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SIMPLE MODULES IN THE AUSLANDER–REITEN QUIVER OF PRINCIPAL BLOCKS WITH ABELIAN DEFECT GROUPS

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Abstract. Given an odd prime p, we investigate the position of simple modules in the stable Auslander–Reiten quiver of the principal block of a finite group with noncyclic abelian Sylow p-subgroups. In particular, we prove a reduction to finite simple groups. In the case that the characteristic is 3, we prove that simple modules in the principal block all lie at the end of their components.

§1. Introduction

The position of simple modules in the stable Auslander–Reiten quiver of the group algebra kG over a field k of characteristic p of a finite group G of order divisible by p is a question that was partially investigated in the 1980s and the 1990s in a series of articles by different authors. We refer the reader in particular to [4, 20–22] and the references therein. The aim of this note is to come back to the following question:

Question A. Let B be a wild p-block of kG. Under which conditions do all simple B-modules lie at the end of their connected components in the stable Auslander–Reiten quiver of kG?

A main reason of interest in this question lies in the fact that a simple kG-module lies at the end of its component if and only if the heart of its projective cover is indecomposable.

In this article, we focus attention on the case in which the principal block $B_0(kG)$ is of wild representation type with abelian defect groups and the prime p is odd. We recall that a p-block is of wild representation type if and only if its defect groups are neither cyclic, nor dihedral, nor semidihedral, nor generalized quaternion (see [18, §8.9 Theorem]). Thus, when p is odd,

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this amounts to requiring that the p-rank of G is at least 2. Question A in the case that $p=2$ was treated by Kawata *et al.* in [21, Theorem 5]. We aim at extending their results and part of their methods to arbitrary primes. Further, we note that the cases when $B_0(kG)$ is of finite or tame representation type are well understood. In the former case, the distance of a simple module to the rim of its connected component (which is a tube of shape $(\mathbb{Z}/e\mathbb{Z})A_m$) is a function of its position in the Brauer tree of the block, while in the latter case the position of the simple modules in their connected components is given by Erdmann's work on tame blocks [10].

Assuming the field k is algebraically closed we prove the following results:

THEOREM B. *Let G be a finite group and $N \trianglelefteq G$ a normal subgroup such that G/N is solvable of p'-order. Let B and b be wild blocks of kG and kN, respectively, such that $1_B = 1_b$. If every simple b-module lies at the end of its connected component in the stable Auslander–Reiten quiver of kN, then every simple B-module lies at the end of its connected component in the stable Auslander–Reiten quiver of kG.*

THEOREM C. *Let p be an odd prime. Let G be a finite group with noncyclic abelian Sylow p-subgroups and $O_{p'}(G)=1$. Write $O^{p'}(G) = Q \times H_1 \times \cdots \times H_m$ $(m \geqslant 0)$, where Q is an abelian p-group and H_i is a nonabelian finite simple group with nontrivial Sylow p-subgroups for each $1 \leqslant i \leqslant m$. Assume that one of the following conditions is satisfied:*

- (i) *$Q \neq 1$; or*
- (ii) *$Q = 1$ and $m \geqslant 2$; or*
- (iii) *$Q = 1$, $m = 1$ and every simple $B_0(kH_1)$-module lies at the end of its connected component in the stable Auslander–Reiten quiver of kH_1.*

Then every simple $B_0(kG)$-module lies at the end of its connected component in the stable Auslander–Reiten quiver of kG.

COROLLARY D. *Let p be an odd prime. Assume that every simple $B_0(kH)$-module lies at the end of its connected component in the stable Auslander–Reiten quiver of kH for every nonabelian finite simple group H with noncyclic abelian Sylow p-subgroups. Then every simple $B_0(kG)$-module lies at the end of its connected component in the stable Auslander–Reiten quiver of kG for any finite group G with noncyclic abelian Sylow p-subgroups.*

We note that if $p = 2$, then the analogues of Theorem C and Corollary D were essentially proven by Kawata *et al.* [21], although not stated in these terms. As a corollary, we also obtain the equivalent of [21, Theorem 5(a)] for the prime 3.

THEOREM E. *Assume* $p = 3$. *Let* G *be a finite group with abelian Sylow* 3*-subgroups. If* $B_0(kG)$ *is a wild* 3*-block, then every simple* $B_0(kG)$*-module lies at the end of its connected component in the stable Auslander–Reiten quiver of* kG.

The paper is organized as follows: in Section 2, we set up the notation and in Section 3 we recall the state of knowledge on the subject and extend a result of Kawata's [20, Theorem 1.5] to describe more precisely the indecomposable summands of the heart of the projective cover of a simple module not lying on the rim of its component. In Section 4, we consider groups having a solvable quotient of p'-order and prove Theorem B. In Sections 5 and 6, we proceed to a reduction of Question A for principal blocks to the case of finite nonabelian simple groups and prove Theorem C and Corollary D. Finally in Section 7 we deal with the case $p = 3$ and prove Theorem E.

§2. Notation and preliminaries on module and block theory

Throughout this paper, unless otherwise stated, we adopt the following notation and conventions. We assume that k is an algebraically closed field of characteristic $p > 0$. All groups are assumed to be finite and we let G denote a finite group of order divisible p. We let $(\mathcal{K}, \mathcal{O}, k)$ be a splitting p-modular system for G and its subgroups, namely $\mathcal{O}$ is a complete discrete vatulation ring, $\mathcal{K}$ its quotient field of characteristic 0, $k = \mathcal{O}/J(\mathcal{O})$ its residue field of characteristic p (where $J(\mathcal{O})$ is the unique maximal ideal of $\mathcal{O}$), and $\mathcal{K}$ and k are both splitting fields for all subgroups of G.

For a p-block B, we write 1_B for the corresponding block idempotent and $\mathrm{IBr}(B)$ for the set of isomorphism classes of simple kB-modules. Furthermore, unless otherwise specified, we assume that $B_0 = B_0(kG)$, the principal block of kG, is wild. Thus, when the defect groups of B are abelian, we may therefore assume that a Sylow p-subgroup of G is noncyclic, or equivalently that the p-rank of G is at least 2.

We write $H \leqslant G$ if H is a subgroup of G, $N \trianglelefteq G$ if N is a normal subgroup of G, $O_{p'}(G)$ for the largest p'-normal subgroup of G, and $O^{p'}(G)$ for the smallest normal subgroup of G with p'-index in G. All modules are assumed to be finitely generated right modules. We denote by k_G the trivial kG-module. If $H \leqslant G$ and X, Y are kG- and kH-modules, respectively, then we write $X{\downarrow}_H$ for the restriction of X to H, and $Y{\uparrow}^G = Y \otimes_{kH} kG$ for the induction of Y from H to G.

We let $J = J(kG)$ denote the Jacobson radical of kG. For a kG-module U, we define $J(kG)^0 = kG$ and for any nonnegative integer $i \geqslant 0$ we let $\mathrm{soc}^i(U) = \{u \in U \mid u\, J^i = \{0\}\}$, then inductively for any $i \geqslant 1$, we write

$$L_i(U) = U\, J^{i-1}/U\, J^i \quad \text{and} \quad S_i(U) = \mathrm{soc}^i(U)/\mathrm{soc}^{i-1}(U)$$

for the ith *Loewy layer* and the ith *socle layer* of U, respectively. We define $\mathrm{soc}(U) = \mathrm{soc}^0(U)$, that is called the socle of U (see [29, Chapter I, Definition 8.1]). For an integer $n \geqslant 1$ and simple kG-modules $S_1, \ldots, S_n$ (possibly $S_i \cong S_j$ for $i \neq j$) we denote by

$$U = \boxed{\begin{matrix} S_1 \\ S_2 \\ \vdots \\ S_n \end{matrix}}$$

a uniserial kG-module of Loewy length n such that $L_i(U) \cong S_i$ for every $1 \leqslant i \leqslant n$. We also recall that the Loewy series does not determine modules up to isomorphism. For instance if $p = 2$ and $G = C_2 \times C_2$, then $kG \cong k[x, y]/(x^2, y^2)$ as k-algebras and there are obviously two *nonisomorphic* uniserial kG-modules U and V of Loewy length 2 ($\dim_k[\mathrm{Ext}^1_{kG}(k, k)] = 2$). We will use throughout the following well-known properties without further mention:

Lemma 2.1. *Assume $N \trianglelefteq G$ of index prime to p.*

(a) *We have $J = \tilde{J}\, kG = kG\, \tilde{J}$ where $\tilde{J} = J(kN)$.*

(b) *Let X be a kG-module and Y a kN-module, then for any $i \geqslant 1$ we have*

$$L_i(X){\downarrow}_N = L_i(X{\downarrow}_N) \quad \textit{and} \quad S_i(X){\downarrow}_N = S_i(X{\downarrow}_N)$$

and

$$L_i(Y){\uparrow}^G = L_i(Y{\uparrow}^G) \quad \textit{and} \quad S_i(Y){\uparrow}^G = S_i(Y{\uparrow}^G).$$

Proof. Part (a) is a well-known result of Villamayor [34] and part (b) follows from (a). ∎

Given a normal subgroup $H \trianglelefteq G$ and $\tilde{b}$ a p-block of H, we will use the group $G[\tilde{b}]$ defined by Dade [7]. A more explicit description of $G[\tilde{b}]$ can also be found in [17, Lemma 3.2]. Roughly speaking $G[\tilde{b}]$ is the stabilizer of $\tilde{b}$ as $k(H \times H)$-module.

