0 is broken (recall that a broken pair may come from several (actually more) direct summands of F_1 (of B_n)), then we can also define $B_j(0)$ but involving more matrix algebras, and similarly for $x = 1$. Then connect $B_j(0)$ and $B_j(1)$, whose total ranks are the same by a similar method to that used above to get B_j, and

$$(\alpha_j)_0(k_1, \dots, k_{r_0}, l_1, \dots, l_{r_1}) = (\alpha_j^0)_0 \oplus (\alpha_j^1)_0.$$

To define $\theta_j \colon TB_j \to TA_m$, by the remark after Lemma 2.2, it suffices to define $\theta_j(\delta_y)$ for $y \in \mathrm{sp}(B_j)$ in such a way that it is continuous on (0,1) and satisfies the right boundary condition. Let

$$\theta(\delta_y) = \sum_{x \in \{0,1\}} \sum_{i=1}^{r_x} \lambda_{x_i} \cdot \delta_{x_i} + \lambda \cdot \mu(y)$$

be the standard form as in Lemma 2.6. In case (i), define $\theta_j = \theta_j^0$. In case (ii), define it as follows:

$$\theta_j(\delta_y) = \begin{cases} \theta_j^0(\delta_y) & [y] = 0, \\ (1 - \frac{y}{\delta})\theta_j^0(\delta_0) + \frac{y}{\delta}\mu(y) & [y] \in (0, \delta), \\ \mu(y) & [y] \in (\delta, 1 - \delta), \\ (\frac{1}{\delta} - \frac{y}{\delta})\mu(y) + (\frac{y}{\delta} + \frac{\delta - 1}{\delta})\theta_j^1(\delta_1) & [y] \in (1 - \delta, 1), \\ \theta_j^1(\delta_y) & [y] = 1. \end{cases}$$

This completes the decomposition.

In the next paragraph, we need to do some grouping of the compatible pairs (κ_j, θ_j). For the number N above (at the beginning), in each block of $B_n(0)$, if we group the resulting generic pairs into batches of N pairs, then there are at most $N - 1$ pairs that remain ungrouped in each block. Also if we group the broken pairs into batches of N pairs, then there are at most $2(N - 1)$ broken pairs ungrouped. Overall, there are at most $(2+s_1+\cdots+s_{r_0})(N-1)$ principal pairs ungrouped for $B_n(0)$. We can do similar grouping for $B_n(1)$. Note that the fractional pairs can be realized by the evaluation maps. For the batches above, apply Theorem 3.2, we obtain N continuous paths $\alpha_k(t)$, and define a homomorphism ϕ_N from A_m by

$$\phi_N(a, f)(t) = \begin{pmatrix} f(\alpha_1(t)) & 0 & \dots & 0 \\ 0 & f(\alpha_2(t)) & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & f(\alpha_N(t)) \end{pmatrix}.$$

Then at $x = 0$, for a given element of a direct summand F_1 of B_n, because the size of each direct summand is greater than or equal to N, if we pick up as many paths $\{\alpha_k(t)\}$ as we need to realize the form of this element then repeat it by the corresponding multiplicity, we will have the required form of $B_n(0)$. Similarly, we can do this at $x = 1$. In other words, we obtain a homomorphism from A_m to the grouped principal pairs; for the fractional pairs, each of them can be realized by an evaluation map. For the ungrouped pairs, choose any homomorphism that induces the right KK-element and also induces the right morphism between tracial state space at boundaries (this can be done since the fibre of B_j at boundary is a fibre of A_m). Then put all the homomorphisms together, and the ungrouped pairs give the tail in the theorem. ∎

This theorem extends naturally to the case where both A, B are finite direct sums of the building blocks.

Corollary 3.4 *Let A be a finite direct sums of the building blocks, $F \subseteq A$ a finite subset, and $\epsilon > 0$ a constant. Then there exists a number $N > 0$ such that if*

(i) *B is a finite direct sums of the building blocks,*
(ii) *$\theta\colon TB \to TA$ is a continuous affine map,*
(iii) *$\alpha \in KK(A, B)$ induces a scaled ordered group homomorphism $\alpha_0\colon K_0(A) \to K_0(B)$ and α_0 is compatible with θ, and $\rho_*(\alpha_0(e)) > N$ for all class $0 < e \in K_0(A)$ and for all nonzero irreducible representations ρ of B,*

then it follows that there is a unital homomorphism $\phi\colon A \to B$ *such that* $KK(\phi) = \alpha$, *and*

$$|\phi^*(\tau)(f) - \theta(\tau)(f)| < \epsilon + max_q \Big\{ \sum_p \frac{s_1 + \cdots + s_{r_0} + t_1 + \cdots + t_{r_1} + 2}{n} N\|f\| \Big\}$$

for all $f \in F$ *and all* $\tau \in TB$, *where* p, q *are the number of direct summands of* A *and* B.

Remark Under the simplicity assumption, as the inductive sequence goes further, the size of each component of F_1 (in the definition of the building blocks) of any summand will explode, hence the tail in Theorem 3.3 will be as small as possible, as will the tail in Corollary 3.4.

