This is a ``preproof'' accepted article for *The Journal of Symbolic Logic*.
This version may be subject to change during the production process.
DOI: 10.1017/jsl.2024.87

THE COMPACTNESS OF GÖDEL LOGIC

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ABSTRACT. If G is any infinite-valued Gödel logic with identity, then the compactness cardinal of G is the least ω_1-strongly compact cardinal.

1. Introduction

Let $\mathcal{L}$ be a logic. A cardinal κ is the compactness cardinal of $\mathcal{L}$ if κ is least for which the following holds: suppose T is an $\mathcal{L}$-theory (in any vocabulary) and every $T' \subset T$ of cardinality $<\kappa$ has an $\mathcal{L}$-model. Then, T has an $\mathcal{L}$-model. Given a closed set $v \subset [0,1]$ of truth values containing 0 and 1, let G_v be v-valued first-order Gödel logic (see §2) with identity (where identities $x = y$ have truth value 1 if and only if $x = y$). We shall prove the following:

Theorem 1. *Suppose $v \subset [0,1]$ is a closed set and let G_v be the Gödel logic given by v. Then,*

1. *Suppose v is finite. Then, the compactness cardinal of G_v is $\aleph_0$.*
2. *Suppose v is infinite. Then, G_v has a compactness cardinal if and only if there is an ω_1-strongly compact cardinal, in which case the compactness cardinal of G_v is the least ω_1-strongly compact cardinal.*

Recall that a cardinal κ is ω_1-strongly compact if any κ-complete (i.e., $<\kappa$-complete) filter can be extended to a countably complete ultrafilter. ω_1-strongly compact cardinals are rather large: clearly, the least ω_1-strongly compact cardinal is at least as large as the least measurable cardinal. It is consistent that the first measurable cardinal is the least ω_1-strongly compact cardinal (e.g., in the model of Magidor [15]), but the latter can also be larger. For more on ω_1-strongly compact cardinals, see Bagaria-Magidor [5, 6].

It follows from the theorem that it is not provable in ZFC that $G_{[0,1]}$ has a compactness cardinal and indeed the existence of such a cardinal implies the consistency of ZFC and disproves Gödel's axiom $V = L$. In contrast, it follows from Baaz-Preining-Zach [3, Theorem 5.1] that if T is a *countable* theory all of whose finite subtheories have $G_{[0,1]}$-models, then T has a $G_{[0,1]}$-model, and this is provable in ZFC. In other words, $\aleph_0$ is a *weak compactness cardinal* for $G_{[0,1]}$. Weak compactness of Gödel logics at $\aleph_0$ has also been investigated by Baaz and Zach [4], Cintula [8], and by Pourmahdian and Tavana [16].

The core of the proof of Theorem 1 is: given a κ-complete filter F, use the compactness of Gödel logic to extend F to a countably complete ultrafilter. This is done by considering a theory T made up of axioms (3.1)–(3.14) in §3 below.

Date: September 10, 2024.
1991 *Mathematics Subject Classification.* 03B52 (primary), 03C07, 03C90.
Key words and phrases. Gödel logic, fuzzy logic, compactness number, strongly compact cardinal.

Theorem 1 is part of a family of results asserting that the least ω_1-strongly compact cardinal is the compactness cardinal of various logics, including the following:

(1) $L_{\omega_1,\omega}$,
(2) L_{ω_1,ω_1},
(3) ω-logic,
(4) β-logic,
(5) $L_{\omega,\omega}(Q_{\omega_1})$,
(6) $L_{\omega,\omega}(Q_{\mathbb{R}})$.

