Glasgow Mathematical Journal (2024), 1–14
doi:10.1017/S0017089524000235

Glasgow Mathematical Journal

RESEARCH ARTICLE

On a weighted anisotropic eigenvalue problem

Nunzia Gavitone[1] and Rossano Sannipoli[2]

[1]Dipartimento di Matematica e Applicazioni "R. Caccioppoli". Università degli Studi di Napoli Federico II, 80126, Napoli, Italy
[2]Dipartimento di Matematica "Tullio Levi-Civita", Universitá degli Studi di Padova, 35131, Padova, Italy
Corresponding author: Nunzia Gavitone; Email: nunzia.gavitone@unina.it

Received: 30 January 2024; **Revised:** 15 May 2024; **Accepted:** 8 July 2024

Keywords: Anisotropic p-Laplacian; Comparison results; Reverse Hölder inequality

2020 Mathematics Subject Classification: *Primary* - 35B51; *Secondary* - 35J62

Abstract
In this paper, we deal with a weighted eigenvalue problem for the anisotropic (p, q)-Laplacian with Dirichlet boundary conditions. We study the main properties of the first eigenvalue and a reverse Hölder type inequality for the corresponding eigenfunctions.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, be an open, bounded, and connected set and let p, q be such that $1 < p$, and $1 < q < p^*$, where $p^* = np/(n-p)$ if $p < n$ and $p^* = \infty$ if $p \geqslant n$. In this paper, we study the following variational problem:

$$\lambda^H_{p,q}(\Omega) = \inf_{\substack{u \in W_0^{1,p}(\Omega),\\ u \neq 0}} \frac{\displaystyle\int_\Omega H(\nabla u)^p\, dx}{\left(\displaystyle\int_\Omega m|u|^q\, dx\right)^{\frac{p}{q}}}, \tag{1.1}$$

where $m \in L^\infty(\Omega)$ is a positive function and $H : \mathbb{R}^n \to [0, +\infty]$ is a $C^1(\mathbb{R}^n \setminus \{0\})$ convex and positively 1-homogeneous function (see Section 2 for more details).
Obviously, $\lambda^H_{p,q}(\Omega)$ depends also on m, but to simplify the notation we will omit its dependence.

The Euler–Lagrange equation associated with the minimization problem (1.1) is the following weighted eigenvalue problem for the anisotropic $(p, q)-$Laplace operator with Dirichlet boundary condition:

$$\begin{cases} -\mathcal{L}_p(u) = \lambda m(x)\|u\|^{p-q}_{q,m}|u|^{q-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\|u\|_{q,m} = \|u\|_{L^q(\Omega,m)}$ is the weighted Lebesgue norm of u and $\mathcal{L}_p$ is the so-called anisotropic p-Laplacian operator defined as follows:

$$\mathcal{L}_p(u) = \operatorname{div}(H(\nabla u)^{p-1}H_\xi(\nabla u)). \tag{1.3}$$

We stress that when $p = q$ and $m(x) \equiv 1$, (1.1) is the first eigenvalue $\lambda^H_p(\Omega)$ of the anisotropic p-Laplacian, and it has been studied by many authors (see for instance [9, 19] and the references therein). In particular, in [9], it is proved that $\lambda^H_p(\Omega)$ is simple for any p, the corresponding eigenfunctions have a sign, and that a suitable Faber–Krahn inequality holds.

When $H = \mathcal{E}$ is the usual Euclidean norm, $\mathcal{L}_p(u)$ is the well-known p-Laplace operator and the eigenvalue problem (1.2) reduces to the following:

$$\begin{cases} -\Delta_p u = \lambda \|u\|_{q,m}^{p-q} |u|^{q-2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

The spectrum of (1.4) and the first eigenvalue $\lambda_{p,q}^{\mathcal{E}}(\Omega)$, when $p \neq q$ and $m \equiv 1$, have been studied for instance in the case $p = 2$ in [11], for any p in [25, 26, 34, 35] and in [20] where the authors study also the weighted case. It is known that $\lambda_{p,q}^{\mathcal{E}}(\Omega)$ is not simple, in general, for any $1 < q < p^*$. Indeed in [27], the authors prove the simplicity for any $1 < q \leqslant p$, while for $p < q < p^*$, $\lambda_{p,q}^{\mathcal{E}}(\Omega)$ could not be necessary simple. This fact has been observed for instance in [11], in the case $p = 2$, and for any p in [32] and [29], where the authors prove that for every fixed $p < q < p^*$ the simplicity fails if Ω is a sufficiently thin spherical shell.

In this paper, we study the main properties of $\lambda_{p,q}^{H}(\Omega)$ and of the corresponding eigenfunctions. In particular, our aim is to prove a reverse Hölder inequality for them.

In the Euclidean case, if $p = q = 2$ and $m \equiv 1$, in [15, 16], Chiti proves the following inequality for the first eigenfunctions v corresponding to the first Dirichlet eigenvalue of the Laplace operator $\lambda_2^{\mathcal{E}}(\Omega) \equiv \lambda(\Omega)$:

$$\|v\|_{L^r(\Omega)} \leqslant C(r, s, n, \lambda(\Omega)) \|v\|_{L^s(\Omega)}, \quad 0 < s < r, \tag{1.5}$$

and the equality case is achieved if and only if Ω is a ball and $v = v^\sharp$, where the symbol "$\sharp$" denotes the Schwarz symmetral of a function (see [30]). In [3], the authors prove (1.5) for the first eigenfunctions of the p-Laplacian. Moreover, in [1], the authors extend the result to the weighted case, and the inequality reads as follows:

$$\|v\|_{L^r(\Omega,m)} \leqslant C(p, r, s, n, \lambda_p^{\mathcal{E}}(\Omega)) \|v\|_{L^s(\Omega,m)}, \quad 0 < s < r. \tag{1.6}$$

The equality sign holds if and only if Ω is a ball, $v = v^\sharp$ and $m = m^\sharp$, modulo translation. In the general case $p \neq q$, in the Euclidean case, a Chiti type inequality is proved in the case $m \equiv 1$ in [14] and [13] when $p = 2$ and for any p, respectively. More precisely in [13], the authors prove the following inequality:

$$\|v\|_{L^r(\Omega)} \leqslant C(p, q, r, n, \lambda_{p,q}^{\mathcal{E}}(\Omega)) \|v\|_{L^q(\Omega)}, \quad q < r. \tag{1.7}$$

Even in this case, the equality sign holds if and only if Ω is a ball and $v = v^\sharp$, modulo translation. The goal of this paper is to prove a Chiti type inequality in the spirit of (1.6) and (1.7) for the first eigenfunctions of the general weighted eigenvalue problem (1.2). We recall that, when $p = q$ and $m \equiv 1$, the result in the anisotropic setting has been proved in [9]. Our main theorem is the following.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded, and connected set. Let $1 < q \leqslant p$, and let u be an eigenfunction corresponding to the first eigenvalue (1.1). Then the following statements hold*

i) There exists a constant $C = C(p, q, r, n, \lambda_{p,q}^{H}(\Omega))$ such that

$$\|u\|_{L^r(\Omega,m)} \leqslant C \ \|u\|_{L^q(\Omega,m)}, \quad q \leqslant r; \tag{1.8}$$

ii) There exists a constant $C = C(p, q, r, n, \lambda_{p,q}^{H}(\Omega))$ such that

$$\|u\|_{L^\infty(\Omega)} \leqslant C \|u\|_{L^r(\Omega,m)} \quad 1 \leqslant r < \infty. \tag{1.9}$$

The equality cases hold if and only if Ω is a Wulff shape and u and m coincide, that is, in Ω with their convex symmetrization, modulo translation.

