The determination of the centre of gravity of a circular arc by reference to a principle of Dynamics.

By G. E. Crawford, M.A.

Mr H. Poincaré, in a recent work (*la Valeur de la Science*) speaks of the interest attaching to proofs of theorems in pure geometry by arguments derived from mechanical or physical considerations. The following is a case in point.

To find the centre of gravity of a circular arc by reference to a principle of dynamics.

Let ABC be a material circular arc rotating freely about its centre O under no forces (Fig. 7).

Then constants are

ω, the uniform angular velocity about O,

r, the radius of the circle,

T, the tension anywhere,

ρ, the line density.

Choose any arc AB to isolate mentally and consider its motion. The non-constants are

$2a$, the angle AB subtends at O,

x, the required distance from O to G the centre of gravity of the arc.

Then we have the equation of acceleration along GO,

$$(2ar\rho)\omega^2 x = 2\mathrm{T}\sin a, \quad \text{or} \quad \frac{xa}{\sin a} = \frac{\mathrm{T}}{\omega^2 r\rho}.$$

We can now take any other arc in which y, β correspond to x, a respectively and we get $\dfrac{y\beta}{\sin\beta} = \dfrac{\mathrm{T}}{\omega^2 r\rho}$ as before.

$$\therefore \quad \frac{xa}{\sin a} = \frac{y\beta}{\sin\beta}.$$

But β can be chosen to vanish, in which case y becomes equal to r and $\dfrac{\beta}{\sin\beta}$ to unity,

$$\therefore \quad x = \frac{r.\sin a}{a}. \qquad \text{Q.E.D.}$$

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On the Reduction of $\int \frac{(Lx+M)dx}{(Ax^2+2Bx+C)^m\sqrt{ax^2+2bx+c}}$.

By D. K. Picken, M.A.

The simplest type of irrational algebraic function of x is given by $f(x, y)$ a rational function of x and y, in which y is a variable whose dependence on x is determined by the equation $\phi(x, y)=0$, where $\phi(x, y)$ is a polynomial of the second degree in x and y.

It is clear that the function can be expressed in the real form

$$\psi(x)+\Sigma\left\{Ax^m+\frac{B}{(x-p)^m}+\frac{Lx+M}{(Ax^2+2Bx+C)^m}\right\}\frac{1}{\sqrt{ax^2+2bx+c}}$$

where $\psi(x)$ is a rational function of x; and for the integration we have to obtain formulæ of reduction for

$$\text{(i)}\ \int\frac{x^m dx}{\sqrt{ax^2+2bx+c}}, \qquad \text{(ii)}\ \int\frac{dx}{(x-p)^m\sqrt{ax^2+2bx+c}},$$

and (iii) $\int\frac{(Lx+M)dx}{(Ax^2+2Bx+C)^m\sqrt{ax^2+2bx+c}}$ (where B^2-AC is negative).

The method of integration by parts can be simply applied to the cases (i) and (ii) and we get reduction formulæ, which may be regarded as the identities obtained by differentiating the functions

$$x^{m-1}\sqrt{ax^2+2bx+c} \quad\text{and}\quad \frac{\sqrt{ax^2+2bx+c}}{(x-p)^{m-1}}.$$

The use of this method is not quite so obvious in case (iii); but the purpose of this note is to show that reduction formulæ can be obtained by differentiations similar to those in cases (i) and (ii).

If $$I_m = \int \frac{dx}{(Ax^2+2Bx+C)^m\sqrt{ax^2+2bx+c}} \equiv \int \frac{dx}{S^m\sqrt{R}},$$

$$I_m' = \int \frac{x\,dx}{S^m\sqrt{R}}, \quad P_m = \frac{\sqrt{R}}{S^{m-1}}, \quad P_m' = \frac{x\sqrt{R}}{S^{m-1}};$$

then

$$\frac{d}{dx}(P_m) = \frac{ax+b}{S^{m-1}\sqrt{R}} - 2(m-1)\,.\,\frac{(Ax+B)(ax^2+2bx+c)}{S^m\sqrt{R}}$$

$$= \frac{ax+b}{S^{m-1}\sqrt{R}} - 2(m-1)\,.\,\frac{(ax+p_1)S+q_1x+r_1}{S^m\sqrt{R}},$$

where p_1, q_1, r_1 are constants obtained from the identity

$$(ax+p_1)(Ax^2+2Bx+C)+q_1x+r_1 \equiv (Ax+B)(ax^2+2bx+c).$$

Hence we can write

$$\frac{d}{dx}(P_m) = \frac{A_mx+B_m}{S^{m-1}\sqrt{R}} + \frac{C_mx+D_m}{S^m\sqrt{R}},$$

where A_m, B_m, C_m, D_m are determinate constants; and, therefore,

$$[P_m] = A_m\,.\,I'_{m-1} + B_m\,.\,I_{m-1} + C_m\,.\,I_m' + D_m\,.\,I_m \ldots\ldots (m).$$

Similarly,

$$\frac{d}{dx}(P_m') = \frac{ax^2+2bx+c+x(ax+b)}{S^{m-1}\sqrt{R}} - 2(m-1)\frac{x(Ax+B)(ax^2+2bx+c)}{S^m\sqrt{R}}$$

$$= \frac{\frac{2a}{A}S+p_2x+q_2}{S^{m-1}\sqrt{R}} - 2(m-1)\frac{\frac{a}{A}S^2+(p_3x+q_3)S+r_3x+s_3}{S^m\sqrt{R}},$$

where the coefficients p_2, q_2, p_3, q_3, r_3, s_3 are determinate as above.

Thus $[P_m'] = E_m\,.\,I_{m-2} + F_m\,.\,I'_{m-1} + G_mI_{m-1} + H_mI_m' + K_m\,.\,I_m \ldots (m)'$.

The equations (m) and $(m)'$ give I_m and I_m' in terms of I_{m-1}, I'_{m-1}, I_{m-2}; thus from the equations (2) and (2)$'$, which form the last pair of the system, we get I_2, I_2' in terms of I_1, I_1' and I_0, and finally we express I_m and I_m' in terms of I_1, I_1' and I_0.

(In numerical cases the ordinary algebraic devices for obtaining the numbers p_1, q_1, ... r_3, s_3 would, of course, be employed.)