This fact for the first four logics is due to Magidor and/or Bagaria and Magidor. Since no proof has appeared in print, it is perhaps worth saying more about the results. Many of the ideas in our proof are likely the same used to prove the corresponding result for these logics. The lower bound for $L_{\omega_1,\omega}$ follows immediately from Theorem 1 and the construction in [1, Section 4], which shows that every Gödel logic can be interpreted in $L_{\omega_1,\omega}$. It can also be proved using the characterizations of ω_1-strong compactness in Bagaria-Magidor [6] (see e.g., Theorem 6.1). The upper bound for L_{ω_1,ω_1} is easily provable by a direct argument using ultraproducts. The result for ω-logic is implicit in our proof (essentially, take T to consist of axioms (3.8)–(3.14)). The result for β-logic is immediate, since it is stronger than ω-logic and weaker than L_{ω_1,ω_1}. The fact that the compactness cardinal of $L_{\omega,\omega}(Q_{\omega_1})$ is the least ω_1-strongly compact cardinal (Q_{ω_1} is Keisler's cardinality quantifier from [14] for $\aleph_1$) can be proved by reducing $L_{\omega_1,\omega}$ to $L_{\omega,\omega}(Q_{\omega_1})$ via an omitting types argument, following Baldwin and Shelah [7] (which in turn makes use of Shelah [17] by essentially first reducing $L_{\omega_1,\omega}$ to ω-logic). It can also be proved directly by adapting part of our argument. The proof is similar to that of $L_{\omega,\omega}(Q_{\mathbb{R}})$. Here, $Q_{\mathbb{R}}$ is the binary quantifier with semantics given by

$$Q_{\mathbb{R}}xy\,\phi(x,y) \leftrightarrow \exists R \subset \mathbb{R}\left(\big\{(x,y) : \phi(x,y)\big\} \cong (R, <_{\mathbb{R}})\right).$$

That is, $Q_{\mathbb{R}}xy\,\phi(x,y)$ holds if the set of pairs (x,y) that satisfy ϕ is isomorphic to a suborder of $\mathbb{R}$. As far as we can tell, the quantifier $Q_{\mathbb{R}}$ has not been considered in the literature, but we mention it since the proof that the compactness number of $L_{\omega,\omega}(Q_{\mathbb{R}})$ is the least ω_1-strongly compact cardinal is also implicit in our argument (take T to contain axioms (3.6)–(3.14) and go from there; the upper bound is again immediate by an ultrapower argument).

1.1. **Acknowledgements.** This work was partially supported by FWF grants ESP-3N and P36837.

2. Preliminaries

Gödel logics are logics intermediate between intuitionistic logic and classical logic derived from intuitionistic logic by assuming that the set of truth values is a suborder of $\mathbb{R}$. They typically do not validate the principle of excluded middle. We consider here first-order Gödel logics G_v, which is defined with the same syntax of classical or intuitionistic first-order logic. They were introduced by Gödel [11] as part of his proof that intuitionistic logic is not finite-valued, and were axiomatized by Dummett [9]. For more background, we refer the reader to Baaz-Preining-Zach [3], Baaz-Preining [2], or Hájek [12]. A characterization of the recursively axiomatizable Gödel logics can be found in [3].

2.1. **Models.** A *model* for Gödel logic consists of

(1) a set M, the *universe* of the model;
(2) functions f^M corresponding to each function symbol f which map tuples of elements of M to elements of M;
(3) predicates P^M corresponding to each predicate symbol P which map tuples of elements of M to values in the set $[0,1]$;
(4) an assignment of elements of M to variables and constants.

A model contains an assignment of truth values $[\![P(\vec{a})]\!]$ to predicate symbols and elements of M. Given a closed $v \subset [0,1]$ containing 0 and 1, the logic G_v is obtained by restricting the possible values of atomic formulae to elements of v.

2.2. **Syntax.** First-order logic contains the binary connectives $\wedge, \vee, \rightarrow$ and the propositional constant $\perp$ (false), as well as quantifiers $\forall, \exists$. For arbitrary formulae ϕ, the truth value is defined by induction according to Figure 1. The set v is assumed to be closed as suprema and infima figure in the definition, and 0 and 1 are respectively interpreted as "false" and "true."