We stress that this result gives, in particular, a Chiti type inequality for the eigenfunctions corresponding to the first weighted eigenvalue of the anisotropic p-Laplacian and extend (1.7) to the weighted case.

https://doi.org/10.1017/S0017089524000235 Published online by Cambridge University Press

The proof is based on symmetrization techniques and a comparison between the eigenfunctions corresponding to the first eigenvalue (1.1) and the first eigenfunctions of a suitable symmetrical eigenvalue problem.

The structure of the paper is the following. In Section 2, we fix some notation, recall some basic properties of the Finsler norms, and give a brief overview about convex symmetrization. In Section 3, we study the main properties of $\lambda^H_{p,q}(\Omega)$ and a Faber–Krahn type inequality. In the last section, we prove Theorem 1.1 by using symmetrization arguments.

2. Notations and preliminaries

Throughout this article, $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^n$, while $\cdot$ is the standard Euclidean scalar product for $n \geqslant 2$. Moreover, we denote by $|\Omega|$ the Lebesgue measure of $\Omega \subseteq \mathbb{R}^n$, by B_R the Euclidean ball centered at the origin with radius R and by ω_n the measure of the unit ball.

Let $E \subseteq \mathbb{R}^n$ be a bounded, open set and $\Omega \subseteq \mathbb{R}^n$ be a measurable set. We recall now the definition of the perimeter of Ω in E in the sense of De Giorgi, that is,

$$P(\Omega; E) = \sup\left\{ \int_\Omega \operatorname{div}\varphi \, dx : \varphi \in C_c^\infty(E;\mathbb{R}^n), \ ||\varphi||_\infty \leq 1 \right\}.$$

The perimeter of Ω in $\mathbb{R}^n$ will be denoted by $P(\Omega)$ and, if $P(\Omega) < \infty$, we say that Ω is a set of finite perimeter. Some results relative to the sets of finite perimeter are contained, for instance, in [31]. Moreover, if Ω has Lipschitz boundary, we have that

$$P(\Omega) = \mathcal{H}^{n-1}(\partial\Omega).$$

2.1. The anisotropic norm

Let $H : \mathbb{R}^n \longrightarrow [0, +\infty[$, $n \geqslant 2$, be a $C^1(\mathbb{R}^n \setminus \{0\})$ convex function which is positively 1-homogeneous, that is,

$$H(t\xi) = |t|H(\xi) \quad \forall \xi \in \mathbb{R}^n, \ \forall t \in \mathbb{R}. \tag{2.1}$$

Moreover, let $0 < \gamma \leqslant \delta$ be positive constants such that

$$\gamma|\xi| \leqslant H(\xi) \leqslant \delta|\xi|. \tag{2.2}$$

These properties guarantee that H is a norm in $\mathbb{R}^n$. Indeed by (2.2), we have that $H(\xi) = 0$ if and only if $\xi = 0$. It is homogeneous by (2.1) and the triangular inequality is a consequence of the convexity of the function H: if $\xi, \eta \in \mathbb{R}^n$, then

$$\frac{H(x+y)}{2} = H\left(\frac{x}{2} + \frac{y}{2}\right) \leqslant \frac{H(x)}{2} + \frac{H(y)}{2}.$$

Because of (2.1), we can assume that the set

$$K = \{\xi \in \mathbb{R}^n : H(\xi) \leqslant 1\}$$

is such that $|K| = \omega_n$, where ω_n is the measure of the unit sphere in $\mathbb{R}^n$. We can define the support function of K as:

$$H^\circ(x) = \sup_{\xi \in K} \langle x, \xi \rangle, \tag{2.3}$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^n$. $H^\circ : \mathbb{R}^n \longrightarrow [0, +\infty]$ is a convex, homogeneous function in the sense of (2.1). Moreover, H and H° are polar to each other, in the sense that

$$H^\circ(x) = \sup_{\xi \neq 0} \frac{\langle x, \xi \rangle}{H(\xi)}$$

https://doi.org/10.1017/S0017089524000235 Published online by Cambridge University Press

and

$$H(x) = \sup_{\xi \neq 0} \frac{\langle x, \xi \rangle}{H^\circ(\xi)}.$$

H is the support function of the set:

$$K^\circ = \{x \in \mathbb{R}^n : H^\circ(x) \leqslant 1\}.$$

The set $\mathcal{W} = \{x \in \mathbb{R}^n : H^\circ(x) < 1\}$ is the so-called Wulff shape centered at the origin. We set $k_n = |\mathcal{W}|$. More generally, we will denote by $\mathcal{W}_R(x_0)$ the Wulff shape centered in $x_0 \in \mathbb{R}^n$ the set $R\mathcal{W} + x_0$, and $\mathcal{W}_R(0) = \mathcal{W}_R$.
The following properties hold for H and H°:

$$H_\xi(\xi) \cdot \xi = H(\xi), \quad H^\circ_\xi(\xi) \cdot \xi = H^\circ(\xi), \tag{2.4}$$

$$H(H^\circ_\xi(\xi)) = H^\circ(H_\xi(\xi)) = 1 \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}, \tag{2.5}$$

$$H^\circ(\xi) H_\xi(H^\circ_\xi(\xi)) = H(\xi) H^\circ_\xi(H_\xi(\xi)) = \xi \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}. \tag{2.6}$$

An example of an anisotropic norm that satisfies the above-mentioned properties is the following. Let $r \in (1, +\infty)$ and let us consider

$$H(\xi) = \left(\sum_{i=1}^n |\xi_i|^r \right)^{\frac{1}{r}},$$

known in literature also as r-norm. With this choice, the highly nonlinear operator $\mathcal{L}_p(u)$, defined in (1.3), becomes

$$\mathcal{L}_p(u) = \sum_{k=1}^n \frac{\partial}{\partial x_k} \left(\left(\sum_{i=1}^n \left| \frac{\partial u}{\partial x_i} \right|^r \right)^{\frac{p-r}{r}} \left| \frac{\partial u}{\partial x_k} \right|^{r-2} \frac{\partial u}{\partial x_k} \right).$$

We stress that for $r = p$, $\mathcal{L}_p(u)$ is the so called pseudo-p-Laplace operator. Examples of non-smooth anisotropic norm can be found in [8] and references therein, where the authors consider a crystalline anisotropy and the associated Wulff shape is a polyhedron.