LEMMA 2.2. (Dade) *Let $H \trianglelefteq G$ such that $p \not| |G/H|$, and let P be a Sylow p-subgroup of H. Let $\tilde{b} = B_0(kH)$ and $B = B_0(kG)$ be the principal blocks of kH and kG, respectively. Set $N = H\,C_G(P)$. Then the following hold:*

(a) *The block $\tilde{b}$ is G-invariant.*
(b) *$N = G[\tilde{b}]$ and $N \trianglelefteq G$.*
(c) *If b denotes the principal block of kN, then $1_B = 1_b$.*

Proof. (a) Obvious since $\tilde{b}$ is the principal block. (b) The first claim follows from [7, Corollary 12.6] since $\tilde{b}$ is the principal block. Moreover, as $\tilde{b}$ is G-invariant, the fact that $G[\tilde{b}] \trianglelefteq G$ follows from [7, Proposition 2.17].

(c) The main argument to prove (c) is given by [27, p. 303 line 10]. We give here a full argument for completeness. As $\tilde{b}$ is G-invariant, $1_{\tilde{b}}$ is an idempotent of $Z(kG)$ and we can write

$$1_{\tilde{b}} = 1_B + 1_{B_1} + \cdots + 1_{B_n}$$

for an integer $n \geqslant 0$ and for distinct nonprincipal blocks $B_1, \ldots, B_n$ of kG. Thus, $1_{\tilde{b}}\, 1_B = 1_B$. Namely,

$$1_B \in 1_{\tilde{b}}\, Z(kG) \subseteq 1_{\tilde{b}}\, C_{kG}(H) =: C.$$

This implies $1_B \in Z(C)$ since $1_B \in Z(kG)$. Hence it follows from [26, Corollary 4] and part (b) that

$$\begin{aligned} 1_B \in C[\tilde{b}] &= Z(\tilde{b}) * G[\tilde{b}] \\ &\subseteq Z(kH) * N \subseteq kN, \end{aligned}$$

where $*$ denotes the crossed product. Thus $1_B \in Z(kN)$. On the other hand, since b is the principal block of kN, we have 1_b is G-invariant, so that $1_b \in Z(kG)$. Hence, as above, we can write

$$1_b = 1_B + 1_{B'_1} + \cdots + 1_{B'_t}$$

where $t \geqslant 0$ is an integer and $B'_1, \cdots, B'_t$ are distinct nonprincipal blocks of kG. Set $\tilde{e} = 1_b - 1_B \in Z(kN)$ (since $1_B \in Z(kN)$). Therefore $1_b = 1_B + \tilde{e}$ is a decomposition of 1_b into orthogonal idempotents of $Z(kN)$. This implies that $\tilde{e} = 0$, and hence $1_b = 1_B$. ∎

Finally, we will need the following well-known properties of relative projectivity and inflation in direct products. We recall that if $N \trianglelefteq G$ and U is a $k(G/N)$-module, then we denote by $\mathrm{Inf}_{G/N}^{G}(U)$ the inflation of U from G/N to G, namely $\mathrm{Inf}_{G/N}^{G}(U) = U$ as k-vector space and becomes a kG-module via the action of G obtained by composition with the canonical epimorphism $G \twoheadrightarrow G/N$. Furthermore, if $G = N \times H$ is the direct product of two finite groups N and H, and U and V, are kN- and kH-modules, respectively, then on the one hand $X = U \otimes_k V$ becomes a kG-module via the action

$$(u \otimes v)(n, h) = un \otimes vh \quad \forall u \in U, v \in V, n \in N, h \in H,$$

and on the other hand, setting $U' = \mathrm{Inf}_{G/H \cong N}^{G}(U)$ and $V' = \mathrm{Inf}_{G/N \cong H}^{G}(V)$, we have that $U' \otimes_k V'$ becomes a kG-module via the diagonal action

$$(u' \otimes v') \cdot g = u'g \otimes v'g \quad \forall u' \in U', v' \in V', g \in G.$$

It is then easily seen that $X \cong U' \otimes_k V'$ as kG-modules.

LEMMA 2.3. *Assume that $G = N \times H$ is the direct product of two finite groups N and H. Let U be a kN-module and V be a kH-module. Set $X = U \otimes_k V$, $U' = \mathrm{Inf}_{G/N}^{G}(U)$ and $V' = \mathrm{Inf}_{G/N}^{G}(V)$. If U is projective as a kL-module, then X is a relatively H-projective kG-module.*

We give a short proof for completeness.

Proof. First, since U is a projective kN-module, it is projective relatively to the trivial subgroup, or equivalently projective relatively to the kN-module $k_{\{1\}}{\uparrow}^{N}$, that is projective relatively to the $k(G/H)$-module $k_{H/H}{\uparrow}^{G/H}$ by using the isomorphism $N \cong G/H$. Therefore, by [30, Lemma 2.1.1(c)], it follows that the inflated kG-module $U' = \mathrm{Inf}_{G/H}^{G}(U)$ is projective relatively to the inflated kG-module

$$\begin{aligned}\mathrm{Inf}_{G/H}^{G}(k_{H/H}{\uparrow}^{G/H}) &= \mathrm{Inf}_{G/H}^{G} \circ \mathrm{Ind}_{H/H}^{G/H}(k_{G/H}) \\ &= \mathrm{Ind}_{H}^{G} \circ \mathrm{Inf}_{H/H}^{H}(k_{H/H}) = k_H{\uparrow}^{G}\end{aligned}$$

(where Ind denotes the induction seen as a functor). Hence U' is projective relatively to H. It follows directly that the tensor product $X \cong U' \otimes_k V'$ is H-projective, because one of the factors is (see e.g., [3, Corollary 3.6.7]). ∎

§3. Background results on the Auslander–Reiten quiver

We recall briefly basic facts concerning the stable Auslander–Reiten quiver of a group algebra, which we will be using in the sequel. For a complete introduction to Auslander–Reiten theory, we refer the reader to the textbooks [2, Chapter IV] and [3, Chapter 4].

To finish setting up our notation, given a kG-module M, we denote by $\Omega^n(M)$ ($n \in \mathbb{Z}$) its nth Heller translate of M. Given a simple kG-module S, we denote by $P(S)$ its projective cover and by $\mathcal{H}(P(S))$ the heart of $P(S)$, that is $\mathcal{H}(P(S)) = P(S)J/\mathrm{soc}\,(P(S))$.

Let M be an indecomposable kG-module. By definition, an *Auslander–Reiten sequence (or AR-sequence)* terminating at M is a nonsplit short exact sequence

$$\mathcal{A}(M): \quad 0 \longrightarrow N \xrightarrow{f} X_M \xrightarrow{g} M \longrightarrow 0$$

satisfying the following conditions: first N is indecomposable, and second for each kG-homomorphism $h : X \to M$ which is not a split epimorphism, there exists a kG-homomorphism $h' : X \to E$ such that $h = gh'$. Given an indecomposable nonprojective kG-module M, there exists always an *AR*-sequence terminating at M, and it is unique up to isomorphism of short exact sequences. Moreover, since kG is a finite-dimensional symmetric Algebra, we have $N \cong \Omega^2(M)$ (see [3, 4.12.8]). In similar fashion, there exists an AR-sequence starting at M, unique up to isomorphism of short exact sequences, with end term isomorphic to $\Omega^{-2}(M)$. For a nonprojective simple kG-module S, the Auslander–Reiten sequence terminating at $\Omega^{-1}(S)$ is of the form

$$\mathcal{A}(\Omega^{-1}(S)): \quad 0 \to \Omega(S) \to \mathcal{H}(P(S)) \oplus P(S) \to \Omega^{-1}(S) \to 0$$

and is called the *standard sequence associated to S*. This is the unique *AR*-sequence in which the PIM $P(S)$ occurs.