$$\begin{aligned}
[\![\perp]\!] &= 0 \\
[\![\varphi \wedge \psi]\!] &= \min\{[\![\varphi]\!], [\![\psi]\!]\} \\
[\![\varphi \vee \psi]\!] &= \max\{[\![\varphi]\!], [\![\psi]\!]\} \\
[\![\varphi \rightarrow \psi]\!] &= \begin{cases} 1, \text{ if } [\![\varphi]\!] \leq [\![\psi]\!], \\ [\![\psi]\!], \text{ if } [\![\varphi]\!] > [\![\psi]\!], \end{cases} \\
[\![\exists x\, \varphi(x)]\!] &= \sup\{[\![\varphi(a)]\!] : a \in M\} \\
[\![\forall x\, \varphi(x)]\!] &= \inf\{[\![\varphi(a)]\!] : a \in M\}.
\end{aligned}$$

FIGURE 1. Definition of truth values for nonatomic formulas in Gödel logic.

We introduce the following defined connectives, which will be used below in the proof:

$$\begin{aligned}
\top &:= \perp \rightarrow \perp \\
\neg\varphi &:= \varphi \rightarrow \perp \\
\varphi \leftrightarrow \psi &:= \varphi \rightarrow \psi \wedge \psi \rightarrow \varphi \\
\varphi \prec \psi &:= (\psi \rightarrow \varphi) \rightarrow \psi.
\end{aligned}$$

Note that it need not be the case that $[\![\varphi(a)]\!] = [\![\exists x\, \varphi(x)]\!]$ for any $a \in M$. We write $M \models \varphi$ if $[\![\varphi]\!]^M = 1$, in which case we say that M is a *model* of φ. Given a set of formulae Γ, we write $M \models \Gamma$ if $M \models \varphi$ for each $\varphi \in \Gamma$.

2.3. **Identity.** We assume identity is part of the language of first-order logic. We require that the following hold for all interpretations:

(1) $[\![a = b]\!] = 1$ if $a = b$, and
(2) $[\![a = b]\!] \neq 1$ if $a \neq b$.

$$
[\![\neg\varphi]\!] = \begin{cases} 1, & \text{if } [\![\varphi]\!] = 0, \\ 0, & \text{if } [\![\varphi]\!] > 0, \end{cases}
$$

$$
[\![\neg\neg\varphi]\!] = \begin{cases} 1, & \text{if } [\![\varphi]\!] > 0, \\ 0, & \text{if } [\![\varphi]\!] = 0, \end{cases}
$$

$$
[\![\varphi \leftrightarrow \psi]\!] = \begin{cases} [\![\psi]\!], & \text{if } [\![\varphi]\!] > [\![\psi]\!], \\ 1, & \text{if } [\![\varphi]\!] = [\![\psi]\!], \\ [\![\varphi]\!], & \text{if } [\![\varphi]\!] < [\![\psi]\!], \end{cases}
$$

$$
[\![\varphi \prec \psi]\!] = \begin{cases} 1, & \text{if } [\![\varphi]\!] < [\![\psi]\!] \\ [\![\psi]\!], & \text{if } [\![\varphi]\!] \geq [\![\psi]\!] \end{cases}
$$

FIGURE 2. Evaluation of defined connectives in Gödel logic.

3. PROOF OF THE THEOREM

Let us first focus on the logic $G_{[0,1]}$ for definiteness and, in particular, on showing that if κ is a compactness cardinal for $G_{[0,1]}$ then κ is ω_1-strongly compact; the other direction has an easier proof and will be treated towards the end. We do remark that the only fact about $[0, 1]$ we will use in the construction is the fact that it contains an infinite subset order-isomorphic to $\mathbb{N}$.

We shall consider a specific theory T in first-order logic consisting of sentences (3.1)–(3.14) below and use a $G_{[0,1]}$-model M of it to extend a given κ-complete filter to a countably complete ultrafilter. We describe the theory in what follows. Its vocabulary will be the set of symbols occurring in (3.1)–(3.14) and will also be described in what follows.

