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ON THE BRAUER GROUP OF ALGEBRAS HAVING A GRADING AND AN ACTION

MORRIS ORZECH

1. Introduction. Beginning with Wall's introduction [**19**] of Z_2-graded central simple algebras over a field K, a number of related generalizations of the Brauer group have been proposed. In [**16**] the field K was replaced by a commutative ring R, building upon the theory developed in [**1**]. The concept of a G-graded central simple K-algebra (G an abelian group) was first defined in [**12**]; this work and that of [**16**] was subsequently unified in [**6**] and [**7**] via the construction and computation of the graded Brauer group $B_\phi(R, G)$ (ϕ a bilinear form from $G \times G$ to $U(R)$, the units of R). In [**13**] Long recently introduced the Brauer group $BD(R, G)$ of R-algebras which have a compatible G-action and G-grading, thus extending not only the previously mentioned work, but the equivariant theory put forth in [**8**]. And in [**14**] Long constructed a generalization of $BD(R, G)$, replacing G (or more precisely the Hopf algebra RG) by a Hopf algebra H to obtain $BD(R, H)$.

After defining the generalized Brauer groups in [**13**] and [**14**] the main thrust of Long's work was to compute these groups in special cases, the most important being that where G is cyclic of prime order p and R is a separably closed field; [**13**] treated the case where $p \neq \text{char}(R)$, [**14**] dealt with $p = \text{char}(R)$. The present paper has two related aims:

1) To extend Long's computations by relaxing considerably the requirement that R be an algebraically closed field and by unifying the distinct treatments in [**13**] and [**14**]; and

2) to develop some of the theory relating to the internal structure of the algebras comprising $BD(R, G)$, a task accomplished in [**7**] and [**16**] for the algebras studied there, but not touched upon in Long's work for R other than a field.

The second aim has been subjugated to the first, and the results derived in Sections 2 and 3 are generally those we need for our computations in Sections 3 and 4. However, we have included some examples relating to $BD(R, Z_2 \times Z_2)$ as these can serve as test cases for extending our work to non-cyclic groups. We have avoided questions which require a Morita theory for G-dimodules (which is being developed by my student M. Beattie) in order to maintain our focus on the computation of $BD(R, G)$, and some of our results in Sections 2 and 3 could be improved with such a theory. Theorems 4.4 and 5.1 contain our main results. The methods of proof represent a slight reformulation of

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Long's methods, one aiming to keep the presentation relevant to finite abelian groups G (rather than cyclic ones) wherever possible.

Throughout this paper R will denote a commutative ring, G a finite abelian group.

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2. The center of a G-Azumaya algebra. We recall some definitions and results from [**5**]: If $f, g : S \to T$ are homomorphisms of commutative rings, they are said to be *strongly distinct* if for every nonzero idempotent e of T, there exists s in S such that $f(s)e \neq g(s)e$. Let G be a finite group of automorphisms of the commutative ring S and let $R = S^G$. The following condition may then be taken as defining S to be a *Galois extension of R with group G*: S is a separable R-algebra and the elements of G are strongly distinct. It follows that S is a projective R-module of finite type. If S is connected (has no nontrivial idempotents) then the condition that the elements of G be strongly distinct may be removed. We begin by sharpening this observation, using an argument introduced in [**10**, Theorem 7].

2.1. LEMMA. *Let R be connected, S a separable R-algebra which is a projective R-module of finite type. Let G be a finite group of R-algebra automorphisms of S such that $S^G = R$.*

(a) *S is a Galois extension of R with group G.*

(b) *Assume G is abelian. Then there is a subgroup H of G, a set of idempotents $\{e_\pi | \pi \in G/H\}$ in S satisfying $\sigma e_\pi = e_{\bar{\sigma}\pi}$ for σ in G, and an R-subalgebra T of S which is a Galois extension of R with group H and for which $S = \prod_{\pi \in G/H} Te_\pi$.*

(c) *With the notation as in* (b), $S^H = \prod_{\pi \in G/H} Re_\pi$.