4 Local Uniqueness Theorem

Let A, B be two unital C^*-algebras, and let $\phi, \psi\colon A \to B$ be two homomorphisms. Recall from [8] that ϕ, ψ are said to be *approximately unitarily equivalent*, denoted by $\phi \sim_{\text{aue}} \psi$ if for any $\epsilon > 0$, and any finite subset $F \subseteq A$, there is a unitary $u \in B$ such that $\|\phi(f) - u^*\psi(f)u\| < \epsilon$ for all $f \in F$. In this section, this equivalence relation among morphisms between two finite direct sums of the building blocks will be discussed.

The same notation will be used as before: $A_m = \mathfrak{A}(\varphi_0, \varphi_1), B_n = \mathfrak{B}(\psi_0, \psi_1)$, where m, n refer to the size of generic fibres.

Lemma 4.1 *Let* $\phi\colon A_m \to B_n$ *be a unital homomorphism. Then there exists*

(i) *a unitary* $u_x \in M_n(\mathbb{C})$ *for each* $x \in [0, 1]$,
(ii) *two sequences of nonnegative numbers* $(\mu_{0_i})_{i=1}^{r_0}$ *and* $(\mu_{1_j})_{j=1}^{r_1}$, *where at least one of the* μ_{0_i} *is strictly less than* s_i, $i = 1, \ldots, r_0$, *and at least one of the* μ_{1_j} *is strictly less than* t_i, $j = 1, \ldots, r_1$,
(iii) *a sequence* $\Lambda = (\lambda_1, \ldots, \lambda_\mu)$ *of continuous paths in* $[0, 1]$ *with* $\lambda_1(x) \le \cdots \le \lambda_\mu(x)$ *for all* $x \in [0, 1]$, *such that*

$$\phi_x(a, f) = u_x^* \begin{pmatrix} \operatorname{diag}(a_i \otimes id_{\mu_{0_i}}) & 0 & \ldots & 0 & 0 \\ 0 & f(\lambda_1) & \ldots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & f(\lambda_\mu) & 0 \\ 0 & 0 & \ldots & 0 & \operatorname{diag}(b_j \otimes id_{\mu_{1_j}}) \end{pmatrix} u_x.$$

Proof Apply Corollary 1.10 to the family of $\phi_x(a, f) = \phi(a, f)(x)$. ∎

Moreover, from Corollary 1.10, the sequence $\Lambda = (\lambda_1, \ldots, \lambda_\mu)$ is uniquely determined by ϕ. So given ϕ, we can define canonically a morphism $\Delta^\phi\colon A_m \to$

$M_n(C[0,1])$ by

$$\Delta^{\phi}(a,f)=\begin{pmatrix} \operatorname{diag}(a_i\otimes \mathrm{id}_{\mu_{0_i}}) & 0 & \dots & 0 & 0\\ 0 & f(\lambda_1) & \dots & 0 & 0\\ \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & \dots & f(\lambda_\mu) & 0\\ 0 & 0 & \dots & 0 & \operatorname{diag}(b_j\otimes \mathrm{id}_{\mu_{1_j}}) \end{pmatrix}.$$

Theorem 4.2 *Let $\phi,\psi\colon A_m\to B_n$ be two unital homomorphisms. If $KK(\phi)=KK(\psi)$ and $\Delta^{\phi}=\Delta^{\psi}$, then $\phi\sim_{\mathrm{aue}}\psi$.*

Proof By Theorem 3.1, ϕ,ψ can be approximated by standard homomorphisms $\phi^{'},\psi^{'}$; with a little bit more work, one can also guarantee that

$$\mathrm{KK}(\phi^{'})=\mathrm{KK}(\phi)=\mathrm{KK}(\psi)=\mathrm{KK}(\psi^{'}),\quad \Delta^{\phi}=\Delta^{\phi^{'}},\quad \Delta^{\psi}=\Delta^{\psi^{'}}.$$

So at the beginning, we can assume that each of ϕ and ψ already has the standard form.

Set

$$\begin{aligned} B_n&=\mathfrak{B}(\psi_0,\psi_1)\\ &=\big\{(b,f)\in F_1\oplus C([0,1],M_n(\mathbb{C})): f(0)=\psi_0(b), f(1)=\psi_1(b)\big\},\\ F_1&=\bigoplus_{i=1}^{r_0}M_{n_i}(\mathbb{C})\oplus\bigoplus_{j=1}^{r_1}M_{n_j}(\mathbb{C}),\ \text{and } \phi_Q,\psi_Q\colon A_m/\mathfrak{I}_m\to B_n/\mathfrak{I}_n, \end{aligned}$$

where $\mathfrak{I}_m,\mathfrak{I}_n$ are the canonical ideals of the building blocks.