The *Auslander–Reiten quiver (or AR-quiver)* of kG (resp. of a p-block B of kG) is the directed graph $\Gamma(kG)$ (resp. $\Gamma(B)$) whose vertices are the isomorphism classes of indecomposable kG-modules (resp. B-modules), and the number of arrows between two indecomposable modules M and N corresponds to the dimension of the space of *irreducible maps* between M and N. We refer the reader to [2, Chapter IV] for a precise definition. Then the *stable Auslander–Reiten quiver* of kG (resp. of B) is obtained from $\Gamma(kG)$ (resp. $\Gamma(B)$) by removing the vertices corresponding to projective modules and all arrows attached to these vertices; it is denoted by $\Gamma_s(kG)$ (resp. $\Gamma_s(B)$). By convention, we use the terminology *AR-component* to refer to a connected component of $\Gamma_s(kG)$, and we denote by $\Gamma_s(M)$ the connected component of $\Gamma_s(kG)$ containing a given indecomposable kG-module M.

Erdmann [11] proved that all components of the stable Auslander–Reiten quiver belonging to a wild block have tree class A_∞, that is of the form $\mathbb{Z}A_\infty$ or infinite tubes $\mathbb{Z}A_\infty/\langle\tau^a\rangle$ of rank a, where $\tau = \Omega^2$ is the Auslander–Reiten shift. In a component with tree class A_∞ an indecomposable nonprojective kG-module M is said to lie *at the end* (or *on the rim)* of its AR-component if the projective-free part of the middle term X_M of the Auslander–Reiten sequence

$$\mathcal{A}(M): \quad 0 \to \Omega^2(M) \to X_M \to M \to 0$$

terminating at M is indecomposable. In this setup, clearly a simple module S lies at the end of its component if and only if $\mathcal{H}(P(S))$ is indecomposable, and S lies in a tube if and only if S is periodic (i.e., Ω-periodic). We also recall that for a selfinjective algebra the shape of the components of the stable Auslander–Reiten quiver is an invariant of its Morita equivalence class. By the above, the property of lying on the rim of its AR-component for a nonprojective simple module is also invariant under Morita equivalence.

Simple kG-modules are known to lie on the rim of their AR-components in the following cases:

THEOREM 3.1. *Let B be a wild p-block of kG. Then every simple B-module lies at the end of its AR-component in all of the following cases:*

(a) *G has a nontrivial normal p-subgroup [20, Theorem 2.1];*
(b) *G is p-solvable [20, Corollary 2.2];*
(c) *G is a perfect finite group of Lie type in the defining characteristic and B has full defect [22, Theorem];*
(d) *G has an abelian Sylow 2-subgroup and B is the principal 2-block [21, Theorem 5];*
(e) *G is a symmetric group or an alternating group [4, Theorems 5.3 and 5.5];*
(f) *$p = 2$ and G is a Schur cover of a symmetric group or of an alternating group [4, Theorems 5.3 and 5.5];*
(g) *$p \neq 2$ and G is a Schur cover of a symmetric group or of an alternating group such that the defect of B is at least 3 [4, Theorems 5.3 and 5.5].*

Moreover, we will use the following computational criterion throughout:

THEOREM 3.2. (Kawata's Criterion on Cartan matrices [20, Theorem 1.5]) *Let B be a wild p-block of kG. Suppose that there exists a simple B-module S lying on the nth row from the end of $\Gamma_s(S)$, where $n \geqslant 2$ is minimal with this property. Then there exist pairwise nonisomorphic simple B-modules $S_2, \ldots, S_n$ with the following properties:*

(a) *for each $2 \leqslant i \leqslant n$, we have that $S_i \cong \Omega^{2(i-2)}(S_2)$ and S_i lies at the end of $\Gamma_s(\Omega(S))$;*
(b) *the projective covers of $P(S_i)$ of the simple modules S_i $(2 \leqslant i \leqslant n)$ are uniserial of length $n+1$ with the following Loewy structure:*

$$P(S_2) = \boxed{\begin{matrix} S_2 \\ S_3 \\ \vdots \\ S_n \\ S \\ S_2 \end{matrix}}\,, P(S_3) = \boxed{\begin{matrix} S_3 \\ \vdots \\ S_n \\ S \\ S_2 \\ S_3 \end{matrix}}\,, \ldots, P(S_n) = \boxed{\begin{matrix} S_n \\ S \\ S_2 \\ \vdots \\ S_{n-1} \\ S_n \end{matrix}}\,.$$

The Cartan matrix of B is given by

$$\begin{pmatrix}
2 & 1 & \cdots & \cdots & 1 & 0 & \cdots & 0 \\
1 & 2 & \ddots & & \vdots & \vdots & & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots & \vdots & & \vdots \\
\vdots & & \ddots & 2 & 1 & 0 & \cdots & 0 \\
1 & \cdots & \cdots & 1 & * & \cdots & \cdots & * \\
0 & \cdots & \cdots & 0 & \vdots & & & \vdots \\
\vdots & & & \vdots & \vdots & & & \vdots \\
0 & \cdots & \cdots & 0 & * & \cdots & \cdots & *
\end{pmatrix},$$

where the columns are labeled by $S_n, \ldots, S_2, S, \ldots$ in this order.

REMARK 3.3.

(a) If the Cartan matrix of a block has the shape of Theorem 3.2(b) above with $n = 2$, then the simple module S corresponding to the second column lies on the 2nd row of its AR-component. Indeed in this case

$P(S_2) = \boxed{\begin{smallmatrix} S_2 \\ S \\ S_2 \end{smallmatrix}}$ and the standard sequence associated to S_2 is

$$0 \to \Omega(S_2) \to S \oplus P(S_2) \to \Omega^{-1}(S_2) \to 0,$$

so that S_2 lies at the end of its AR-component and S on the 2nd row of its AR-component.

A converse to Kawata's Criterion need not be true in general for an $n \geqslant 3$.

(b) The above was used to produce two counterexamples of simple modules not lying at the end of their AR-components. Namely, the group $F_4(2)$ for $p = 5$ has a simple module in the principal block of dimension 875823 lying on the 2nd row of its AR-component, and the group $2.Ru$ for $p = 3$ has a faithful simple module also lying on the 2nd row. See [22, §4]. Both counterexamples are obtained thanks to the decomposition matrices of these groups computed by Hiß.

We can now improve Kawata's result by describing more accurately the structure of the heart of the projective cover of the simple module S lying on the nth row of its AR-component.

COROLLARY 3.4. *With the assumptions and the notation of Theorem 3.2, we have that the heart of the projective cover of the simple module S is decomposable and has a uniserial indecomposable summand of length $n - 1$. More precisely*

$$\mathcal{H}(P(S)) = \boxed{\begin{matrix} S_2 \\ S_3 \\ \vdots \\ S_n \end{matrix}} \oplus V,$$

where V is an indecomposable kG-module.

Proof. First, since by assumption the module S does not lie at the end of its AR-component, which is of tree class A_∞, it is clear that the heart of the projective cover of S must be the direct sum of two indecomposable direct summands, say

$$\mathcal{H}(P(S)) \cong U \oplus V.$$

Using [20, Proposition 1.4], we have that the part of the component $\Gamma_s(S)$ below S is as follows:

$$\begin{tikzcd}
 & & S \arrow[dr, hook] & & \\
 & \boxed{\begin{smallmatrix} S \\ S_2 \end{smallmatrix}} \arrow[ur, two heads] \arrow[dr, hook] & & \boxed{\begin{smallmatrix} S_n \\ S \end{smallmatrix}} \arrow[dr, hook] & \\
\cdots \arrow[ur, two heads] & & \cdots \arrow[ur, two heads] & & \cdots
\end{tikzcd}$$

Therefore, using the fact that the Heller operator Ω induces a graph isomorphism from $\Gamma_s(S)$ to $\Gamma_s(\Omega(S))$, we have that in $\Gamma_s(\Omega(S))$ the diamond corresponding to the standard sequence associated to S is as follows:

$$\begin{tikzcd}
 & V \arrow[dr] & \\
\Omega(S) \arrow[ur] \arrow[dr] & & \Omega^{-1}(S) \\
 & \Omega\left(\boxed{\begin{smallmatrix} S_n \\ S \end{smallmatrix}}\right) = U \arrow[ur] &
\end{tikzcd}$$

Hence it suffices to compute $\Omega\left(\boxed{\begin{smallmatrix} S_n \\ S \end{smallmatrix}}\right)$.