If $E \subset \mathbb{R}^n$ is an open, bounded set with Lipschitz boundary and Ω is an open subset of $\mathbb{R}^n$, we can give a generalized definition of perimeter of Ω with respect to the anisotropic norm as follows (see for instance [6]):

$$P_H(\Omega, E) = \int_{\partial^* \Omega \cap E} H(\nu)\, d\mathcal{H}^{n-1},$$

where $\partial^* \Omega$ is the reduced boundary of Ω (for the definition see [21]), ν is its Euclidean outer normal, and $\mathcal{H}^{n-1}$ is the $(n-1)-$dimensional Hausdorff measure in $\mathbb{R}^n$. Clearly, if E is open, bounded and Lipschitz, then the outer unit normal exists almost everywhere and

$$P_H(E, \mathbb{R}^n) := P_H(E) = \int_{\partial E} H(\nu)\, d\mathcal{H}^{n-1}. \tag{2.7}$$

By (2.2), we have that

$$\gamma P(E) \leqslant P_H(E) \leqslant \delta P(E).$$

In [5], it is shown that if $u \in W^{1,1}(\Omega)$, then for, that is, $t > 0$

$$-\frac{d}{dt} \int_{\{u>t\}} H(\nabla u)\, dx = P_H(\{u > t\}, \Omega) = \int_{\partial^* \{u>t\} \cap \Omega} \frac{H(\nabla u)}{|\nabla u|}\, d\mathcal{H}^{n-1}. \tag{2.8}$$

Moreover, an isoperimetric inequality for the anisotropic perimeter holds (for instance see [2, 12, 18, 23, 24])

$$P_H(E) \geqslant n k_n^{\frac{1}{n}} |E|^{1-\frac{1}{n}}. \tag{2.9}$$

2.2. Convex symmetrization

Let $\Omega \subset \mathbb{R}^n$ be an open, bounded, and connected set. Let $f:\Omega \longrightarrow [0, +\infty]$ be a measurable function. The decreasing rearrangement f^* of f is defined as follows:

$$f^*(s) = \inf\{t \geqslant 0 : \mu(t) < s\} \quad s \in [0, |\Omega|],$$

where

$$\mu(t) = |\{x \in \Omega : |f(x)| > t\}|,$$

is the distribution function of f. We recall that the Schwarz symmetrand of f is a radially spherically function defined as follows:

$$f^\sharp(x) = f^*(\omega_n |x|^n) \qquad x \in \Omega^\sharp.$$

where $\Omega^\sharp$ is the ball centered at the origin such that $|\Omega^\sharp| = |\Omega|$. The convex symmetrization $f^\star$ of f, instead, is a function symmetric with respect to H° defined as follows:

$$f^\star(x) = f^*(k_n(H^\circ(x))^n) \qquad x \in \Omega^\star,$$

where $\Omega^\star$ is a Wulff shape centered at the origin and such that $|\Omega^\star| = |\Omega|$ (see [2]). We stress that both $f^\star$ and $f^\sharp$ are defined by means the decreasing rearrangement f^*, but they have different symmetry. In particular, it is well known that the functions $f, f^*, f^\sharp$, and $f^\star$ are equimeasurable, that is,

$$|\{f > t\}| = |\{f^\sharp > t\}| = |\{f^* > t\}| = |\{f^\star > t\}| \quad t \geqslant 0.$$

As a consequence, if $f \in L^p(\Omega)$, $p \geqslant 1$, then

$$\|f\|_{L^p(\Omega)}\| = \|f^\sharp\|_{L^p(\Omega^\sharp)} = \|f^*\|_{L^p([0,|\Omega|])} = \|f^\star\|_{L^p(\Omega^\star)}. \tag{2.10}$$

Regarding the norm of the gradient, a generalized version of the well-known Pólya–Szegö inequality holds and it states (see for instance [2])

Theorem 2.1. (Pólya–Szegö principle). *If $w \in W_0^{1,p}(\Omega)$ for $p \geqslant 1$, then we have that*

$$\int_\Omega H(\nabla u)^p \, dx \geqslant \int_{\Omega^\star} H(\nabla u^\star)^p \, dx.$$

where $\Omega^\star$ is the Wulff Shape such that $|\Omega^\star| = |\Omega|$.

For the sake of completeness, we will state the result concerning the equality case of the Pólya–Szegö inequality, whose proof is contained in [22] for the generic anisotropic case and in [38] for the Euclidean case.

Theorem 2.2. *Let u be a non-negative function in $W^{1,p}(\mathbb{R}^n)$, for $1 < p < +\infty$, such that*

$$|\{|\nabla u^\star| = 0\} \cap \{0 < u^\star < \operatorname{ess\,sup} u\}| = 0.$$

Then

$$\int_{\mathbb{R}^n} H(\nabla u)^p \, dx = \int_{\mathbb{R}^n} H(\nabla u^\star)^p \, dx$$

if and only if $u = u^\star$ a.e. in $\mathbb{R}^n$, up to translations.

Obviously, Theorem 2.2 can holds true in the case of a $W_0^{1,p}(\Omega)$ function.
We conclude this section by recalling some known properties about rearrangements that we will use in the proof of the main theorem. The following result is the well-known Hardy–Littlewood inequality (see [30]):

$$\int_\Omega |f(x)g(x)| \, dx \leqslant \int_0^{|\Omega|} f^*(s)g^*(s) \, ds. \tag{2.11}$$

https://doi.org/10.1017/S0017089524000235 Published online by Cambridge University Press

So, if we consider g as the characteristic function of the set $\{x \in \Omega : u(x) > t\}$, for some measurable function $u : \Omega \to \mathbb{R}$ and $t \geqslant 0$, then we get

$$\int_{\{u>t\}} f(x)\, dx \leqslant \int_0^{\mu(t)} f^*(s)\, ds, \tag{2.12}$$

where, again, $\mu(t)$ is the distribution function of u. Finally, we recall the definition of dominated rearrangements (see for instance [4] and [17]).

Definition 2.3. *Let $f, g \in L^1(\Omega)$ be nonnegative functions. We say that g is dominated by f and write $g \prec f$ if the following two statements hold*

(i) $\displaystyle\int_0^s g^*(t)\, dt \leqslant \int_0^s f^*(t)\, dt;$

(ii) $\displaystyle\int_0^{|\Omega|} g^*(t)\, dt = \int_0^{|\Omega|} f^*(t)\, dt.$

In [4], the authors prove the following result:

Proposition 2.4. *Let f, g, h be positive and such that $hf, hg \in L^1(\Omega)$. Let F be a convex, nonnegative function such that $F(0) = 0$. If $hg \prec hf$ Then*

$$\int_0^{|\Omega|} h^* F(g^*)\, dt \leqslant \int_0^{|\Omega|} h^* F(f^*)\, dt.$$

Moreover, if F is strictly convex, the equality holds if and only if $f^* \equiv g^*$, that is, in $[0, |\Omega|]$.