The vocabulary of the theory T contains a binary relation symbol $\in$. We add excluded middle for membership:

$$
\forall x\, \forall y\, \big(x \in y \vee \neg(x \in y)\big) \tag{3.1}
$$

Thus, $\in$ is crisp and behaves classically. We will use $x \notin y$ as shorthand for $\neg(x \in y)$. T also contains axioms for

$$
\mathsf{ZFC}^* \tag{3.2}
$$

for the relation symbol $\in$. Here, ZFC^* is some large enough finite fragment of ZFC, such as ZFC with Replacement restricted to Σ_{2025} formulas. Since ZFC proves the consistency of all its finite subtheories and we are working in ZFC, ZFC^* has many (classical) models, and in fact some of the form V_η, for η a cardinal. If necessary to avoid confusion, we will use $\in^M$ to refer to the interpretation of $\in$ in the (eventual) model $M \models T$. In continuing the description of T, we shall speak of objects such as $\mathbb{N}^M$ or the interval $[0, 1]^M$ as shorthand for the unique element satisfying the usual definition of $\mathbb{N}$ or $[0, 1]$, using the prediate $\in^M$. We may also simply write $\mathbb{N}$ or $[0, 1]$, but will use superscripts if necessary to avoid confusion with $\mathbb{N}^V$ and $[0, 1]^V$. Later, we will also identify natural numbers $n \in \mathbb{N}^V$ with numerals that denote them. Note that, according to T, $\mathbb{N}$ and $[0, 1]$ must exist, by ZFC^*, and $\mathbb{N}^M$ is a model of PA.

The theory T also makes use of a unary relation symbol R. It also contains the following axiom:

$$\forall x\, \forall y \left[\Big(\neg\neg R(x) \wedge \neg\neg R(y) \Big) \rightarrow \Big(\big(R(x) \leftrightarrow R(y) \big) \rightarrow x = y \Big) \right]. \tag{3.3}$$

The predicate R will be the only relation symbol for which excluded middle does not hold in T, other than identity. The axiom asserts that if R attains the same non-zero value at two arguments x, y, then they must be equal. We also have

$$\forall x \Big(x \in \mathbb{N} \leftrightarrow \neg\neg R(x) \Big), \tag{3.4}$$

which asserts that the elements of $\mathbb{N}^M$ are precisely the objects at which the predicate R attains non-zero values. We also add:

$$\forall x\, \forall y \left[\Big(\neg\neg R(x) \wedge \neg\neg R(y) \Big) \rightarrow \Big(x <_{\mathbb{N}} y \rightarrow R(x) \prec R(y) \Big) \right]. \tag{3.5}$$

Observe that $R(x) \prec R(y)$ attains value 1 if and only if $R(x)$ has strictly smaller truth value than $R(y)$ or else if both $R(x)$ and $R(y)$ have truth value 1 (see Figure 2). We have:

Claim 2. *Suppose that*

$$M = \left(M, \in^M, \mathbb{N}^M \right)$$

is a model of (3.1)–(3.5)*. Then,* $\mathbb{N}^M$ *embeds into the unit interval* $(0, 1)$*.*

Proof. The embedding is given by mapping $x \mapsto [\![R(x)]\!]$. The domain of this mapping is $\mathbb{N}^M$ by (3.4). The mapping is injective by (3.3).

Note that for no $y \in \mathbb{N}^M$ do we have $[\![R(y)]\!] = 1$, for otherwise we would have

$$\Big[\!\!\Big[\neg\neg R(y) \wedge \neg\neg R(y+1) \Big]\!\!\Big] = 1 \text{ and } \Big[\!\!\Big[y <_{\mathbb{N}} y+1 \Big]\!\!\Big] = 1,$$

but by (3.3) we cannot have $[\![R(y)]\!] = [\![R(y+1)]\!]$ and hence we would have

$$[\![R(y+1)]\!] < [\![R(y)]\!] = 1,$$

which would force (3.5) to have truth value < 1. Thus, the range of the mapping is contained in $(0, 1)$.