Given any $\epsilon>0$ and any finite subset $F\subseteq A_m$, we are going to find a unitary $U\in B_n$ satisfying the requirement. Since $\mathrm{KK}(\phi)=\mathrm{KK}(\psi)$, we have $K_0(\phi_Q)=K_0(\psi_Q)$. By a standard argument as in the AF-algebra case, there exists a unitary $v\in F_1$, such that

$$\begin{aligned} &\psi_0(v)\phi(a,f)(0)\psi_0(v)^*=\psi(a,f)(0),\\ &\psi_1(v)\phi(a,f)(1)\psi_1(v)^*=\psi(a,f)(1),\forall(a,f)\in A_m. \end{aligned}$$

So we need to extend $\psi_0(v),\psi_1(v)$ to a unitary $U\in B_n$ such that

$$U(0)=\psi_0(v),\quad U(1)=\psi_1(v).$$

Because $\Delta^{\phi}=\Delta^{\psi}$, there is a unitary $\widetilde{V}(t)\in M_n(C[0,1])$ such that

$$\widetilde{V}(t)\phi(a,f)(t)\widetilde{V}^*(t)=\psi(a,f)(t),t\in[0,1].$$

For $t=1$, we have

$$\widetilde{V}(1)\phi(a,f)\widetilde{V}^*(1)=\psi(a,f)(1),\quad \psi_1(v)\phi(a,f)(1)\psi_1(v)^*=\psi(a,f)(1),$$

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so
$$\widetilde{V}^*(1)\psi_1(v)\phi(a,f)(1)\psi_1(v)^*\widetilde{V}(1)=\psi(a,f)(1);$$
namely, $\phi(a,f)(1)$ commutes with $\widetilde{V}^*(1)\psi_1(v)$.

Let σ be a small number to be specified later. Connect the identity I_n and $\widetilde{V}^*(1)\psi_1(v)$ on $[1-\sigma,1]$ in the unitary group of the commutant of $\{\phi(a,f)(1)\mid (a,f)\in A_m\}$. Extend this unitary path to $[\sigma,1]$ by the constant I_n and denote it by $X(t)$. Let σ be small enough such that for any $(a,f)\in F$, the following inequalities hold:
$$\|\phi(a,f)(t)-\phi(a,f)(1)\|<\frac{\epsilon}{4},\qquad \|\psi(a,f)(t)-\psi(a,f)(1)\|<\frac{\epsilon}{4},$$
$$\|\phi(a,f)(s)-\phi(a,f)(0)\|<\frac{\epsilon}{4},\qquad \|\psi(a,f)(s)-\psi(a,f)(0)\|<\frac{\epsilon}{4},$$
for all $t\in[1-\sigma,1]$ and all $s\in[0,\sigma]$.

On $[\sigma,1]$, denote $\widetilde{V}(t)X(t)$ by $U(t)$; then on $[\sigma,1-\sigma]$, we have
$$U(t)=\widetilde{V}(t),\quad U(1)=\widetilde{V}(1)X(1)=\widetilde{V}(1)\widetilde{V}^*(1)\psi_1(v)=\psi_1(v).$$
It follows that
$$\begin{aligned}
&\|U(t)\phi(a,f)(t)U^*(t)-\psi(a,f)(t)\|\\
&\quad\le \|U(t)\phi(a,f)(t)U^*(t)-\psi(a,f)(1)\|+\|\psi(a,f)(1)-\psi(a,f)(t)\|\\
&\quad\le \|\widetilde{V}(t)X(t)\phi(a,f)(t)X^*(t)\widetilde{V}^*(t)-\widetilde{V}(t)X(t)\phi(a,f)(1)X^*(t)\widetilde{V}^*(t)\|\\
&\qquad+\|\widetilde{V}(t)\phi(a,f)(1)\widetilde{V}^*(t)-\psi(a,f)(1)\|+\frac{\epsilon}{4}\\
&\quad\le \|\widetilde{V}(t)X(t)(\phi(a,f)(t)-\phi(a,f)(1))X^*(t)\widetilde{V}^*(t)\|\\
&\qquad+\|\widetilde{V}(t)(\phi(a,f)(1)-\phi(a,f)(t))\widetilde{V}^*(t)\|\\
&\qquad+\|\widetilde{V}(t)\phi(a,f)(t)\widetilde{V}^*(t)-\psi(a,f)(1)\|+\frac{\epsilon}{4}\\
&\quad\le \|\phi(a,f)(t)-\phi(a,f)(1))\|+\|\phi(a,f)(1)-\phi(a,f)(t))\|\\
&\qquad+\|\psi(a,f)(t)-\psi(a,f)(1))\|+\frac{\epsilon}{4}\\
&\quad\le \frac{\epsilon}{4}+\frac{\epsilon}{4}+\frac{\epsilon}{4}+\frac{\epsilon}{4}=\epsilon,
\end{aligned}$$
for all $t\in[1-\sigma,1]$ and all $(a,f)\in F$.