3. The (p,q)-anisotropic Laplacian

In this section, we study the main properties of (1.1) and the corresponding minimizers. Let $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$ be an open, bounded, and connected set. Let $m \in L^\infty(\Omega)$ be a positive function and p, q be such that $1 < p < \infty$ and $1 < q < p^*$, where $p^* = np/(n-p)$, if $p < n$, and $p^* = \infty$, if $p \geqslant n$. A function $v \in W_0^{1,p}(\Omega)$ is a weak solution to the problem (1.2) corresponding to λ if

$$\int_\Omega (H(\nabla v))^{p-1} H_\xi(\nabla v) \cdot \nabla \varphi\, dx = \lambda \|v\|_{q,m}^{p-q} \int_\Omega m(x)\, |v|^{q-2}\, v\, \varphi\, dx, \tag{3.1}$$

for every $\varphi \in W_0^{1,p}(\Omega)$. By standard argument of calculus of variations, it is not difficult to prove the following result:

Theorem 3.1. *Let $n \geqslant 2$ and $\Omega \subset \mathbb{R}^n$, be an open, bounded set and let p, q and m be as above. Then $\lambda^H_{p,q}(\Omega)$, defined in (1.1), is strictly positive and actually a minimum. Moreover, any minimizer is a weak solution to the problem (1.2), with $\lambda = \lambda^H_{p,q}(\Omega)$, and has constant sign on every connected component.*

As regard the simplicity, we have

Theorem 3.2. *Let $n \geqslant 2$ and $\Omega \subset \mathbb{R}^n$, be an open, bounded, and connected set and let p and m be as above and let $1 < q \leqslant p$. Then $\lambda^H_{p,q}(\Omega)$ is simple, that is, there exists a unique corresponding eigenfunction up to multiplicative constants.*

https://doi.org/10.1017/S0017089524000235 Published online by Cambridge University Press

The proof of the previous result is contained in [29], where the authors consider a more general class of quasilinear operators. We stress that this result was already proved when $H=\mathcal{E}$ and $m\equiv 1$ in the paper [28] and in the case of a positive and essentially bounded weight in [33]. Finally, we have the following:

Theorem 3.3. *Let $n\geqslant 2$ and let $\Omega\subset\mathbb{R}^n$ be an open, bounded, and connected set. Let p and m be as above and let $1<q\leqslant p$. Any nonnegative function $v\in W_0^{1,p}(\Omega)$, which is a weak solution to the problem (1.2), for some $\lambda>0$, is a first eigenfunction, that is $\lambda=\lambda^H_{p,q}(\Omega)$.*

Proof. The proof is similar to the one contained in [10, Theorem 5.1], and it follows standard arguments and a general Picone inequality. For the reader convenience and sake of completeness, we write the main steps. Let v be a non-negative weak solution to the problem (1.2) corresponding to λ. By the strong maximum principle, we have that $v>0$ in Ω. Let u be the first positive eigenfunction corresponding to $\lambda^H_{p,q}(\Omega)$ such that

$$\|u\|_{L^q(\Omega,m)}=\|v\|_{L^q(\Omega,m)}. \tag{3.2}$$

Then,

$$\int_\Omega (H(\nabla u))^p\,dx=\lambda^H_{p,q}(\Omega)\left(\int_\Omega m(x)\,u^q\,dx\right)^{\frac{p}{q}}. \tag{3.3}$$

Being v a weak positive solution to (1.2) corresponding to λ, we can chose $\varphi=\dfrac{u^q}{v^{q-1}}$ as test function in (3.1) obtaining

$$\begin{aligned}\int_\Omega ((H(\nabla v))^{p-1}\,H_\xi(\nabla v)\cdot\nabla\left(\frac{u^q}{v^{q-1}}\right)\,dx&=\lambda\|m^{\frac{1}{q}}v\|_q^{p-q}\int_\Omega m(x)\,u^q\,dx\\&=\lambda\left(\int_\Omega m(x)\,u^q\,dx\right)^{\frac{p}{q}},\end{aligned} \tag{3.4}$$

where last equality follows by (3.2). In the left-hand side, we can apply the general Picone inequality (see Proposition 2.9 in [10]) and we have

$$\int_\Omega (H(\nabla v))^q\,(H(\nabla u))^{p-q}\,dx\geqslant\lambda\left(\int_\Omega m(x)\,u^q\,dx\right)^{\frac{p}{q}}.$$

By the Hölder inequality, the normalization (3.2) and (3.3) we get that $\lambda^H_{p,q}(\Omega)\geqslant\lambda$, that implies $u=v$. □

3.1. The case $\Omega=\mathcal{W}_R$

In this subsection, we study the problem (1.2) when Ω is a Wulff shape. In this case, the eigenfunctions inherit some symmetry properties. Let be $\Omega=\mathcal{W}_R$ and let $m\in L^\infty(\mathcal{W}_R)$ be a positive function such that $m(x)=m^\star(x)$. Then problem (1.2) becomes

$$\begin{cases}-\mathcal{L}_p(v)=\lambda m^\star(x)\|v\|^{p-q}_{q,m^\star}|v|^{q-2}v & \text{in } \mathcal{W}_R\\ v=0 & \text{on } \partial\mathcal{W}_R.\end{cases} \tag{3.5}$$

The following result holds

Proposition 3.4. *Let $1<p<\infty$ and $1<q\leqslant p$. Let $v\in C^1(\overline{\Omega})\cap C^{1,\alpha}(\Omega)$ be a first positive eigenfunction to the problem (3.5). Then there exists a decreasing function $\rho(r)$, $r\in[0,R]$, such that $\rho\in C^\infty((0,R))\cap C^1([0,R])$, $\rho'(0)=0$, and $v(x)=\rho(H^o(x))$.*

https://doi.org/10.1017/S0017089524000235 Published online by Cambridge University Press

Proof. By the simplicity, we can assume that $\|v\|_{L^q(\mathcal{W}_R,m^\star)}=1$. Let B_R be the ball centered at the origin with radius $R>0$, and let us consider the weighted p-Laplace eigenvalue problem in B_R:

$$\begin{cases} -\Delta_p z=\lambda\tilde{m}(|x|)\|z\|_{q,m^\star}^{p-q}|z|^{q-2}z & \text{in } B_R \\ z=0 & \text{on } \partial B_R, \end{cases} \tag{3.6}$$

where $\tilde{m}(r)=m^*(k_n r^n)$, $0\leqslant r\leqslant R$. Let z be the positive eigenfunction corresponding to the first eigenvalue $\lambda^{\mathcal{E}}_{p,q}(B_R)$ to the problem (3.6), such that $\|z\|_{L^q(B_R,\tilde{m})}=\|v\|_{L^q(\mathcal{W}_R,m^\star)}=1$. Then uniqueness guarantees that z is radially symmetric, which means that there exists a positive one-dimensional function $\rho_p : r\in[0,R]\to\mathbb{R}^+$ such that $z(x)=\rho_p(|x|)$, and ρ_p solves the following problem:

$$\begin{cases} -(p-1)|\rho_p'|^{p-2}\rho_p''+\frac{n-1}{r}|\rho_p'|^{p-1}=\lambda^{\mathcal{E}}_{p,q}(B_R)\tilde{m}|\rho_p|^{q-2}\rho_p, & r\in(0,R) \\ \rho_p'(0)=\rho_p(R)=0. \end{cases} \tag{3.7}$$

In particular, integrating equation (3.7), it is possible to see that ρ_p' is zero only when $r=0$ and consequently that ρ_p is strictly decreasing in $[0,R]$. Now we can come back to the anisotropy. Indeed if we consider $w=\rho_p(H^\circ(x))$, then using properties (2.4)-(2.6) and the regularity of H, by construction, we obtain that $w(x)$ is a solution to problem (3.5), which is positive and radial with respect to the anisotropic norm. The simplicity and Theorem 3.3 imply that $v=w$, and this concludes the proof. □

Remark 3.5. *We stress that the proof of the previous result shows that the first eigenvalue $\lambda^H_{p,q}(\mathcal{W}_R)$ coincides with the first eigenvalue of problem* (3.6).