According to (3.5), for all $x, y \in \mathbb{N}^M$, $x <_{\mathbb{N}}^M y$ implies that $[\![R(x)]\!] < [\![R(y)]\!]$, since $[\![R(y)]\!] = 1$ is impossible. Hence, the mapping is strictly order-preserving, and thus an embedding, as desired. $\square$

We add to the vocabulary continuum-many unary function symbols $\{f_x\}_{x \in (2^{\mathbb{N}})^V}$. We add an axiom

$$f_x : \mathbb{N}^M \rightarrow \mathbb{N}^M \tag{3.6}$$

for each $x \in (2^{\mathbb{N}})^V$. (This axiom is expressed using membership $\in$.)

Moreover, whenever $x, y \in 2^{\mathbb{N}}$ and $n \in \mathbb{N}$ are such that $x \upharpoonright n \neq y \upharpoonright n$, we add the axiom:

$$\forall k \in \mathbb{N}^M \left(n <_{\mathbb{N}} k \rightarrow f_x(k) \neq f_y(k) \right). \tag{3.7}$$

Here, (3.7) is expressed using the numeral n, which in turn is expressed using membership $\in$. Similarly, the inequality $n <_{\mathbb{N}} k$ is expressed using membership.

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Claim 3. *Suppose that*

$$M = \big(M, \in^M, \mathbb{N}^M, \{f_x^M : x \in 2^{\mathbb{N}}\}\big)$$

is a model of (3.1)–(3.7). *Then, M is an ω-model.*

Proof. Suppose otherwise and let $k \in \mathbb{N}^M$ be non-standard. Let $x, y \in (2^{\mathbb{N}})^V$ be different. Then, there is some standard $n \in \mathbb{N}$ such that $x \restriction n \neq y \restriction n$ and so by (3.7), we have $f_x(k) \neq f_y(k)$. By (3.6), it follows that

$$|\mathbb{N}^M| \geq 2^{\aleph_0}.$$

In particular, $\mathbb{N}^M$ is uncountable. By Claim 2, $\mathbb{N}^M$ embeds into $(0,1)$. However, it is a theorem of Smoryński [18] that no uncountable model of PA embeds into $\mathbb{R}$. (To see this, suppose otherwise and let $e : \mathbb{N}^M \to \mathbb{R}^V$ be an order-preserving embedding. For each $m \in \mathbb{N}^M$, let

$$I_m = \big(e(m), e(m+1)\big).$$

Then, $\{I_m : m \in \mathbb{N}^M\}$ is an uncountable family of pairwise disjoint nontrivial intervals of real numbers, which is absurd). Since clearly $\mathbb{N}^M \models \mathsf{PA}$, this is a contradiction, so the claim is proved. □

Now, let F be a κ-complete filter on some set. Without loss of generality, let us assume that F is a filter on some cardinal λ with $\kappa \leq \lambda$. We add to T a family of 2^λ-many unary predicates $\dot{A}$, one for each $A \subset \lambda$. We also add a constant c. We add axioms:

$$\dot{A}(c) \vee \neg \dot{A}(c), \text{ whenever } A \subset \lambda \tag{3.8}$$

$$\dot{A}(c), \text{ whenever } A \in F \tag{3.9}$$

$$\dot{A}(c) \leftrightarrow \neg \dot{B}(c), \text{ whenever } A = \lambda \setminus B \tag{3.10}$$

$$\dot{A}(c) \wedge \dot{B}(c) \leftrightarrow \dot{C}(c), \text{ whenever } C = A \cap B \subset \lambda \tag{3.11}$$

$$\dot{A}(c) \to \dot{B}(c), \text{ whenever } A \subset B \subset \lambda. \tag{3.12}$$

In addition, for each ω-sequence $s : \mathbb{N} \to \mathcal{P}(\lambda)$ of subsets of λ, we add a binary relation symbol $\dot{A}^s$ and axioms:

$$\dot{A}^s(i, c) \leftrightarrow \dot{A}(c), \text{ whenever } A = s(i) \tag{3.13}$$

$$\forall x \Big(x \in \mathbb{N}^M \to \dot{A}^s(x, c)\Big) \leftrightarrow \dot{A}(c), \text{ whenever } A = \bigcap_{i \in \mathbb{N}} s(i). \tag{3.14}$$

As before, (3.13) is expressed using the numeral i.