Since the top of the uniserial module $\boxed{\begin{smallmatrix} S_n \\ S \end{smallmatrix}}$ is S_n, the projective cover of $\boxed{\begin{smallmatrix} S_n \\ S \end{smallmatrix}}$ must be $P(S_n)$. Hence taking the kernel of the canonical surjection $P(S_n) \twoheadrightarrow \boxed{\begin{smallmatrix} S_n \\ S \end{smallmatrix}}$, we obtain from the Loewy series of the PIM $P(S_n)$ in Theorem 3.2(c) that U has the following Loewy series

$$U = \Omega\left(\boxed{\begin{smallmatrix} S_n \\ S \end{smallmatrix}}\right) = \boxed{\begin{smallmatrix} S_2 \\ S_3 \\ \vdots \\ S_n \end{smallmatrix}}. \qquad \square$$

§4. Groups having a solvable quotient of p'-order

Throughout this section, we will assume that the following hypotheses hold:

HYPOTHESIS 4.1. *Assume that:*

(a) *G is a finite group of order divisible by p and $N \trianglelefteq G$ is a normal subgroup such that $|G/N| =: q$ is a prime number with $q \neq p$, and we set $G/N =: \langle gN \rangle$ for an element $g \in G \backslash N$.*
(b) *B and b are wild blocks of kG and kN, respectively, such that $1_B = 1_b$.*

LEMMA 4.2. *Assume Hypothesis 4.1 holds. Let $\zeta \in k^{\times}$ be a primitive qth root of unity in k, and for each $1 \leqslant j \leqslant q$ let Z_j be the one-dimensional $k(G/N)$-module defined by $Z_j = \langle \alpha_j \rangle_k$ and $\alpha_j \cdot gN = \zeta^{j-1} \alpha_j$, so that in particular $Z_1 = k_{G/N}$. The following holds:*

(a) *If $\mathcal{S} \in \mathrm{IBr}(B)$ is such that $\mathcal{S}{\downarrow_N}$ is not simple, then for each $1 \leqslant j \leqslant q$,*

$$\mathcal{S} \otimes_k Z_j \cong \mathcal{S}$$

as kG-modules, where we see Z_j as a kG-module via inflation.
(b) *There are integers $m \geqslant 1$ and $\ell \geqslant 0$ such that*

$$\mathrm{IBr}(B) = \{\mathcal{S}_{ij} \mid 1 \leqslant i \leqslant m; 1 \leqslant j \leqslant q\} \bigsqcup \{\mathcal{S}_i \mid m+1 \leqslant i \leqslant m+\ell\} \quad \text{and}$$

$$\mathrm{IBr}(b) = \{\mathcal{T}_i \mid 1 \leqslant i \leqslant m\} \bigsqcup \{\mathcal{T}_{ij} \mid m+1 \leqslant i \leqslant m+\ell, 1 \leqslant j \leqslant q\},$$

where for each $1 \leqslant i \leqslant m$ and each $1 \leqslant j \leqslant q$,

$$\mathcal{S}_{ij}{\downarrow_N} = \mathcal{T}_i \quad \text{and} \quad \mathcal{T}_i{\uparrow^G} = \mathcal{S}_{i1} \oplus \cdots \oplus \mathcal{S}_{iq},$$

and for each $m+1 \leqslant i \leqslant m+\ell$ and each $1 \leqslant j \leqslant q$,

$$\mathcal{S}_i{\downarrow_N} = \mathcal{T}_{i1} \oplus \mathcal{T}_{i2} \oplus \cdots \oplus \mathcal{T}_{iq} \quad \text{and} \quad \mathcal{T}_{ij}{\uparrow^G} = \mathcal{S}_i$$

where we may assume that $\mathcal{T}_{ij} = {\mathcal{T}_{i1}}^{g^{j-1}}$.
Moreover, we can assume that for each $1 \leqslant j \leqslant q$,

$$\mathcal{S}_{ij} = \mathcal{S}_{i1} \otimes_k Z_j.$$

Proof. (a) Let $1 \leqslant j \leqslant q$. By assumption and Clifford's theory we have that

$$\begin{aligned}(\mathcal{S} \otimes_k Z_j){\downarrow_N} &= \mathcal{S}{\downarrow_N} \otimes_k k_N \cong \mathcal{S}{\downarrow_N}. \\ &= \mathcal{T} \oplus \mathcal{T}^g \oplus \cdots \oplus \mathcal{T}^{g^{q-1}}\end{aligned}$$

for some $\mathcal{T} \in \mathrm{IBr}(b)$. Hence $\mathcal{T}{\uparrow^G} \cong \mathcal{S}$, and $\mathcal{T}{\uparrow^G} \cong \mathcal{S} \otimes_k Z_j$ for each $1 \leqslant j \leqslant q$.

(b) As by Hypothesis 4.1 the quotient G/N is cyclic, the claim follows from the result of Schur and Clifford [31, Chapter 3, Corollary 5.9 and Problem 11(i)]. ∎

Lemma 4.3. *Assume Hypothesis 4.1 holds. Let* $\mathcal{S} \in \mathrm{IBr}(B)$.

(a) *If* $\mathcal{S}{\downarrow_N} =: \mathcal{T}$ *is simple, then* $P(\mathcal{S}){\downarrow_N} \cong P(\mathcal{T})$ *and* $\mathcal{H}(P(\mathcal{S})){\downarrow_N} \cong \mathcal{H}(P(\mathcal{T}))$.

(b) *If* $\mathcal{S}{\downarrow_N}$ *is not simple, then we can write* $\mathcal{S}{\downarrow_N} = \mathcal{T}_1 \oplus \mathcal{T}_2 \oplus \cdots \oplus \mathcal{T}_q$ *with* $\mathcal{T}_j = {\mathcal{T}_1}^{g^{j-1}}$ *for each* $1 \leqslant j \leqslant q$ *and we have that*

$$P(\mathcal{S}){\downarrow_N} \cong P(\mathcal{T}_1) \oplus \cdots \oplus P(\mathcal{T}_q) \quad \textit{and} \quad \mathcal{H}(P(\mathcal{S})){\downarrow_N} \cong \bigoplus_{j=1}^{q} \mathcal{H}(P(\mathcal{T}_j)).$$

Proof. (a) Obviously

$$\begin{aligned}\mathcal{T} = \mathcal{S}{\downarrow_N} &= (P(\mathcal{S})/P(\mathcal{S})J){\downarrow_N} = P(\mathcal{S}){\downarrow_N}/(P(\mathcal{S})J){\downarrow_N} \\ &= P(\mathcal{S}){\downarrow_N}/(P(\mathcal{S})\, kG\tilde{J}) \quad \text{by Lemma 2.1} \\ &= P(\mathcal{S}){\downarrow_N}/P(\mathcal{S}){\downarrow_N}\, \tilde{J}.\end{aligned}$$

Hence the top of $P(\mathcal{S}){\downarrow_N}$ is $\mathcal{T}$, which implies that $P(\mathcal{S}){\downarrow_N} \cong P(\mathcal{T})$. Therefore,

$$\mathcal{H}(P(\mathcal{S})){\downarrow_N} = (P(\mathcal{S})\, J/\mathcal{S}){\downarrow_N} = \mathcal{H}(P(\mathcal{T})).$$

(b) Similar to (a). ∎

Proposition 4.4. *Assume Hypothesis 4.1 holds. If every simple module* $T \in \mathrm{IBr}(b)$ *lies at the end of its AR-component, then every simple module* $S \in \mathrm{IBr}(B)$ *lies at the end of its AR-component.*

Proof. Let $S \in \mathrm{IBr}(B)$ be a simple module. First assume that $S{\downarrow_N} =: T \in \mathrm{IBr}(b)$ is simple. Then by Lemma 4.3(a)

$$\mathcal{H}(P(S)){\downarrow_N} \cong \mathcal{H}(P(T)).$$

But by assumption $\mathcal{H}(P(T))$ is indecomposable, therefore so is $\mathcal{H}(P(S))$.

We assume now for the rest of the proof that $S{\downarrow_N}$ is not simple. If S lies at the end of its AR-component, then there is nothing to do. Therefore we now also assume that S lies on the nth row from the bottom of $\Gamma_s(S)$ for an integer $n \geqslant 2$, minimal (as in Kawata's Criterion on Cartan matrices). By Lemma 4.2(b),

$$S{\downarrow_N} = T_{11} \oplus \cdots \oplus T_{1q} \quad \text{and} \quad T_{1j}{\uparrow^G} = S \quad \text{for each } 1 \leqslant j \leqslant q,$$

where $T_{1j} = {T_{11}}^{g^{j-1}}$ for $1 \leqslant j \leqslant q$ are nonisomorphic simple modules in $\mathrm{IBr}(b)$. We also set $T_1 = T_{11}$.