3.2. A Faber–Krahn type inequality

Theorem 3.6. *Let $\Omega\in\mathbb{R}^n$, $n\geqslant 2$, be an open, bounded, and connected set and let $1<q\leqslant p$. Then*

$$\lambda^H_{p,q}(\Omega)\geqslant\lambda^H_{p,q}(\Omega^\star), \tag{3.8}$$

where $\Omega^\star$ is the Wulff shape such that $|\Omega^\star|=|\Omega|$. The equality case holds if and only if $\Omega=\Omega^\star$ and $m=m^\star$, that is, in Ω, up to translations, where $m^\star$ is the convex symmetrization of m.

Proof. We argue as in [9]. We observe that $\lambda^H_{p,q}(\Omega^\star)$ has the following variational characterization:

$$\lambda^H_{p,q}(\Omega^\star)=\inf_{\substack{w\in W_0^{1,p}(\Omega^\star),\\ w\neq 0}}\frac{\displaystyle\int_{\Omega^\star}H(\nabla w)^p\,dx}{\left(\displaystyle\int_{\Omega^\star}m^\star|w|^q\,dx\right)^{\frac{p}{q}}}. \tag{3.9}$$

The Faber–Krahn inequality is a straightforward application of the Pólya–Szegö principle and the Hardy–Littlewood inequality. Indeed if u is a positive eigenfunction corresponding to $\lambda^H_{p,q}(\Omega)$, then

$$\lambda^H_{p,q}(\Omega)=\frac{\displaystyle\int_\Omega H(\nabla u)^p\,dx}{\left(\displaystyle\int_\Omega mu^q\,dx\right)^{\frac{p}{q}}}\geqslant\frac{\displaystyle\int_{\Omega^\star}H(\nabla u^\star)^p\,dx}{\left(\displaystyle\int_{\Omega^\star}m^\star(u^\star)^q\,dx\right)^{\frac{p}{q}}}\geqslant\lambda^H_{p,q}(\Omega^\star). \tag{3.10}$$

Let us now consider the equality case. From (3.10), Pólya–Szegö principle and Hardy–Littlewood inequality, we get

$$1\leqslant\frac{\displaystyle\int_\Omega H(\nabla u)^p\,dx}{\displaystyle\int_{\Omega^\star}H(\nabla u^\star)^p\,dx}=\frac{\left(\displaystyle\int_{\Omega^\star}m^\star(u^\star)^q\,dx\right)^{\frac{p}{q}}}{\left(\displaystyle\int_\Omega mu^q\,dx\right)^{\frac{p}{q}}}\leqslant 1.$$

https://doi.org/10.1017/S0017089524000235 Published online by Cambridge University Press

It follows that

$$\int_{\Omega} H(\nabla u)^p \, dx \, dx = \int_{\Omega^\star} H(\nabla u^\star)^p \, dx, \tag{3.11}$$

and

$$\left(\int_{\Omega} m u^q \, dx \right)^{\frac{p}{q}} = \left(\int_{\Omega^\star} m^\star (u^\star)^q \, dx \right)^{\frac{p}{q}}. \tag{3.12}$$

The thesis follows from (3.11), Theorem 2.2 and (3.12). □

4. A Chiti type inequality

In this section, we prove a reverse Hölder inequality for the eigenfunctions corresponding to $\lambda^H_{p,q}(\Omega)$. We first prove the following proposition as in the spirit of the Talenti result contained in [37] (see also [1–3, 7, 36]).

Proposition 4.1. *Let $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, be an open, bounded, and connected set, $1 < q \leqslant p$, and let $m \in L^\infty(\Omega)$ be a positive function. Let u be a positive eigenfunction corresponding to $\lambda^H_{p,q}(\Omega)$. Then we have that*

$$(-u^{*\prime}(s))^{p-1} \leqslant n^{-p} k_n^{-\frac{p}{n}} \lambda^H_{p,q}(\Omega) \|u\|^{p-q}_{q,m} \int_0^s m^* u^*(r)^{q-1} \, dr, \qquad s \in [0, |\Omega|]. \tag{4.1}$$

In particular, the equality case holds if and only if $\Omega = \Omega^\star$ and $m = m^\star$, that is, in Ω, up to translations, where $m^\star$ is the convex symmetrization of m.

Proof. We argue exactly as in the proof of [9, Lemma 3.6]. Let u be a weak solution to the problem (1.2) corresponding to the first eigenvalue $\lambda^H_{p,q}(\Omega)$, that is,

$$\begin{cases} -\mathcal{L}_p(u) = \lambda^H_{p,q}(\Omega) m \|u\|^{p-q}_{q,m} u^{q-1} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Let $t, h > 0$ and let us choose as a test function in (3.1) the following function in $W^{1,p}_0(\Omega)$:

$$\varphi_h = \begin{cases} 0 & u \leqslant t \\ u - t & t < u \leqslant t + h \\ h & u > t + h. \end{cases}$$

By standard arguments, we have

$$-\frac{d}{dt} \int_{\{u>t\}} H(\nabla u)^p \, d\mathcal{H}^{n-1} = \lambda^H_{p,q}(\Omega) \|u\|^{p-q}_{q,m} \int_{\{u>t\}} m u^{q-1} \, dx. \tag{4.2}$$

Recalling that the anisotropic perimeter can be written as follows:

$$-\frac{d}{dt} \int_{\{u>t\}} H(\nabla u) \, d\mathcal{H}^{n-1} = P_H(\{u > t\}),$$

by Hölder inequality, we get

$$P_H(\{u > t\}) \leqslant \left(-\frac{d}{dt} \int_{\{u>t\}} H(\nabla u)^p \, d\mathcal{H}^{n-1} \right)^{\frac{1}{p}} (-\mu'(t))^{1-\frac{1}{p}}.$$

Therefore, the isoperimetric inequality (2.9) gives

$$(-\mu'(t))^{1-p} \left(-\frac{d}{dt} \int_{\{u>t\}} H(\nabla u)^p \, d\mathcal{H}^{n-1} \right) \geqslant n^p k_n^{\frac{p}{n}} \mu(t)^{p-\frac{p}{n}}$$

https://doi.org/10.1017/S0017089524000235 Published online by Cambridge University Press