Claim 4. *Suppose that*

$$M = \Big(M, \in^M, \mathbb{N}^M, \big\{f_x^M : x \in 2^{\mathbb{N}}\big\}, \big\{\dot{A}^M : A \subset \lambda\big\}, \big\{(\dot{A}^s)^M : s \in \mathcal{P}(\lambda)^{\mathbb{N}}\big\}, c^M\Big)$$

is a model of (3.1)–(3.14). *Let*

$$U = \Big\{A \subset \lambda : M \models \dot{A}(c)\Big\}.$$

Then, U is an ω_1-complete ultrafilter extending F.

Proof. U extends F by (3.9). It is a filter by (3.11) and (3.12). Moreover, it is an ultrafilter by (3.8) and (3.10). It remains to verify that it is ω_1-complete. Let $s = \{A_i : i \in \mathbb{N}\}$ be an ω-sequence of subsets of λ in U. By (3.13), we have

$$M \models \dot{A}^s(i, c)$$

for all $i \in \mathbb{N}$. By Claim 3, M is an ω-model, so we have

$$M \models \forall x \left(x \in \mathbb{N} \to \dot{A}^s(x, c) \right).$$

By (3.14), we then have

$$M \models \dot{A}(c),$$

where $A = \bigcap_{i \in \mathbb{N}} A_i$, so $A \in U$. This proves the claim. □

In order to complete the proof of the theorem, it remains to show:

Claim 5. *Suppose $T' \subset T$ has cardinality $\gamma < \kappa$. Then, T' has a $G_{[0,1]}$-model.*

Proof. We will in fact find a model M of all sentences (3.1)–(3.8) and (3.10)–(3.14), together with any collection of size $\gamma < \kappa$ of sentences of the form (3.9). We begin the description of the model by setting $(M, \in^M) = (V_\eta, \in)$, where η is large enough so that $\lambda < \eta$ and $V_\eta \models \mathsf{ZFC}^*$. For each $n \in \mathbb{N}$, we let

$$[\![R(n)]\!] = 1 - \frac{1}{2+n}. \tag{3.15}$$

For elements $x \in V_\eta \setminus \mathbb{N}$, we put $[\![R(x)]\!] = 0$. Identity is defined as usual for all x which are not natural numbers. For $n, m \in \mathbb{N}$, we put $[\![n = m]\!] = 1$ if $n = m$ and otherwise we put

$$[\![n = m]\!] = \min \left\{ 1 - \frac{1}{2+n}, 1 - \frac{1}{2+m} \right\}.$$

Axioms (3.1)–(3.5) are thus satisfied. In particular, note that $(M, \in)$ satisfies extensionality: if two sets $x, y \in M$ have the same elements (this is an antecedent with value in $\{0, 1\}$), then we really have $x = y$ and thus $[\![x = y]\!] = 1$. Perhaps the only axiom that needs further arguing is (3.3) for elements $n, m \in \mathbb{N}$, in which case the antecedent of the implication attains truth value 1. If $n = m$, then the conclusion trivially attains truth value 1, so the implication is true. Otherwise, by (3.15) we have

$$[\![R(n)]\!] \neq [\![R(m)]\!],$$

so according to Figure 2, we have

$$\begin{aligned} \left[\!\!\left[R(n) \leftrightarrow R(m) \right]\!\!\right] &= \min \left\{ \left[\!\!\left[R(n) \right]\!\!\right], \left[\!\!\left[R(m) \right]\!\!\right] \right\} \\ &= \min \left\{ 1 - \frac{1}{2+n}, 1 - \frac{1}{2+m} \right\} \\ &= [\![n = m]\!], \end{aligned}$$

so the implication is true.