Similarly, one can define $U(t)$ on $[0,\sigma]$ and $U(0)=\psi_0(v)$, then it is done. ∎

Lemma 4.3 *Let $\phi,\psi\colon A_m\to B_n$ be two unital homomorphisms, assume $KK(\phi)=KK(\psi)$, then for Δ^ϕ and Δ^ψ defined above, there exist two unitaries $U_0,U_1\in M_n(\mathbb{C})$ such that*
$$U_0\Delta^\phi(a,f)(0)U_0^*=\psi_0(b_0),\qquad U_0\Delta^\psi(a,f)(0)U_0^*=\psi_0(b_0'),$$
$$U_1\Delta^\phi(a,f)(1)U_1^*=\psi_1(b_1),\qquad U_1\Delta^\psi(a,f)(1)U_1^*=\psi_1(b_1'),$$

for some $b_1, b_1^{'} \in \bigoplus_{j=1}^{r_1} M_{n_{1_j}}(\mathbb{C})$*, for some* $b_0, b_0^{'} \in \bigoplus_{i=1}^{r_0} M_{n_{0_i}}(\mathbb{C})$*, and for all* $(a, f) \in A_m$.

Proof Since KK (ϕ) = KK (ψ), from Lemma 4.1, there are two unitaries $u_0, v_0 \in \mathrm{M}_n(\mathbb{C})$, such that

$$\Delta^{\phi}(a,f)(0) = u_0^* \phi(a,f)(0) u_0$$

$$= \begin{pmatrix} \operatorname{diag}(a_i \otimes \mathrm{id}_{\mu_{0_i}}) & 0 & \dots & 0 & 0 \\ 0 & f(\lambda_1^{\phi}(0)) & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & f(\lambda_{\mu}^{\phi}(0)) & 0 \\ 0 & 0 & \dots & 0 & \operatorname{diag}(b_j \otimes \mathrm{id}_{\mu_{1_j}}) \end{pmatrix},$$

$$\Delta^{\psi}(a,f)(0) = v_0^* \psi(a,f)(0) v_0$$

$$= \begin{pmatrix} \operatorname{diag}(a_i \otimes \mathrm{id}_{\mu_{0_i}}) & 0 & \dots & 0 & 0 \\ 0 & f(\lambda_1^{\psi}(0)) & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & f(\lambda_{\mu}^{\psi}(0)) & 0 \\ 0 & 0 & \dots & 0 & \operatorname{diag}(b_j \otimes \mathrm{id}_{\mu_{1_j}}) \end{pmatrix}.$$

Consider

$$\phi\colon A_m \to B_n \xrightarrow{V_{0_i}^{B_n}} \mathrm{M}_{n_{0_i}}(\mathbb{C}), [V_{0_i}^{B_n} \circ \phi] = \eta[\phi]([V_{0_i}^{B_n}]),$$

where η is the isomorphism between KK (A_m, B_n) and $\operatorname{Hom}(\mathrm{K}^0(B_n), \mathrm{K}^0(A_m))$. Let

$$\zeta([V_{0_i}^{B_n} \circ \phi]) = (p_{0_1}^i, \dots, p_{0_{r_0}}^i, d^i, q_{1_1}^i, \dots, q_{1_{r_1}}^i),$$

where $(y_1^1, \dots, y_{d^1}^1, y_1^2, \dots, y_{d^2}^2, \dots, y_1^{r_0}, \dots, y_{d^{r_0}}^{r_0}) \subseteq (0,1)$ is a strictly increasing sequence.

Define $\rho_i\colon A_m \to \mathrm{M}_{n_{0_i}}(\mathbb{C})$ by

$$\rho_i(a,f) = \begin{pmatrix} \operatorname{diag}(a_k \otimes \mathrm{id}_{p_{0_k}^i}) & 0 & \dots & 0 & 0 \\ 0 & f(y_1^i) & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & f(y_{d^i}^i) & 0 \\ 0 & 0 & \dots & 0 & \operatorname{diag}(b_s \otimes \mathrm{id}_{p_{1_s}^i}) \end{pmatrix}.$$

Assume

$$\phi(a,f)(0) = \psi_0(c)$$

$$= \begin{pmatrix} (V_{0_1}^{B_n} \circ \phi)(a,f) \otimes \mathrm{id}_{s_1} & 0 & \dots & 0 \\ 0 & (V_{0_2}^{B_n} \circ \phi)(a,f) \otimes \mathrm{id}_{s_2} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & (V_{0_{r_0}}^{B_n} \circ \phi)(a,f) \otimes \mathrm{id}_{s_{r_0}} \end{pmatrix}$$

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for some $c \in \bigoplus_{i=1}^{r_0} M_{n_{0_i}}(\mathbb{C})$, where $\{s_i\}$ is associated with ψ_0. Then define a homomorphism $\rho_0^\phi\colon A_m \to M_n(\mathbb{C})$ by

$$\rho_0^\phi = \begin{pmatrix} \rho_1 \otimes \mathrm{id}_{s_1} & 0 & \dots & 0 \\ 0 & \rho_2 \otimes \mathrm{id}_{s_2} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \rho_{r_0} \otimes \mathrm{id}_{s_{r_0}} \end{pmatrix},$$

then $\rho_0^\phi(A_m) \subseteq B_n(0)$. Because $[\rho_0^\phi] = [\phi(a,f)(0)] \in \mathrm{K}^0(A_m)$, there exists a unitary $U_0 \in M_n(\mathbb{C})$ such that