Since $\mu'(t) = \frac{1}{u^{*\prime}(\mu(t))}$ and (4.2) holds true, we have

$$(-u^{*\prime}(\mu(t)))^{p-1} \leqslant n^{-p} k_n^{-\frac{p}{n}} \lambda_{p,q}^H(\Omega) \|u\|_{q,m}^{p-q} \mu(t)^{\frac{p}{n}-p} \int_{\{u>t\}} m(x) u^{q-1}\, d\mathcal{H}^{n-1}.$$

Using (2.12) and calling $s = \mu(t)$, we have

$$(-u^{*\prime}(s))^{p-1} \leqslant n^{-p} k_n^{-\frac{p}{n}} \lambda_{p,q}^H(\Omega) \|u\|_{q,m}^{p-q}\, s^{\frac{p}{n}-p} \int_0^s m^*(u^*)^{q-1}\, dr.$$

An application of the Hardy–Littlewood inequality gives the desired result. □

The main tool we use in order to prove Theorem 1.1 is a suitable comparison result between u and an eigenfunction z of a suitable eigenvalue problem. More precisely, let $\Omega_\lambda^\star$ be the Wulff shape centered at the origin such that $\lambda_{p,q}^H(\Omega)$ is the first eigenvalue to the following symmetric problem:

$$\begin{cases} -\mathcal{L}_p(z) = \mu m^\star \|z\|_{q,m^\star}^{p-q} z^{q-1} & \text{in } \Omega_\lambda^\star \\ z = 0 & \text{on } \partial\Omega_\lambda^\star, \end{cases} \tag{4.3}$$

We stress that the Faber–Krahn inequality (3.8) implies that

$$|\Omega| \geqslant |\Omega_\lambda^\star|, \tag{4.4}$$

and hence $m^\star$ is well defined in $\Omega_\lambda^\star$.

Let z be a positive eigenfunction for the problem (4.3) corresponding to the first eigenvalue $\lambda_{p,q}^H(\Omega)$, and we observe that repeating the same argument as before, by Proposition 3.4, for any $1 < q \leqslant p$ we have

$$(-z^{*\prime}(s))^{p-1} = n^{-p} k_n^{-\frac{p}{n}} \lambda_{p,q}^H(\Omega) \|z^*\|_{q,m^*}^{p-q} \int_0^s m^* z^*(r)^{q-1}\, dr. \tag{4.5}$$

The following proposition gives a comparison result between the eigenfunctions u and z when they are normalized with respect to the L^∞ norm.

Proposition 4.2. *Let $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, be an open, bounded, and connected set, $1 < q \leqslant p$ and let $m \in L^\infty(\Omega)$ be a positive function. Let u be a positive solution to the problem (1.2) corresponding to $\lambda_{p,q}^H(\Omega)$ and let z be a positive eigenfunction to the problem (4.3) corresponding to $\lambda_{p,q}^H(\Omega)$ such that*

$$\|u\|_{L^\infty(\Omega)} = \|z\|_{L^\infty(\Omega_\lambda^\star)}$$

Then

$$u^*(s) \geqslant z^*(s), \qquad \forall s \in [0, |\Omega_\lambda^\star|],$$

where u^* and z^* are, respectively, the decreasing rearrangements of u and z. The equality case holds if and only if $\Omega = \Omega_\lambda^\star$ and $m = m^\star$, that is, in Ω, up to translations, where $m^\star$ is the convex symmetrization of m.

Proof. First of all we stress that, if $|\Omega| = |\Omega_\lambda^\star|$, then there is nothing to prove, since Faber–Krahn inequality implies that $u^*(s) = z^*(s)$.
Moreover, we have $u^*(|\Omega_\lambda^\star|) > z^*(|\Omega_\lambda^\star|) = 0$. Then, the following definition is well posed:

$$s_0 = \inf\{s \in [0, |\Omega_\lambda^\star|] : u^*(t) \geqslant z^*(t),\ \forall t \in [s, |\Omega_\lambda^\star|]\}.$$

By definition, $u^*(s_0) = z^*(s_0)$, and we want to prove that $s_0 = 0$. We proceed by contradiction supposing that $s_0 > 0$. Then under this assumption, u^* and z^* coincide in 0 and s_0 and we have

$$\begin{cases} u^*(s) < z^*(s) & s \in (0, s_0) \\ u^*(s) \geqslant z^*(s) & s \in (s_0, |\Omega_\lambda^\star|). \end{cases} \tag{4.6}$$

https://doi.org/10.1017/S0017089524000235 Published online by Cambridge University Press

By (4.1), (4.5), and (4.6), we have that

$$-u^{*\prime}(t) \leqslant -z^{*\prime}(t), \qquad \text{for every } t \in (0, s_0).$$

Integrating between $(0, s)$, with $s \in (0, s_0)$, being $u^*(0) = z^*(0)$, we get

$$u^*(s) \geqslant z^*(s), \qquad \forall s \in (0, s_0),$$

which is in contradiction with the definition of s_0. Hence, $s_0 = 0$, and the proof is completed. □

As an immediately consequence of the previous result, we get the following scale-invariant inequality for any $r > 0$:

$$\frac{\|u\|_{L^r(\Omega,m)}}{\|u\|_{L^\infty(\Omega)}} \geqslant \frac{\|z\|_{L^r(\Omega_\lambda^\star,m^\star)}}{\|z\|_{L^\infty(\Omega_\lambda^\star)}}. \tag{4.7}$$

When the functions u and z are normalized with respect to the weighted L^q-norm, we get the following comparison result.

Theorem 4.3. *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded, and connected set, $1 < q \leqslant p$ and let $m \in L^\infty(\Omega)$ be a positive function. Let u be a positive solution to the problem (1.2) corresponding to $\lambda_{p,q}^H(\Omega)$ and let z be a positive eigenfunction to the problem (4.3) corresponding to $\lambda_{p,q}^H(\Omega)$ such that*

$$\int_\Omega m\, u^q \, dx = \int_{\Omega_\lambda^\star} m^\star z^q \, dx. \tag{4.8}$$

Then we have

$$\int_0^s m^* \, (u^*)^r \, dt \leqslant \int_0^s m^* \, (z^*)^r \, dt, \qquad s \in [0, |\Omega_\lambda^\star|], \qquad q \leqslant r \tag{4.9}$$

where u^*, m^*, and z^* are, respectively, the decreasing rearrangements of u, m, and z, and $m^\star$ is the convex symmetrization of m. The equality case holds if and only if $\Omega = \Omega^\star$, $z = u = u^*$, and $m = m^\star$, that is, Ω, up to translations.