Let $S_2, \ldots, S_n$ be the simple modules given by Theorem 3.2.

Claim 1. If the modules $S_2{\downarrow_N}, \ldots, S_n{\downarrow_N}$ are all nonsimple, then we have a contradiction.

Proof of Claim 1. By assumption and Lemma 4.2, we can write

$$S_i{\downarrow_N} = T_{i1} \oplus T_{i2} \oplus \cdots \oplus T_{iq}.$$

For each $2 \leqslant i \leqslant n$ we define $T_i \in \mathrm{IBr}(b)$ by $T_{ij} = T_i^{g^{j-1}}$, where $1 \leqslant j \leqslant q$. We claim that

$$P(T_2) = \boxed{\begin{matrix} T_2 \\ T_3 \\ \vdots \\ T_n \\ T_1 \\ T_2 \end{matrix}}, \; P(T_3) = \boxed{\begin{matrix} T_3 \\ \vdots \\ T_n \\ T_1 \\ T_2 \\ T_3 \end{matrix}}, \ldots, P(T_n) = \boxed{\begin{matrix} T_n \\ T_1 \\ T_2 \\ \vdots \\ T_{n-1} \\ T_n \end{matrix}}.$$

Indeed, we know by Theorem 3.2(b) and Lemma 4.3(b) that

$$\begin{aligned} P(T_2) \oplus P(T_2)^g \oplus \cdots \oplus P(T_2)^{g^{q-1}} &= P(S_2){\downarrow_N} \\ &= \boxed{\begin{matrix} S_2 \\ S_3 \\ \vdots \\ S_n \\ S \\ S_2 \end{matrix}}{\downarrow_N} = \boxed{\begin{matrix} T_2 & T_2{}^g & \cdots & T_2{}^{g^{q-1}} \\ T_3 & T_3{}^g & \cdots & T_3{}^{g^{q-1}} \\ & \cdots & & \\ T_n & T_n{}^g & \cdots & T_n{}^{g^{q-1}} \\ T_1 & T_1{}^g & \cdots & T_1{}^{g^{q-1}} \\ T_2 & T_2{}^g & \cdots & T_2{}^{g^{q-1}} \end{matrix}}, \end{aligned}$$

where the boxes mean the Loewy and socle series of the kN-modules. Since the left-hand side is a direct sum of exactly q indecomposable kN-modules that are $\langle g\rangle$-conjugate to each other, by interchanging the indices of $T_3, \ldots, T_n, T_1$ desired, we may assume that the PIM $P(T_2)$ has the desired structure. Then automatically the structures of $P(T_3), \ldots, P(T_n)$ are as claimed. ∎

Now, using a similar argument as above, we also obtain

$$P(T_1) \oplus P(T_1)^g \oplus \cdots \oplus P(T_1)^{g^{q-1}} = P(S){\downarrow_N} = \boxed{\begin{array}{c} \boxed{T_1 \oplus {T_1}^g \oplus \cdots \oplus {T_1}^{g^{q-1}}} \\ \\ \boxed{\boxed{\begin{array}{c} S_2 \\ S_3 \\ \vdots \\ S_n \end{array}}{\downarrow_N} \; \bigoplus \; V{\downarrow_N}} \\ \\ \boxed{T_1 \oplus {T_1}^g \oplus \cdots \oplus {T_1}^{g^{q-1}}} \end{array}},$$

for a kG-module V where the last equality holds by Corollary 3.4. Hence we have

$$\begin{aligned} &\mathcal{H}(P(T_1)) \oplus \mathcal{H}(P(T_1))^g \oplus \cdots \oplus \mathcal{H}(P(T_1))^{g^{q-1}} \\ &= \boxed{\begin{array}{c} T_2 \oplus {T_2}^g \oplus \cdots \oplus {T_2}^{g^{q-1}} \\ T_3 \oplus {T_3}^g \oplus \cdots \oplus {T_3}^{g^{q-1}} \\ \vdots \\ T_n \oplus {T_n}^g \oplus \cdots \oplus {T_n}^{g^{q-1}} \end{array}} \oplus V{\downarrow_N} \\ &= \left(\boxed{\begin{array}{c} T_2 \\ T_3 \\ \vdots \\ T_n \end{array}} \oplus \boxed{\begin{array}{c} T_2 \\ T_3 \\ \vdots \\ T_n \end{array}}^{g} \oplus \cdots \oplus \boxed{\begin{array}{c} T_2 \\ T_3 \\ \vdots \\ T_n \end{array}}^{g^{q-1}} \right) \oplus V{\downarrow_N} \end{aligned}$$

since $P(T_2), \ldots, P(T_n)$ are uniserial by the above.

But we are assuming that $T_2, \ldots, T_n$ lie at the end of their AR-components, so that $\mathcal{H}(P(T_2)), \ldots, \mathcal{H}(P(T_n))$ are indecomposable. Further $\mathcal{H}(P(T_1))$ is also indecomposable since T_1 lies at the end of its AR-connected component. Therefore the right-hand side term in the latter equation has exactly q indecomposable direct summands. This implies that $V = \{0\}$, hence a contradiction.

Claim 2. If the modules $S_2{\downarrow_N}, \ldots, S_n{\downarrow_N}$ are all simple, then we have a contradiction.

Proof of Claim 2. Set $T_i = S_i{\downarrow_N}$ for $2 \leqslant i \leqslant n$. We have

$$S{\downarrow_N} = T_1 \oplus {T_1}^g \oplus \cdots \oplus {T_1}^{g^{q-1}}.$$

$\square$

By the assumption and Lemma 4.2, for each $2 \leqslant i \leqslant n$ we can write $T_i{\uparrow^G} = S_{i1} \oplus \cdots \oplus S_{iq}$ with $S_{ij} = S_{i1} \otimes_k Z_j$ for $1 \leqslant j \leqslant q$. In particular $S_{i1} = S_i$ for each $2 \leqslant i \leqslant n$. By Theorem 3.2(b)

$$P(S_2) = \boxed{\begin{matrix} S_2 \\ S_3 \\ \vdots \\ S_n \\ S \\ S_2 \end{matrix}},$$

so Lemma 4.2(a) implies that

$$P(S_{2j}) = P(S_2) \otimes_k Z_j = \boxed{\begin{matrix} S_{2j} \\ S_{3j} \\ \vdots \\ S_{nj} \\ S \\ S_{2j} \end{matrix}} \quad \text{for } 1 \leqslant j \leqslant q.$$

These yield that $P(S)$ has q distinct uniserial submodules

$$W_1 = \boxed{\begin{matrix} S_2 \\ S_3 \\ \vdots \\ S_n \\ S \end{matrix}}, \quad W_j = \boxed{\begin{matrix} S_{2j} \\ S_{3j} \\ \vdots \\ S_{nj} \\ S \end{matrix}} \quad \text{of Loewy length } n \text{ for } j = 2, \cdots, q.$$

Set $W = W_1 + W_2 + \cdots + W_q \subseteq P(S)$. Then $\operatorname{soc}(W) = S$, and the Loewy and socle structure of W is as follows:

$$W=\boxed{\begin{matrix}\boxed{\begin{matrix}S_2\\S_3\\\vdots\\S_n\end{matrix}}\oplus\boxed{\begin{matrix}S_{22}\\S_{32}\\\vdots\\S_{n2}\end{matrix}}\oplus\cdots\oplus\boxed{\begin{matrix}S_{2q}\\S_{3q}\\\vdots\\S_{nq}\end{matrix}}\\ \\ S\end{matrix}}$$

with simple socle isomorphic to S. Therefore W/S has a proper uniserial submodule

$$U=\boxed{\begin{matrix}S_2\\S_3\\\vdots\\S_n\end{matrix}}.$$

Now by Corollary 3.4, $U|\mathcal{H}(P(S))$, so that by Lemma 4.2(a)

$$\begin{aligned}\boxed{\begin{matrix}S_{2j}\\S_{3j}\\\vdots\\S_{nj}\end{matrix}}&=\boxed{\begin{matrix}S_2\\S_3\\\vdots\\S_n\end{matrix}}\otimes_k Z_j\\ &=(U\otimes_k Z_j)\mid(\mathcal{H}(P(S))\otimes_k Z_j)\cong\mathcal{H}(P(S\otimes_k Z_j))\cong\mathcal{H}(P(S))\end{aligned}$$

for each $1\leqslant j\leqslant q$. Therefore $q=2$ since $\mathcal{H}(P(S))$ has exactly two nonprojective indecomposable direct summands by the assumption that S does not lie at the end of its AR-component. Notice that this already provides a contradiction in case the characteristic of k is 2, since we assume $q\neq p$. So we now assume that $p\geqslant 3$. Then, the Loewy and socle structures of PIMs $P(S)$, $P(S_i)$ and $P(S_{i2})$ for $2\leqslant i\leqslant n$ are:

$$\boxed{\begin{matrix}S\\\boxed{\begin{matrix}S_2\\S_3\\\vdots\\S_n\end{matrix}}\ \boxed{\begin{matrix}S_{22}\\S_{32}\\\vdots\\S_{n2}\end{matrix}}\\S\end{matrix}},\ \boxed{\begin{matrix}S_2\\S_3\\\vdots\\S_n\\S\\S_2\end{matrix}},\ \boxed{\begin{matrix}S_{22}\\S_{32}\\\vdots\\S_{n2}\\S\\S_{22}\end{matrix}},\ \boxed{\begin{matrix}S_3\\\vdots\\S_n\\S\\S_2\\S_3\end{matrix}},\ \boxed{\begin{matrix}S_{32}\\\vdots\\S_{n2}\\S\\S_{22}\\S_{32}\end{matrix}},\ldots,\boxed{\begin{matrix}S_n\\S\\S_2\\\vdots\\S_{n-1}\\S_n\end{matrix}},\ \boxed{\begin{matrix}S_{n2}\\S\\S_{22}\\\vdots\\S_{n-1,2}\\S_{n,2}\end{matrix}}.$$

Now considering the restrictions $S{\downarrow}_N$ and $S_i{\downarrow}_N$ for $2 \leqslant i \leqslant n$, we obtain by Lemma 4.3 that the Loewy and socle structures of the PIMs $P(T_1)$, $P(T_1^g)$ and $P(T_i)$ for each $2 \leqslant i \leqslant n$ are

$$\boxed{\begin{array}{c} T_1 \\ T_2 \\ T_3 \\ \vdots \\ T_n \\ T_1 \end{array}}, \quad \boxed{\begin{array}{c} {T_1}^g \\ T_2 \\ T_3 \\ \vdots \\ T_n \\ {T_1}^g \end{array}}, \quad \boxed{\begin{array}{c} T_2 \\ T_3 \\ \vdots \\ T_n \\ T_1 \oplus {T_1}^g \\ T_2 \end{array}}, \quad \boxed{\begin{array}{c} T_3 \\ \vdots \\ T_n \\ T_1 \oplus {T_1}^g \\ T_2 \\ T_3 \end{array}}, \ldots, \boxed{\begin{array}{c} T_n \\ T_1 \oplus {T_1}^g \\ T_2 \\ T_3 \\ \vdots \\ T_n \end{array}}$$

since $T_1 \ncong {T_1}^g$. Now, as the dimension of any PIM for kN is divisible by $|N|_p =: p^a$ for an integer $a \geqslant 1$, and since $\dim\ T_1 = \dim\ {T_1}^g$, we have for each $2 \leqslant i \leqslant n$

$$0 \equiv \dim\ P(T_i) - \dim\ P(T_1) = \dim\ T_i (\mathrm{mod}\ p^a),$$

so that

$$\begin{aligned} 0 &\equiv \dim\ P(T_1) \\ &\equiv \dim\ P(T_1) - (\dim\ T_2 + \dim\ T_3 + \cdots + \dim\ T_n) = 2 \cdot \dim\ T_1 (\mathrm{mod}\ p^a). \end{aligned}$$

This implies that

$$\dim\ T_1 \equiv 0 (\mathrm{mod}\ p^a)$$

since $p \neq 2$ (since $q = 2$). Thus, $\dim\ T_i \equiv 0 \pmod{p^a}$ for any $1 \leqslant i \leqslant n$. Now, looking at the composition factors of the PIMs $P(T_1), P({T_1}^g), P(T_2), \ldots, P(T_n)$, we know that $\mathrm{IBr}(b) = \{T_1, {T_1}^g, T_2, \ldots, T_n\}$, which implies that $p^a \mid \dim\ \mathcal{T}$ for any $\mathcal{T} \in \mathrm{IBr}(b)$. Now it follows from Brauer's result [31, Chapter 3 Theorem 6.25] that there is a simple $\mathcal{T} \in \mathrm{IBr}(b)$ such that $\nu_p(\dim\ \mathcal{T}) = a - d(b)$ (where $d(b)$ is the defect of b). Hence we have a contradiction since b is a wild block, i.e., of positive defect.

Claim 3.

(a) If there is an integer $2 \leqslant m \leqslant n-1$ such that $S_2{\downarrow}_N, \ldots, S_m{\downarrow}_N$ are not simple and $S_{m+1}{\downarrow}_N$ is simple, then we have a contradiction.

(b) If there is an integer $2 \leqslant m \leqslant n-1$ such that $S_2{\downarrow}_N, \ldots, S_m{\downarrow}_N$ are simple and $S_{m+1}{\downarrow}_N$ is not simple, then we have a contradiction.

Proof of Claim 3. (a) Set $T_{m+1} = S_{m+1}{\downarrow_N}$. By Lemma 4.2 there exists a simple module $T_m \in \mathrm{IBr}(b)$ with $S_m{\downarrow_N} = T_m \oplus {T_m}^g \oplus \cdots \oplus {T_m}^{g^{q-1}}$. Then, by Lemma 4.2,

$$T_{m+1}{\uparrow^G} = S_{m+1} \oplus S_{m+1,2} \oplus \cdots \oplus S_{m+1,q}$$

where $S_{m+1,j} = S_{m+1} \otimes_k Z_j$ for each $1 \leqslant j \leqslant q$ and $T_m{\uparrow^G} = S_m$. By the structure of $P(S)$, we have that $\mathrm{Ext}^1_{kG}(S_m, S_{m+1}) \neq 0$. Therefore by Eckmann–Shapiro's lemma we have that $\mathrm{Ext}^1_{kN}(T_m, T_{m+1}) \neq 0$. Thus there exists a kN-module with Loewy structure

$$\boxed{\begin{matrix} T_m \\ T_{m+1} \end{matrix}}.$$

So it follows from Lemma 2.1 that

$$\boxed{\begin{matrix} T_m \\ T_{m+1} \end{matrix}}{\uparrow^G} = \boxed{\begin{matrix} S_m \\ S_{m+1} \oplus S_{m+1,2} \oplus \cdots \oplus S_{m+1,q} \end{matrix}}$$

where the right-hand side box is the Loewy and socle series. But $P(S_m)$ is uniserial by Theorem 3.2(b), so applying again Lemma 2.1, we must have $q = 1$, which contradicts the assumption that q is a prime.

(b) follows in a similar fashion using a dual argument. □

Altogether, Claims 1–3 prove that the simple modules $S_2, \ldots, S_n$ cannot exist, therefore S must lie at the end of its AR-component. □

As a consequence of the above discussion we obtain Theorem B of the Introduction.

Proof of Theorem B. Because G/N is solvable of order prime to p, it follows by induction on $|G/N|$, that we may assume that $|G/N|$ is a prime distinct from p. Then Proposition 4.4 yields the result. □

§5. The principal block of $O^{p'}(G)$

From now on, we assume that $p \geqslant 3$ and G is a finite group with nontrivial abelian Sylow p-subgroups. Because we consider the principal block only, we assume that $O_{p'}(G) = 1$ since $B_0(kG) \cong B_0\,(\,k(G/O_{p'}(G))$ as k-algebras.

The structure of $O^{p'}(G)$ can be obtained using the classification of finite simple groups and a result of Fong and Harris [12, 5A–5C].

LEMMA 5.1. [9, Theorem 1.7] *Let p be an odd prime. Let G be a finite group with a nontrivial abelian Sylow p-subgroup. Then*

$$O^{p'}(G/O_{p'}(G)) \cong Q \times H_1 \times \cdots \times H_m,$$

where m is a nonnegative integer (i.e., possibly $O^{p'}(G/O_{p'}(G)) \cong Q$),$Q$ is an abelian p-group, and for each $1 \leqslant i \leqslant m$, H_i is a nonabelian simple group with nontrivial Sylow p-subgroups.

Therefore, we fix the notation $O^{p'}(G) = Q \times H_1 \times \cdots \times H_m$, where Q is an abelian p-group, and $H_1, \ldots, H_m$ are nonabelian simple groups with nontrivial Sylow p-subgroups as given by Lemma 5.1.