$$U_0^* \rho_0^\phi U_0 = \begin{pmatrix} \mathrm{diag}(a_i \otimes \mathrm{id}_{\mu_{0_i}}) & 0 & \dots & 0 & 0 \\ 0 & f(Y_1^\phi) & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & f(Y_\mu^\phi) & 0 \\ 0 & 0 & \dots & 0 & \mathrm{diag}(b_j \otimes \mathrm{id}_{\mu_{1_j}}) \end{pmatrix},$$

where $Y_1^\phi \leq Y_2^\phi \leq \cdots \leq Y_\mu^\phi$ come from $\{0\}, \{1\}$, and $\{y_j^i\}$, and U_0 represents the procedure of exchanging elements on the diagonal. Then $U_0 \Delta^\phi(a,f)(0) U_0^*$ has the special form of $B_n(0)$; namely, there is a $b_0 \in \bigoplus_{i=1}^{r_0} M_{n_{0_i}}(\mathbb{C})$ such that $\psi_0(b_0) = U_0 \Delta^\phi(a,f)(0) U_0^*$.

This is because

$$[U_0 \Delta^\phi(a,f)(0) U_0^*] = [\phi(a,f)(0)] = [\rho_0^\phi] \in \mathrm{K}^0(A_m),$$

so $[U_0^* \Delta^\phi(a,f)(0) U_0] - [U_0^* \rho_0^\phi U_0] = 0$. By Lemma 1.9, we have the conclusion that if $Y_i^\phi = 0$, then $\lambda_i^\phi(0) = 0$. Define $(a,g) \in A_m$ as follows:

$$g(0) = f(0), \quad g(1) = f(1), \quad g(Y_i^\phi) = f(\lambda_i^\phi(0)).$$

For all $Y_i^\phi \neq 0$, connect these points to get g. Then one obtains

$$\begin{aligned} U_0 &\Delta^\phi(a,f)(0) U_0^* \\ &= U_0 \begin{pmatrix} \mathrm{diag}(a_i \otimes \mathrm{id}_{\mu_{0_i}}) & 0 & \dots & 0 & 0 \\ 0 & f(\lambda_1^\phi(0)) & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & f(\lambda_\mu^\phi(0)) & 0 \\ 0 & 0 & \dots & 0 & \mathrm{diag}(b_j \otimes \mathrm{id}_{\mu_{1_j}}) \end{pmatrix} U_0^* \\ &= U_0 \begin{pmatrix} \mathrm{diag}(a_i \otimes \mathrm{id}_{\mu_{0_i}}) & 0 & \dots & 0 & 0 \\ 0 & g(Y_1^\phi) & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & g(Y_\mu^\phi) & 0 \\ 0 & 0 & \dots & 0 & \mathrm{diag}(b_j \otimes \mathrm{id}_{\mu_{1_j}}) \end{pmatrix} U_0^* \\ &= \rho_0^\phi(a,g), \end{aligned}$$

which has the special form of $B_n(0)$.

Since KK $(\phi) = $ KK (ψ),

$$\zeta\big([V_{0_i}^{B_n}\circ\phi]\big) = \zeta\big(\eta[\phi](V_{0_i}^{B_n})\big) = \zeta\big(\eta[\psi](V_{0_i}^{B_n})\big) = \zeta\big([V_{0_i}^{B_n}\circ\psi]\big).$$

So we can define

$$\rho_0^{\psi} = \begin{pmatrix} \rho_1\otimes \mathrm{id}_{s_1} & 0 & \dots & 0 \\ 0 & \rho_2\otimes \mathrm{id}_{s_2} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \rho_{r_0}\otimes \mathrm{id}_{s_{r_0}} \end{pmatrix}.$$

We also want to write it in the compressed diagonalization form. Note that

$$(y_1^1,\dots,y_{d^1}^1,y_1^2,\dots,y_{d^2}^2,\dots,y_1^{r_0},\dots,y_{d^{r_0}}^{r_0})$$

is strictly increasing, and $[\rho_0^{\psi}] = [\psi(a,f)(0)]$. Hence by a method similar to that used above, one can prove that $U_0\Delta^{\psi}(a,f)(0)U_0^*$ has the special form of $B_n(0)$; namely, there is a $b_0' \in \bigoplus_{i=1}^{r_0}\mathrm{M}_{n_{0_i}}(\mathbb{C})$ such that $\psi_0(b_0') = U_0\Delta^{\psi}(a,f)(0)U_0^*$.