Proof. If $|\Omega| = |\Omega_\lambda^\star|$, the conclusion is trivial. Let be $|\Omega| > |\Omega_\lambda^\star|$, since u and z verify (4.8), by (4.7) it holds that

$$u^*(0) = \|u\|_{L^\infty(\Omega)} \leqslant \|z\|_{L^\infty(\Omega_\lambda^\star)} = z^*(0),$$

If $u^*(0) = z^*(0)$, then Proposition 4.2 and the normalization (4.8) imply that $u^*(s) = z^*(s)$ for every $s \in [0, |\Omega_\lambda^\star|]$ and than the claim follows trivially.
Let $u^*(0) < z^*(0)$. Since $u^*(|\Omega_\lambda^\star|) > z^*(|\Omega_\lambda^\star|)$, we can consider

$$s_0 = \sup\{s \in (0, |\Omega_\lambda^\star|) : u^*(t) \leqslant z^*(t) \text{ for } t \in [0, s]\}.$$

Obviously, $0 < s_0 < |\Omega_\lambda^\star|$, $u^*(s_0) = z^*(s_0)$ and $u^* \leqslant z^*$ in $[0, s_0]$. We want to show that $u^* > z^*$ in $[s_0, |\Omega_\lambda^\star|]$. Indeed, if we suppose by contradiction that there exists $s_1 > s_0$ such that $u^*(s_1) = z^*(s_1)$ and $u^*(s) > z^*(s)$ for $s \in (s_0, s_1)$, we can construct the following function:

$$w^*(s) = \begin{cases} z^*(s) & s \in [0, s_0] \cup [s_1, |\Omega_\lambda^\star|] \\ u^*(s) & s \in [s_0, s_1]. \end{cases}$$

It is straightforward to check that

$$\int_\Omega H(\nabla w)^p \, dx = n^p k_n^p \int_\Omega (-w^{*\prime}(k_n H^\circ(x)^n))^p H^\circ(x)^{p(n-1)} \, dx.$$

Applying Coarea Formula and considering the change of variables $s = k_n t^n$, we get

$$\int_\Omega H(\nabla w)^p \, dx = n^p k_n^{\frac{p}{n}} \int_0^{|\Omega_\lambda^\star|} s^{p-\frac{p}{n}} (-w^{*\prime}(t))^p \, dt.$$

Thanks to the normalization (4.8) and the definition of w, we have that

$$\|u\|_{L^q(\Omega,m)} = \|z\|_{L^q(\Omega^\star_\lambda,m^\star)} \leqslant \|w\|_{L^q(\Omega^\star_\lambda,m^\star)},$$

then by (4.1) and (4.5), we have that

$$(-w^{*\prime}(s))^{p-1} \leqslant n^{-p}k_n^{-\frac{p}{n}}\lambda^H_{p,q}(\Omega)\|w^*\|^{p-q}_{q,m^*}s^{\frac{p}{n}-p}\int_0^s m^*(r)(w^*)^{q-1}(r)\,dr. \tag{4.10}$$

Multiplying (4.10) by $-w'$, rearranging the terms and integrating between 0 and $|\Omega^\star_\lambda|$, we get

$$n^p k_n^{\frac{p}{n}}\int_0^{|\Omega^\star_\lambda|} s^{p-\frac{p}{n}}(-w^{*\prime}(s))^p\,ds \leqslant$$

$$\leqslant \lambda^H_{p,q}(\Omega)\|w^*\|^{p-q}_{q,m^*}\int_0^{|\Omega^\star_\lambda|}(-w^{*\prime}(s))\int_0^s m^*(r)(w^*)^{q-1}(r)\,dr\,ds.$$

An integration by parts allows us to conclude that

$$\frac{\displaystyle\int_{\Omega^\star_\lambda} H(\nabla w)^p\,dx}{\left(\displaystyle\int_{\Omega^\star_\lambda} m^\star w^q\,dx\right)^{\frac{p}{q}}} = \frac{n^p k_n^{\frac{p}{n}}\displaystyle\int_0^{|\Omega^\star_\lambda|} s^{p-\frac{p}{n}}(-w'(s))^p\,ds}{\left(\displaystyle\int_0^{|\Omega^\star_\lambda|} m^*(s)(w^*)^q(s)\,ds\right)^{\frac{p}{q}}} \leqslant \lambda^H_{p,q}(\Omega) = \lambda^H_{p,q}(\Omega^\star_\lambda).$$

By the minimality and the simplicity of $\lambda^H_{p,q}$, and the definition of w^*, it must be $w^*(s) = z^*(s)$ for every $s \in [0, |\Omega^\star_\lambda|]$, but this is a contradiction since in (s_0, s_1) we have that $u^*(s) > z^*(s)$.
In this way, we have proved that there exists a unique point s_0 where u^* and z^* can cross each other, and such that

$$\begin{cases} u^*(s) \leqslant z^*(s) & s \in [0, s_0] \\ u^*(s) \geqslant z^*(s) & s \in [s_0, |\Omega^\star_\lambda|]. \end{cases} \tag{4.11}$$

If we extend z^* to be zero in $[|\Omega^\star_\lambda|, |\Omega|]$, by (4.8) and (4.11) then we have that for every $s \in [0, |\Omega|]$

$$\int_0^s m^*(t)(u^*(t))^q\,dt \leqslant \int_0^s m^*(t)(z^*(t))^q\,dt. \tag{4.12}$$

Indeed (4.11) implies that the function:

$$G(s) = \int_0^s m^*(t)((z^*))^q - (u^*)^q\,dt, \qquad s \in [0, |\Omega|]$$

has a maximum in s_0 and cannot be negative in any point. This proves (4.12). Finally, inequality (4.9) follows easily by (4.12) by using Proposition 2.4 being $m^\star(u^\star)^q \prec m^\star z^q$. □

Proof of Theorem 1.1. The proof of statement i) follows directly from (4.8) and (4.9), indeed we have

$$\left(\int_\Omega mu^r\,dx\right)^{\frac{1}{r}} \leqslant \left(\int_{\Omega^\star_\lambda} m^\star z^r\,dx\right)^{\frac{1}{r}} = \frac{\left(\displaystyle\int_{\Omega^\star_\lambda} m^\star z^r\,dx\right)^{\frac{1}{r}}}{\left(\displaystyle\int_{\Omega^\star_\lambda} m^\star z^q\,dx\right)^{\frac{1}{q}}}\left(\int_\Omega mu^q\,dx\right)^{\frac{1}{q}}.$$

Therefore, we have

$$\|u\|_{L^r(\Omega,m)} \leqslant C\ \|u\|_{L^q(\Omega,m)}, \quad q \leqslant r < +\infty;$$

with

$$C = \frac{\|z\|_{L^r(\Omega^\star_\lambda,m^\star)}}{\|z\|_{L^q(\Omega^\star_\lambda,m^\star)}}.$$

https://doi.org/10.1017/S0017089524000235 Published online by Cambridge University Press

The proof of statement ii) follows immediately by Proposition 4.2 with $C = \frac{\|z\|_\infty}{\|z\|_{L^r(\Omega^\star,m^\star)}}$ and the theorem is completely proved. □

Acknowledgments. This work has been partially supported by MiUR PRIN 2017 grant "Qualitative and quantitative aspects of nonlinear PDEs," the MiUR PRIN-PNRR 2022 grant "Linear and Nonlinear PDE's: New directions and Applications," and GNAMPA of INdAM. The second author acknowledges the MIUR Excellence Department Project awarded to the Department of Mathematics, University of Pisa, CUP I57G22000700001.

Conflicts of interest. The authors declare that there is no conflict of interest.