5.1 Simple modules in infinite tubes $\mathbb{Z}A_\infty/\langle\tau^a\rangle$

LEMMA 5.2. ([21, Lemma 5.2] generalized version) *Let $H = \tilde{H}_1 \times \cdots \times \tilde{H}_m$ $(m \geqslant 1)$ be a finite group such that $p \mid |\tilde{H}_i|$ for each $1 \leqslant i \leqslant m$. If $B_0(kH)$ is a wild block and contains a periodic simple module, then $m = 1$.*

Proof. Let S be a simple periodic $B_0(kH)$-module. Then we may write $S = S_1 \otimes_k \cdots \otimes_k S_m$ where S_i is a simple $B_0(k\tilde{H}_i)$-module for each $1 \leqslant i \leqslant m$. Then, by iterating [21, Lemma 2.2], there exists an index $1 \leqslant i_0 \leqslant m$ such that S_{i_0} is periodic and S_j is a projective $k\tilde{H}_j$-module for each $1 \leqslant j \neq i_0 \leqslant m$. But $B_0(k\tilde{H}_j)$ cannot contain a simple projective module, since we assume that $p \mid |\tilde{H}_i|$ for each $1 \leqslant i \leqslant m$. Hence this forces $H = \tilde{H}_{i_0}$, i.e., $m = 1$. ∎

As a consequence, the existence of simple periodic modules in the principal block lying in tubes drastically restricts the possible structure of $O^{p'}(G)$.

COROLLARY 5.3. *If $B_0(kG)$ contains a periodic simple module, then $O^{p'}(G) = H_1$ is a nonabelian finite simple group with noncyclic abelian Sylow p-subgroups.*

Proof. By Lemma 5.2, either $O^{p'}(G) = Q$ or $O^{p'}(G) = H_1$. But the former cannot happen. Indeed, the indecomposable direct summands of the restriction to $O^{p'}(G)$ of a simple periodic kG-module are all simple periodic modules, however the unique simple kQ-module is the trivial module, which is not periodic since we assume that $B_0(kG)$ is wild, and hence Q is noncyclic. This leaves only the possibility $O^{p'}(G) = H_1$, and the p-rank of H_1 must be at least 2 again because we assume that $B_0(kG)$ is wild. ∎

This immediately leads to the following reduction to nonabelian simple groups:

COROLLARY 5.4. *Assume that every periodic simple* $B_0(kH)$*-module lies at the end of its AR-component for every nonabelian finite simple group* H *with noncyclic abelian Sylow* p*-subgroups. Then every simple periodic* $B_0(kO^{p'}(G))$*-module lies at the end of its AR-component for any finite group* G *with* $O_{p'}(G)=1$ *and noncyclic abelian Sylow* p*-subgroups.*

5.2 Simple modules in $\mathbb{Z}A_\infty$-components

LEMMA 5.5. *Let* $H=\tilde{H}_1\times\cdots\times\tilde{H}_m$ $(m\geqslant 1)$ *be a finite group with abelian Sylow* p*-subgroups such that* $p\mid|\tilde{H}_i|$ *for each* $1\leqslant i\leqslant m$*. If* $B_0(kH)$ *is a wild block containing a nonperiodic simple module* S *not lying at the end of its AR-component, then* $m=1$*.*

This lemma and its proof below generalize parts of the proof of [21, Theorem 5(i)].

Proof. Assume that $m\geqslant 2$. Then by Theorem 3.2(b), there exists a simple $B_0(kH)$-module T lying at the end of $\Gamma_s(\Omega(S))$. By Knörr's Theorem [23, 3.7 Corollary], we know that the vertices of the simple modules in $B_0(kH)$ are the Sylow p-subgroups of H, because they are abelian. Now by assumption $\Gamma_s(S)\cong\mathbb{Z}A_\infty$ by [11], which implies that all the modules in $\Gamma_s(S)$ and $\Gamma_s(\Omega(S))$ have the Sylow p-subgroups as their vertices by [32, Theorem]. So all the modules in $\Gamma_s(S)$ and $\Gamma_s(\Omega(S))$ are not projective relatively to the subgroup $N=\tilde{H}_1\times\cdots\times\tilde{H}_{m-1}$ as it does not contain a Sylow p-subgroup of H. Thus, as $p\neq 2$, all the simple direct summands of $S{\downarrow}_N$ belong to blocks of defect zero by [21, Lemma 1.4]. But

$$B_0(kH)=B_0(kN)\otimes_k B_0(k\tilde{H}_m)$$

and hence there exist a simple $B_0(kN)$-module S_0 and a simple $B_0(k\tilde{H}_m)$-module S_m such that

$$S=\mathrm{Inf}^{H}_{N\times\tilde{H}_m/1\times\tilde{H}_m}(S_0)\otimes_k\mathrm{Inf}^{H}_{N\times\tilde{H}_m/N\times 1}(S_m).$$

Because $S{\downarrow}_N\cong(\dim_k S_m)S_0$, the module S_0 is a projective kN-module by the above. Hence by Lemma 2.3 S is relatively $\tilde{H}_m$-projective. This contradicts the fact that the vertices of S are the Sylow p-subgroups of H. Thus we conclude that S must lie at the end of $\Gamma_s(S)$. ∎

PROPOSITION 5.6. *Let G be a finite group with $O_{p'}(G)=1$ and noncyclic abelian Sylow p-subgroups. Assume moreover that one of Conditions (i), (ii), or (iii) of Theorem C is satisfied. Then every nonperiodic simple $B_0(kO^{p'}(G))$-module lies at the end of its AR-component.*

Proof. We have $O^{p'}(G)=Q$ or $O^{p'}(G)=Q\times H_1\times\cdots\times H_m$, where Q is an abelian p-group and H_i is a nonabelian finite simple group with nontrivial Sylow p-subgroups for each $1\leqslant i\leqslant m$.

If (i) holds, that is $Q\neq 1$, then by Theorem 3.1(a), all simple $B_0(kO^{p'}(G))$-modules lie at the end of their AR-components. Therefore, we assume for the rest of the proof that $Q=1$.

Next if (ii) holds, that is $m\geqslant 2$, the claim follows from Lemma 5.5.

Finally if (iii) holds, that is $O^{p'}(G)=H_1$, then H_1 must have a noncyclic Sylow p-subgroup, therefore all simple $B_0(kO^{p'}(G))$-modules lie at the end of their AR-components by assumption. □

§6. Reduction to $O^{p'}(G)$

We continue assuming that G is a finite group with noncyclic abelian Sylow p-subgroups such that $O_{p'}(G)=1$, unless otherwise stated. We now prove that an answer to Question A is detected by restriction to the normal subgroup $O^{p'}(G)$ of G.

We set $H=O^{p'}(G)$, let $P\in\mathrm{Syl}_p(H)$ be a Sylow p-subgroup, and set $N=HC_G(P)$. Moreover we set $B=B_0(kG)$, $b=B_0(kN)$ and $\tilde{b}=B_0(kH)$. Then N is Dade's Group $G[\tilde{b}]$ and $N\trianglelefteq G$, see Lemma 2.2.

First of all Question A has an affirmative answer for the group N if and only if it has an affirmative answer for the group H.

LEMMA 6.1. *With the above notation, every simple b-module lies at the end of its AR-component if and only if every simple $\tilde{b}$-module lies at the end of its AR-component.*

Proof. By the Alperin–Dade theorem [8, Theorem] (see [1]), the blocks b and $\tilde{b}$ are isomorphic as k-algebras, hence Morita equivalent. But for a simple module, lying at the end of its AR-component is a property preserved by Morita equivalence. □

PROPOSITION 6.2. *If every simple $\tilde{b}$-module lies at the end of its AR-component, then every simple B-module lies at the end of its AR-component.*

Proof. Let S be a simple B-module and let T be a simple direct summand of $S{\downarrow_H}$. Then T is periodic if and only if S is. Therefore $\Gamma_s(S)\cong\mathbb{Z}A_\infty$ if and only if $\Gamma_s(T) \cong \mathbb{Z}A_\infty$, and $\Gamma_s(S)$ is an infinite tube with tree class A_∞ if and only if $\Gamma_s(T)$ is an infinite tube with tree class A_∞.

In case $\Gamma_s(S) \cong \mathbb{Z}A_\infty$, then S lies at the end of $\Gamma_s(S)$ if and only if T lies at the end of $\Gamma_s(T)$ by [21, Lemma 1.5].

In case $\Gamma_s(S)$ is an infinite tube with tree class A_∞, then by Corollary 5.3, H is a nonabelian finite simple group with noncyclic abelian Sylow p-subgroups. Now, by Schreier's conjecture (now proven by the Classification of Finite Simple Groups, see [15, Definition 2.1] and [16, Theorem 7.1.1]), we know that G/H is a solvable p'-subgroup of $\mathrm{Out}(H)$. Now by Lemma 6.1, we may assume $H = N$ and by Lemma 2.2(c) we have $1_B = 1_b$. Therefore Theorem B implies that S lies at the end of $\Gamma_s(S)$ because every simple b-module lies at the end of its AR-component. ∎

As a corollary, we obtain Theorem C of the Introduction.