Similarly, one can prove the lemma for the endpoint 1. ∎

Before the statement of the main theorem in this section, let us recall some preliminaries. First, define the test functions $h_{d,r}\in C[0,1]$. For any integer $r>0$ and $0\le d<r$, let $h_{d,r}(x)$ be the function on $[0,1]$ that is 0 on $[0,d/r]$, 1 on $[(d+1)/r,1]$ and linear on $[d/r,(d+1)/r]$. These functions are due to Elliott, who developed a fundamental technique for comparing the eigenvalues of two homomorphisms within small distance. It involves the spectral distribution property, denoted by $sdp(r,\delta)$ for some $\delta>0$. Recall that a homomorphism $\phi\colon C[0,1]\to \mathrm{M}_n(C[0,1])$ is said to have $sdp(r,\delta)$ if $|\operatorname{sp}\phi_y\cap T|\ge\delta|\operatorname{sp}\phi_y|$ for any $y\in[0,1]$ and any open interval $T\subset[0,1]$ with length $\frac{1}{r}$. In the present case, the KK-condition indicates the part of the fractional spectra, and by a standard argument of Elliott, one can prove the following result.

Lemma 4.4 *Let $\phi,\psi\colon A_m\to B_n\subseteq M_n(C[0,1])$ be two standard unital homomorphisms with KK $(\phi)=$ KK (ψ). If both ϕ and ψ have $sdp(r,\delta)$ for some integer $r>0$ and some $\delta>0$, and the maps $\phi^*,\psi^*\colon TB_n\to TA_m$ agree to strictly within δ on all the central elements corresponding to the test functions $h_{d,r}$, $d=0,\dots,r-1$, then the fractional parts of the eigenvalues of ϕ_x and ψ_x are the same for any $x\in(0,1)$, and the principal parts of the eigenvalues of ϕ_x and ψ_x can be paired within $\frac{3}{r}$.*

Proof It suffices to prove that the fractional spectra of the two homomorphisms are the same. The other statement follows by Elliott's standard argument. Recall the remark after Theorem 1.7, Lemma 1.8, and Lemma 1.9, the index

$$(\mu_{0_1},\dots,\mu_{0_{r_0}},\mu,\mu_{1_1},\dots,\mu_{1_{r_1}}) = \zeta([\phi_x]) = \zeta(\eta(\mathrm{KK}(\phi))([V_x^{B_n}])).$$

Now since KK $(\phi)=$ KK (ψ), so the associated indices of the two homomorphisms are the same. ∎

Theorem 4.5 (Local Uniqueness) *Let $\phi, \psi\colon A_m \to B_n$ be two unital homomorphisms. Given any finite subset $F \subseteq A_m$, and any $\epsilon > 0$, if there is an integer $r > 0$ such that*

(i) *$KK(\phi) = KK(\psi)$,*
(ii) *both ϕ and ψ have $sdp(r, \delta)$ for some $\delta > 0$,*
(iii) *the maps $\phi^*, \psi^*\colon TB_n \to TA_m$ agree to strictly within δ on all the central elements corresponding to the test functions $h_{d,r}$, $d = 0, \dots, r-1$,*
(iv) *$\|f(z) - f(y)\| < \epsilon/3$ for all $(a, f) \in F$ and for any $y, z \in [0,1]$ with $|z - y| < 3/r$,*

then it follows that there is a unitary $u \in B_n$ such that $\|\psi(a, f) - u^*\phi(a, f)u\| < \epsilon$ for all $(a, f) \in F$.

Proof Since KK $(\phi) =$ KK (ψ), from Lemma 4.4 we have

$$\Delta^\phi(a, f) = \begin{pmatrix} \operatorname{diag}(a_i \otimes \mathrm{id}_{\mu_{0_i}}) & 0 & \dots & 0 & 0 \\ 0 & f(\lambda_1^\phi) & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & f(\lambda_\mu^\phi) & 0 \\ 0 & 0 & \dots & 0 & \operatorname{diag}(b_j \otimes \mathrm{id}_{\mu_{1_j}}) \end{pmatrix}$$

$$\Delta^\psi(a, f) = \begin{pmatrix} \operatorname{diag}(a_i \otimes \mathrm{id}_{\mu_{0_i}}) & 0 & \dots & 0 & 0 \\ 0 & f(\lambda_1^\psi) & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & f(\lambda_\mu^\psi) & 0 \\ 0 & 0 & \dots & 0 & \operatorname{diag}(b_j \otimes \mathrm{id}_{\mu_{1_j}}) \end{pmatrix},$$

where $\Lambda^\phi = (\lambda_1^\phi, \dots, \lambda_\mu^\phi)$, $\Lambda^\psi = (\lambda_1^\psi, \dots, \lambda_\mu^\psi)$ are the continuous sequences for ϕ, ψ. Moreover, from conditions (ii) and (iii), by Lemma 4.4, we get $\|\lambda_i^\phi - \lambda_i^\psi\| < 3/r$, then by (iv), we have $\|\Delta^\phi(a, f) - \Delta^\psi(a, f)\| < \epsilon/3$ for all $(a, f) \in F$.