Data availability statement. Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

References

[1] A. Alberico, A. Ferone and R. Volpicelli, Some properties for eigenvalues and eigenfunctions of nonlinear weighted problems, *Rend. Mat. Appl* **19**(1) (1999), 45–63.
[2] A. Alvino, V. Ferone, P.-L. Lions and G. Trombetti, Convex symmetrization and applications, *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **14**(2) (1997), 275–293.
[3] A. Alvino, V. Ferone and G. Trombetti, On the properties of some nonlinear eigenvalues, *SIAM J. Math. Anal.* **29**(2) (1998), 437–451.
[4] A. Alvino, P.-L. Lions and G. Trombetti, On optimization problems with prescribed rearrangements, *Nonlinear Anal. Theory Methods Appl.* **13**(2) (1989), 185–220.
[5] A. Alvino, G. Trombetti and P.-L. Lions, Comparison results for elliptic and parabolic equations via Schwarz symmetrization, *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **7**(2) (1990), 37–65.
[6] M. Amar and G. Bellettini, A notion of total variation depending on a metric with discontinuous coefficients, *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **11**(1) (1994), 91–133.
[7] V. Amato, R. Barbato, A. L. Masiello and G. Paoli, The Talenti comparison result in a quantitative form (2023). arXiv preprint.
[8] G. Bellettini, V. Caselles, A. Chambolle and M. Novaga, The volume preserving crystalline mean curvature flow of convex sets in $\mathbb{R}^N$, *J. Math. Pures Appl. (9)* **92**(5) (2009), 499–527.
[9] M. Belloni, V. Ferone and B. Kawohl, Isoperimetric inequalities, Wulff shape and related questions for strongly nonlinear elliptic operators, *Zeitschrift fur Angewandte Mathematik und Physik (ZAMP)* **54**(5) (2003), 771–783.
[10] L. Brasco and G. Franzina, Convexity properties of Dirichlet integrals and Picone-type inequalities, *Kodai Math. J.* **37**(3) (2014), 769–799.
[11] L. Brasco and G. Franzina, An overview on constrained critical points of Dirichlet integrals, *Rend. Semin. Mat. Univ. Politec. Torino* **78**(2) (2020), 7–50.
[12] H. Busemann, The isoperimetric problem for Minkowski area, *Am. J. Math.* **71**(4) (1949), 743–762.
[13] T. Carroll, M. M. Fall and J. Ratzkin, On the rate of change of the sharp constant in the Sobolev-Poincaré inequality, *Math. Nachr.* **290**(14-15) (2017), 2185–2197.
[14] T. Carroll and J. Ratzkin, A reverse Hölder inequality for extremal sobolev functions, *Potential Anal.* **42**(1) (2015), 283–292.
[15] G. Chiti, An isoperimetric inequality for the eigenfunctions of linear second order elliptic operators, *Boll. Un. Mat. Ital.* **1**(1) (1982), 145–151.
[16] G. Chiti, A reverse Hölder inequality for the eigenfunctions of linear second order elliptic operators, *Zeitschrift für Angewandte Mathematik und Physik ZAMP* **33**(1) (1982), 143–148.
[17] K. M. Chong and N. M. Rice, Equimeasurable rearrangements of functions, *Queen's Paper Pure Appl. Math.* **28** (1971).Queen's University, Ontario.
[18] B. Dacorogna and C.-E. Pfister, Wulff theorem and best constant in Sobolev inequality, *J. Math. Pures et Appl.* **71**(2) (1992), 97–118.
[19] F. Della Pietra and N. Gavitone, Sharp bounds for the first eigenvalue and the torsional rigidity related to some anisotropic operators, *Math. Nachr.* **287**(2-3) (2013), 194–209.
[20] P. Drábek, A. Kufner and F. Nicolosi, *Quasilinear elliptic equations with degenerations and singularities*, De Gruyter Series in Nonlinear Analysis and Applications, vol. **5** (Walter de Gruyter & Co, Berlin, 1997).
[21] L. C. Evans and R. F. Gariepy, Measure theory and fine properties of functions, in *Textbooks in mathematics* (CRC Press, Boca Raton, FL, 2015).
[22] A. Ferone and R. Volpicelli, Convex rearrangement: equality cases in the Pólya-Szegö inequality, *Calc. Var. Partial Differ. Equations* **21**(3) (2004), 259–272.

[23] A. Figalli, F. Maggi and A. Pratelli, A mass transportation approach to quantitative isoperimetric inequalities, *Invent. Math.* **182**(1) (2010), 167–211.

[24] I. Fonseca and S. Müller, A uniqueness proof for the Wulff theorem, *Proc. R. Soc. Edinburgh Sect. A Math.* **119**(1-2) (1991), 125–136.

[25] G. Franzina and P. D. Lamberti, Existence and uniqueness for a p-Laplacian nonlinear eigenvalue problem, *Electron. J. Differ. Equations* **26** (2010), 1–10.

[26] J. P. García Azorero and I. Peral Alonso, Existence and nonuniqueness for the p-Laplacian: nonlinear eigenvalues, *Commun. Partial Differ. Equations* **12**(12) (1987), 1389–1430.

[27] T. Idogawa and M. Ôtani, The first eigenvalues of some abstract elliptic operators, *Funkcialaj Ekvacioj* **38**(1) (1995), 1–9.

[28] B. Kawohl, Symmetry results for functions yielding best constants in Sobolev-type inequalities, *Discrete Contin. Dyn. Syst.* **6**(3) (2000), 683–690.

[29] B. Kawohl, M. Lucia and S. Prashanth, Simplicity of the principal eigenvalue for indefinite quasilinear problems, *Adv. Differ. Equations* **12**(4) (2007), 407–434.

[30] S. Kesavan, *Symmetrization and applications*, in: Series in Analysis, vol. 3 (World Scientific Publishing Co. Pte. Ltd., New Jersey, 2006).

[31] F. Maggi, Sets of finite perimeter and geometric variational problems, in *Cambridge studies in advanced mathematics*, vol. 135 (Cambridge University Press, Cambridge, 2012).

[32] A. I. Nazarov, The one-dimensional character of an extremum point of the Friedrichs inequality in spherical and plane layers, *J. Math. Sci.* **102**(5) (2000), 4473–4486.

[33] A. I. Nazarov, On the symmetry of extremals in the weight embedding theorem, *J. Math. Sci. (New York)* **107**(3) (2001), 3841–3859.

[34] M. Ôtani, On certain second order ordinary differential equations associated with Sobolev-Poincaré-type inequalities, *Nonlinear Anal. Theory Methods Appl.* **8**(11) (1984), 1255–1270.

[35] M. Ôtani, Existence and nonexistence of nontrivial solutions of some nonlinear degenerate elliptic equations, *J. Funct. Anal.* **76**(1) (1988), 140–159.

[36] R. Sannipoli, Comparison results for solutions to the anisotropic Laplacian with Robin boundary conditions, *Nonlinear Anal.* **214** (2022), 112615.

[37] G. Talenti, Elliptic equations and rearrangements, *Ann. Scuola Norm-SCI* **3**(4) (1976), 697–718.

[38] W. P. Ziemer and J. E. Brothers, Minimal rearrangements of Sobolev functions, Journal für die reine und angewandte Mathematik 384 (1988), 153–179.