Proof of Theorem C. Let G be a finite group with noncyclic abelian Sylow p-subgroups. As $B_0(kG) \cong B_0(k\,G/O_{p'}(G))$ as k-algebras, we may assume that $O_{p'}(G) = 1$. Therefore, by Proposition 6.2, every simple $B_0(kG)$-module lies at the end of its AR-component if every simple $B_0(k\,O^{p'}(G))$-module lies at the end of its AR-component. Now if $B_0(kG)$ contains a periodic simple module, then by Corollary 5.3 we must have that $O^{p'}(G) = H_1$ is a nonabelian finite simple group with noncyclic abelian Sylow p-subgroups, then the claim holds by Corollary 5.4. Therefore we may assume that $B_0(kG)$, and hence $B_0(k\,O^{p'}(G))$, contains no periodic simple module. In this case, if one of Conditions (i), (ii), or (iii) holds, then the claim follows from Proposition 5.6. ∎

Now Corollary D is a direct consequence of Theorem C.

§7. Principal 3-blocks

We now fix $p = 3$, and continue assuming that G is a finite group with noncyclic Sylow 3-subgroups, so that $B_0(kG)$ is wild. We may also assume that $O_{3'}(G) = 1$.

We start by investigating principal 3-blocks of nonabelian finite simple groups with abelian defect group. To this aim, we recall that the list of nonabelian finite simple groups with abelian Sylow 3-subgroups is known by the classification of finite simple groups and was determined by Paul Fong (in an unpublished manuscript).

PROPOSITION 7.1. [25, Proposition 4.3] *If G is a nonabelian finite simple group with noncyclic abelian Sylow 3-subgroup, then G is one of:*

(i) $\mathfrak{A}_7$, $\mathfrak{A}_8$, M_{11}, M_{22}, M_{23}, HS, $O'N$;
(ii) $\mathrm{PSL}_3(q)$ *for a prime power q such that* $3||(q-1)$;
(iii) $\mathrm{PSU}_3(q^2)$ *for a prime power q such that* $3||(q+1)$;
(iv) $\mathrm{PSp}_4(q)$ *for a prime power q such that* $3|(q-1)$;
(v) $\mathrm{PSp}_4(q)$ *for a prime power q such that $q>2$ and* $3|(q+1)$;
(vi) $\mathrm{PSL}_4(q)$ *for a prime power q such that $q>2$ and* $3|(q+1)$;
(vii) $\mathrm{PSU}_4(q^2)$ *for a prime power q such that* $3|(q-1)$;
(viii) $\mathrm{PSL}_5(q)$ *for a prime power q such that* $3|(q+1)$;
(ix) $\mathrm{PSU}_5(q^2)$ *for a prime power q such that* $3|(q-1)$; *or*
(x) $\mathrm{PSL}_2(3^n)$ *for an integer* $n \geqslant 2$.

As a consequence we obtain:

PROPOSITION 7.2. *If G is a nonabelian finite simple group with noncyclic abelian Sylow 3-subgroups, then every simple $B_0(kG)$-modules lies at the end of its component in $\Gamma_s(B_0(kG))$.*

Proof. Let $P \in \mathrm{Syl}_3(G)$, and set $N = N_G(P)$ and $B_0 = B_0(kG)$. We go through the list of groups in Proposition 7.1.

In case (i), in all cases all simple B_0-modules lie at the end of their component in $\Gamma_s(B_0)$ by Theorem 3.2(b): indeed if G is one of $\mathfrak{A}_8$, M_{22} or $O'N$, then one checks from GAP [14] that the Cartan matrix of B_0 has no diagonal entry equal to 2. If G is one of $\mathfrak{A}_7$, M_{11}, M_{23}, or HS, then one checks from GAP [14] that the Cartan matrix of B_0 does not have the shape of Theorem 3.2(b) either.

In case (ii), then the Cartan matrix of B_0 is computed in [28, Table 2] and does not satisfy Theorem 3.2(b).

Next if G is one of the groups listed in Proposition 7.1(iii), (iv), (vii), or (ix), then it is proven in [25, Lemma 3.7] that B_0 is Puig equivalent to $B_0(kN)$. But N has a nontrivial normal Sylow 3-subgroup, therefore all simple $B_0(kN)$-modules lie at the end of their components in $\Gamma_s(B_0(kN))$ by Theorem 3.1(a), and therefore so do the simple B_0-modules via the latter Puig (Morita) equivalence.

In case (v), the decomposition numbers of B_0 were computed by White and Okuyama–Waki. If q is even then we read from [36, Table II] that each column of the decomposition matrix of B_0 has at least 3 positive entries. If q is odd, then the decomposition matrix of B_0 is given in [35, Theorem 4.2] up to two parameters α and β. But [33, Theorem 2.3] proves that $\alpha \in \{1, 2\}$. This is enough to see that each column of the decomposition matrix of B_0 has at least 3 positive entries. Therefore in both cases all the diagonal entries of the Cartan matrix of B_0 are at least 3.

In cases (vi) and (viii), we proceed as follows. For $n \in \{4, 5\}$ fixed, we may regard $B_0(k\mathrm{PSL}_n(q))$ as the principal block of $\mathrm{SL}_n(q)$ as $3 \nmid |Z(\mathrm{SL}_n(q))|$. Then we check that the Cartan matrix of $B_0(k\mathrm{GL}_n(q))$ does not satisfy Theorem 3.2(b). To this end we use the information on the decomposition numbers of $B_0(k\mathrm{GL}_n(q))$ provided in [19, Appendix I]. In both cases, it is enough to consider only the square submatrix $\Delta_{n,0}$ of the decomposition matrix of $B_0(k\mathrm{GL}_n(q))$ whose rows are indexed by the unipotent characters. Both in case $n = 4$ and $n = 5$, there are five modular characters in the principal block (using [13]) and

$$\Delta_{4,0} = \left[\begin{array}{c|ccccc} (4) & 1 & & & & \\ (31) & 1 & 1 & & & \\ (2^2) & & 1 & 1 & & \\ (21^2) & 1 & 1 & 1 & 1 & \\ (1^4) & 1 & & & 1 & 1 \end{array}\right] \qquad \Delta_{5,0} = \left[\begin{array}{c|ccccc} (5) & 1 & & & & \\ (32) & & 1 & & & \\ (31^2) & 1 & 1 & 1 & & \\ (2^21) & & 1 & 1 & 1 & \\ (1^5) & 1 & & 1 & & 1 \end{array}\right].$$

(See e.g., [24, Propositions 3.1 and 4.1].) It follows that the Cartan integers of $B_0(\mathrm{GL}_n(q))$ have lower bounds given by the entries of the following matrices:

$$^{T}\Delta_{4,0}\Delta_{4,0} = \begin{bmatrix} 4 & 2 & 1 & 2 & 1 \\ 2 & 3 & 2 & 1 & 0 \\ 1 & 2 & 2 & 1 & 0 \\ 2 & 1 & 1 & 2 & 1 \\ 1 & 0 & 0 & 1 & 1 \end{bmatrix} \qquad ^{T}\Delta_{5,0}\Delta_{5,0} = \begin{bmatrix} 3 & 1 & 2 & 0 & 1 \\ 1 & 3 & 2 & 1 & 0 \\ 2 & 2 & 3 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Therefore the Cartan matrix of $B_0(k\mathrm{GL}_n(q))$ cannot satisfy Theorem 3.2(b), and we conclude that all simple $B_0(k\mathrm{GL}_n(q))$-modules lie at the end of their AR-components. Now, from the known values of the unipotent characters of $\mathrm{GL}_n(q)$, we easily check using CHEVIE [6] that the dimensions of the simple modules in $B_0(k\mathrm{GL}_n(q))$ are prime to 3. Hence they cannot be periodic by [5], as $3^{(a-1)}$ must divide the dimension of any simple periodic module, where $a =$ the p-rank of the group, but in our case $a \geqslant 2$. Therefore every simple $B_0(k\mathrm{SL}_n(q))$-module lies at the end of its AR-component by [21, Lemma 1.5].

Finally, if $G = \mathrm{PSL}_2(3^n)$ for some integer $n \geqslant 2$, then the claim follows from Theorem 3.1(c) as G is a finite simple group of Lie type in defining characteristic. ∎

As a corollary we obtain Theorem E of the Introduction.

Proof of Theorem E. The claim now follows from Corollary D together with Proposition 7.1. ∎

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