By Lemma 4.3, there are unitaries $U_0, U_1 \in \mathrm{M}_n(\mathbb{C})$ such that

$$U_0\Delta^\phi(a, f)(0)U_0^* = \psi_0(b_0), \quad U_0\Delta^\psi(a, f)(0)U_0^* = \psi_0(b_0^{'}),$$

for some $b_0, b_0^{'} \in \bigoplus_{i=1}^{r_0} \mathrm{M}_{n_{0_i}}(\mathbb{C})$,

$$U_1\Delta^\phi(a, f)(1)U_1^* = \psi_1(b_1), \quad U_1\Delta^\psi(a, f)(1)U_1^* = \psi_1(b_1^{'}),$$

for some $b_1, b_1^{'} \in \bigoplus_{j=1}^{r_1} \mathrm{M}_{n_{1_j}}(\mathbb{C})$.

Set $b = b_0 \oplus b_1$, $b^{'} = b_0^{'} \oplus b_1^{'}$, then

$$U_0\Delta^\phi(a, f)(0)U_0^* = \psi_0(b), \qquad U_1\Delta^\phi(a, f)(1)U_1^* = \psi_1(b),$$
$$U_0\Delta^\psi(a, f)(0)U_0^* = \psi_0(b^{'}), \qquad U_1\Delta^\psi(a, f)(1)U_1^* = \psi_1(b^{'}).$$

Choose a continuous unitary path $U(t) \in \mathrm{M}_n(\mathbb{C})$ that connects U_0 and U_1 and define

$$\bar\phi(a, f)(t) = U(t)\Delta^\phi(a, f)(t)U^*(t), \quad \bar\psi(a, f)(t) = U(t)\Delta^\psi(a, f)(t)U^*(t).$$

Then $\bar{\phi}$ and $\bar{\psi}$ are two homomorphisms from A_m to B_n. By Theorem 4.2, there are unitaries $v, w \in B_n$ such that

$$\|\phi(a, f) - v^*\bar{\phi}(a, f)v\| < \epsilon/3, \quad \|\psi(a, f) - w^*\bar{\psi}(a, f)w\| < \epsilon/3, \quad \forall (a, f) \in F.$$

Let $u = v^*w$, then

$$\begin{aligned}
&\|\psi(a, f) - u^*\phi(a, f)u\| \\
&\quad \le \|\psi(a, f) - w^*\bar{\psi}(a, f)w\| \\
&\qquad + \|w^*\bar{\psi}(a, f)w - w^*\bar{\phi}(a, f)w\| + \|w^*\bar{\phi}(a, f)w - w^*v\phi(a, f)v^*w\| \\
&\quad \le \frac{\epsilon}{3} \times 3 = \epsilon, \quad \forall (a, f) \in F.
\end{aligned}$$
■

The theorem above extends naturally to the case where $A = \bigoplus_i A_i, B = \bigoplus_j B_j$ are two finite direct sums of the basic building blocks. To state the result, let us introduce some notation: $TA = \bigoplus_i TA_i$, for each i, define the test function $h_{i,d,r} = (0, \dots, 0, h_{d,r}, 0, \dots, 0)$, where $h_{d,r}$ is on the i-th position.

Corollary 4.6 *Let $A = \bigoplus_i A_i, B = \bigoplus_j B_j$ be two finite direct sums of the basic building blocks, and let $\phi, \psi\colon A \to B$ be two unital homomorphisms. For any finite subset $F \subseteq A$, any $\epsilon > 0$, if there is an integer $r > 0$, such that*

(i) *$KK(\phi) = KK(\psi)$,*
(ii) *both ϕ and ψ have $sdp(r, \delta)$ for some $\delta > 0$,*
(iii) *the maps $\phi^*, \psi^*\colon TB \to TA$ agree to strictly within $\delta/2$ on all the central elements of each direct summand of A corresponding to the test functions $h_{i,d,r}$ for all i and $0 \le d < r$,*
(iv) *$\|f(z) - f(y)\| < \epsilon/3$ for all $(a, f) \in F$ and for any $y, z \in [0, 1]$ with $|z - y| < 3/r$,*

then it follows that there is a unitary $u \in B$, such that $\|\psi(a, f) - u^*\phi(a, f)u\| < \epsilon$ for all $(a, f) \in F$.

5 Main Classification Theorem

In this section, the main classification theorem will be proved.

First, a basic fact about the connecting homomorphisms can be proved in a standard way as in [14]; see also [23, Section 2.2].

Lemma 5.1 *Let A be a unital and simple inductive limit of finite direct sums of the basic building blocks, then there exists another inductive limit sequence (A_n, ϕ_n), where each A_n is a finite direct sums of the basic building blocks and each ϕ_n is unital and injective, such that $A = \varinjlim(A_n, \phi_n)$.*

The main classification theorem is as follows:

Theorem 5.2 *Let A and B be two unital simple inductive limits of finite direct sums of the basic building blocks. If $\alpha_0\colon K_0(A) \to K_0(B)$ is a scaled ordered group homomorphism, $\alpha_1\colon K_1(A) \to K_1(B)$ is a group homomorphism, and $\theta\colon TB \to TA$ is a*

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continuous affine map that is compatible with α_0, then it follows that there is a homomorphism $\rho\colon A \to B$ that induces α_0, α_1 and θ.

If $\alpha_0, \alpha_1, \theta$ are isomorphisms, then ρ can be chosen to be an isomorphism between the algebras.