For the functions f_x, notice that for each $k \in \mathbb{N}$ there are only 2^k different sequences of the form $x \upharpoonright k$, for $x \in 2^{\mathbb{N}}$. Thus, we define all f_x by induction on k so that $f_x(k)$ depends only on $x \upharpoonright k$ and for each distinct sequence $x \upharpoonright k$, $f_x(k)$ takes a different value in

$$\{2^k, 2^k + 1, \dots, 2^{k+1} - 1\}.$$

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Thus axioms (3.6)–(3.7) are satisfied.

For each $A \subset \lambda$, we interpret $\dot{A}$ as A. Fix any collection $\mathcal{A} = \{A_\iota : \iota < \gamma\}$ of sets in F, where $\gamma < \kappa$. Since F is κ-complete, there is some ordinal $\delta < \lambda$ with

$$\delta \in \bigcap_{\iota<\gamma} A_\iota.$$

We interpret the constant c as δ. The model then satisfies axiom (3.8) and axioms (3.10)–(3.12), as well as all axioms of the form (3.9) given by the γ-many sets A_ι from the family $\mathcal{A}$.

Finally, for each ω-sequence $s : \mathbb{N} \to \mathcal{P}(\lambda)$, we interpret $\dot{A}^s$ as the graph of s. Then, all axioms (3.13)–(3.14) are satisfied. This completes the proof of the claim. □

With Claim 4 and Claim 5, one direction of the proof of the theorem is complete. For the other direction, we simply observe that by the argument in [1, Section 4], every Gödel logic G_v can be interpreted in $L_{\omega_1,\omega}$. Since the compactness number of $L_{\omega_1,\omega}$ is the least ω_1-strongly compact cardinal (see the proof of [10, Theorem 5.4]), this is an upper bound for that of G_v, for any set of truth values $v \subset [0,1]$. (This only needs the easy direction: that if κ is ω_1-strongly compact, then $L_{\omega_1,\omega}$ is κ-compact, and this can be proved directly using the usual ultrapower argument.)

Let us now indicate how to modify the proof given above for other sets of truth values $v \subset [0,1]$. Note that the fact that $v = [0,1]$ was not used at any point in the proof other than in Claim 5. It is used in Claim 5 in order to ensure that the model constructed satisfies axiom (3.5), since $\mathbb{N}^V$ needs to embed into the set of truth values, for which it suffices that v contain a subset order-isomorphic to $\mathbb{N}$ (which in our case was the set of all numbers of the form

$$1 - \frac{1}{2+n},$$

and these served as the truth values of formulas $R(n)$ for $n \in \mathbb{N}^M$ and for identities $n = m$ between natural numbers). If v is infinite, it must contain a subset order-isomorphic to one of $\mathbb{N}$ or $\mathbb{Z}_{\leq 0}$. In the second case, we modify axiom (3.5) to the following "dual" form:

$$\forall x \, \forall y \left[\Big(\neg\neg R(x) \wedge \neg\neg R(y) \Big) \to \Big(y <_{\mathbb{N}} x \to R(x) \prec R(y) \Big) \right].$$

With this modification, everything works.

Finally, if $v \subset [0,1]$ is a finite set of truth values, say $|v| = n$, then given a first-order theory T, let T' be the theory containing the following axioms:

(1) ZFC^*,
(2) $\dot{M}$ is an n-valued model in the Gödel logic G_v,
(3) $\dot{M} \models \phi$, for each $\phi \in T$.

If T is finitely satisfiable, then so is T'. If so, by the compactness theorem for classical logic, T' has a model, and the interpretation of $\dot{M}$ in this model will be a model of T.

Remark 6. The usual arguments (see e.g., Jech [13]) show that if $G_{v,\kappa}$ is any Gödel logic augmented with infinitary connectives of length $\eta < \kappa$ for some κ, then κ is

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the compactness cardinal of $G_{v,\kappa}$ if and only if κ is strongly compact. Similarly, κ is a weak compactness cardinal for $G_{v,\kappa}$ if and only if κ is weakly compact. In $G_{v,\kappa}$, the truth values of infinite conjunctions and disjunctions are given by infima and suprema, respectively.

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https://doi.org/10.1017/jsl.2024.87 Published online by Cambridge University Press