Proof The proof uses the standard (approximate) intertwining argument of Elliott, and follows closely the same strategy as in [8]. Therefore it is only sketched for the homomorphism part, and the reader is referred to [8] for detail.

Assume $A = \varinjlim(A_n, \phi_n)$, $B = \varinjlim(B_n, \psi_n)$, by Lemma 5.1, one can assume the connecting map is unital and injective. In the proof below, it is often necessary to replace an inductive sequence by a proper subsequence; this process is called a compression.

First, the morphisms on the invariant level can be lifted. From the UCT, there is a KK-element κ that induces α_0 and α_1. With the help of [31, Proposition 7.13 and Theorem 1.14], and also after necessary compressions, there is a $\kappa_n \in \mathrm{KK}\,(A_n, B_n)$ for each n such that

$$\Psi_{n_*}(\kappa_n) = \Phi_n^*(\kappa_n) \in \mathrm{KK}\,(A_n, B), \quad \psi_{n_*}(\kappa_n) = \phi_n^*(\kappa_{n+1}) \in \mathrm{KK}\,(A_n, B_{n+1}),$$

where $\Psi_n\colon B_n \to B$, $\Phi_n\colon A_n \to A$ are the canonical homomorphisms. Hence one obtains a lift of the KK-element κ.

The continuous affine map θ can also be lifted approximately by approximating the spectrum of A_m with finitely many points (as in [8]); namely, after another compression, one gets continuous affine maps $\theta_n\colon TB_n \to TA_n$, which make the diagram

$$\begin{tikzcd}
TA_1 & TA_2 \arrow[l, "\phi^*_{1,2}"'] & TA_3 \arrow[l, "\phi^*_{2,3}"'] & \cdots TA \arrow[l] \\
TB_1 \arrow[u, "\theta_1"'] & TB_2 \arrow[l, "\psi^*_{1,2}"'] \arrow[u, "\theta_2"'] & TB_3 \arrow[l, "\psi^*_{2,3}"'] \arrow[u, "\theta_3"'] & \cdots TB \arrow[l] \arrow[u, "\theta"']
\end{tikzcd}$$

almost commute in the sense that there are finite subsets $F_n \subseteq A_n$ such that $\cup_n \phi_n(F_n)$ is dense in A and

$$\left| \left\langle \phi_{n,n+1}(a, f), \theta_{n+1}(t) \right\rangle - \left\langle (a, f), \theta_n \circ \psi^*_{n,n+1}(t) \right\rangle \right| < \frac{1}{2^n},$$

for all $(a, f) \in F_n$ and all $t \in TB_{n+1}$. Moreover, each θ_n is compatible with the induced map $\kappa_{n,0}\colon \mathrm{K}_0 A_n \to \mathrm{K}_0 B_n$.

Second, for each n, we need to realize the $\kappa_{n,0}, \theta_n$ by a homomorphism $\rho_n\colon A_n \to B_n$. Let $e \in B_n$ be a nonzero projection. By the simplicity of B, it follows that for any given $N > 0$, there is an integer $m > N$ such that $\sigma_*[\psi_{n,m}(e)] > N$ for any nonzero irreducible representation σ of B_m. So, given any A_n, and any $N > 0$, there exists an integer $m > n$ such that $\kappa_n \otimes_{B_n} [\psi_{n,m}] \in \mathrm{KK}\,(A_n, B_m)$ satisfies the condition of Corollary 3.4, and hence by a compression, for any finite subset $F_n \subseteq TB_n$, there is a homomorphism $\rho_n\colon A_n \to B_n$ such that $[\rho_n] = \kappa_n$ and

$$\left| \left\langle (a, f), \theta_n(t) \right\rangle - \left\langle \rho_n(a, f), t \right\rangle \right| < \frac{1}{2^n},$$

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for all $(a, f) \in F_n$ and for all $t \in TB_n$.

In other words, one gets the following diagram:

$$\begin{tikzcd}
A_1 \arrow[r, "\phi_{1,2}"] \arrow[d, "\rho_1"] & A_2 \arrow[r, "\phi_{2,3}"] \arrow[d, "\rho_2"] & A_3 \arrow[r] \arrow[d, "\rho_3"] & \cdots \arrow[r] & A \\
B_1 \arrow[r, "\psi_{1,2}"] & B_2 \arrow[r, "\psi_{2,3}"] & B_3 \arrow[r] & \cdots \arrow[r] & B
\end{tikzcd}$$

However, the diagram above may fail to commute approximately, and this can be fixed closely as in [8]: by the injectivity of the connecting homomorphisms and simplicity of the algebras. After necessary compressions, the condition of Corollary 4.6 can be satisfied for any $n > 0$ and for any pair of maps $\rho_{n+1} \circ \phi_{n,n+1}$ and $\psi_{n,n+1} \circ \rho_n$, then one could modify each ρ_n by an inner automorphism of B_n to make the diagram above to be approximately commutative. So by Elliott's intertwining argument, there is a homomorphism $\rho\colon A \to B$ that satisfies the requirement. ∎

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