Proof. (a) Since S has a well-defined rank over R, it decomposes as a product of connected R-algebras which are necessarily R-separable [**15**, Corollary 4.5]:

$$S = \prod_{i=1}^{k} S_i.$$

Let $1 = \sum e_i$ with e_i in S_i. The e_i are the minimal idempotents of S, hence are permuted by the elements of G. Let $E = \{e_1, \ldots, e_k\}$ and let $E_1, \ldots, E_m$ be the orbits in E with respect to the action of G. Let f_i be the sum of the elements of E_i, $i = 1, \ldots, m$. Clearly f_i is in $S^G = R$, and $\sum f_i = 1$. Since R is connected, we conclude that there is only one orbit, i.e. G acts transitively on E. Let $H = \{\sigma | \sigma e_1 = e_1\}$; then H is the set of elements of G fixing e_i as well, for $i = 1, \ldots, k$. There is a bijection $G/H \leftrightarrow E$ and we may write

$$E = \{e_\pi | \pi \in G/H\}, \ \sigma e_\pi = e_{\bar{\sigma}\pi}$$

for σ in G.

To show S is a Galois extension of R we need to show that any two elements of G are strongly distinct. Since any idempotent in S is a sum of certain of the e_π, it suffices to show that $\sigma(s)e_\pi = se_\pi$ for all s in S implies $\sigma = 1$. Replacing s by $\tau^{-1}(s)$ in the last equality, and applying τ, yields $se_{\bar{\tau}\pi} = \sigma(s)e_{\bar{\tau}\pi}$. Summing as τ ranges over a set of coset representatives of H in G, we conclude that $s = \sigma s$.

(b) Let $1 = \sigma_1, \ldots, \sigma_k$ be a set of coset representatives of H in G. Write e_i for $e_{\bar{\sigma}_i}$ and define

$$T = \left\{ \sum_{i=1}^{k} \sigma_i(s)e_i | s \text{ in } S \right\}.$$

There is a well-defined action of H on T given by $\eta(\sum \sigma_i(s)e_{\bar{\sigma}_i}) = \sum \sigma_i\eta(s)e_i$. To show that T is a Galois extension of R we must verify that H is a group of automorphisms of T, and that $T^H = R$.

Suppose η in H is the identity on T. Then $\eta(s)e_1 = se_1$ for all s in S, and $\eta = 1$ since the elements of G are strongly distinct.

Suppose $x = \sum \sigma_i(s)e_i$ is in T^H. For σ in G let $\sigma = \sigma_k\eta$, with η in H. Then x is in R since

$$\sigma x = \textstyle\sum_i \sigma_k\sigma_i\eta(s)e_{\overline{\sigma_k\sigma_i}} = \sum_j \sigma_j(s)e_j = x.$$

It is clear that $S = \prod_{\pi \in G/H} Te_\pi$.

(c) Let $x = \sum t_\pi e_\pi$ be in S^H, with t_π in T. Then $\eta(t_\pi)e_\pi = t_\pi e_\pi$ for all η in H. But $t_\pi e_\pi = t_\pi$ for t_π in T, and since $T^H = R$, it follows that $S^H = \prod Re_\pi$.

The objects of main interest to us will be G-Azumaya algebras, which we proceed to define, following the terminology of [**13**]. A G-dimodule algebra A is an R-algebra which is graded by G ($A = \oplus_{\sigma \in G} A_\sigma$, with $A_\sigma A_\tau \subseteq A_{\sigma\tau}$) and on which G acts as R-algebra automorphisms (not necessarily faithfully) in such a way that $\sigma A_\tau \subseteq A_\tau$ for σ, τ in G. For A, B two G-dimodule algebras, their smash product $A \# B$ is defined to be the R-module $A \otimes B$ given an R-algebra structure satisfying

$$(a \# b)(c \# d) = \textstyle\sum_\sigma a\sigma(c) \# b_\sigma d,$$

and having diagonal G-action and the usual (codiagonal) G-grading. We shall abbreviate formulas such as the above to

$$(a \# b)(c \# d) = a^b c \# bd$$

this sort of expression being interpreted as valid for homogeneous elements. The algebra $\bar{A}$ is defined to be $\{\bar{a} | a \text{ in } A\}$, with multiplication $\bar{a}\bar{b} = \overline{{}^a b a}$ and natural G-action and grading. Maps

$$\mu : A \# \bar{A} \rightarrow \operatorname{End}_R(A), \quad \eta : \bar{A} \# A \rightarrow \operatorname{End}_R(A)^{op}$$

are defined by $\mu(a \# \bar{b})(x) = a^b xb$ and $\eta(\bar{a} \# b) = f^{op}$, where $f(x) = {}^x axb$. These maps are G-dimodule algebra homomorphisms, where $\operatorname{End}_R(A)$ has G-action given by $(\sigma h)(x) = \sigma(h(\sigma^{-1}x))$ (see [**13**] for more details).

The G-dimodule algebra A is said to be G-Azumaya if it is a faithful projective R-module of finite type and μ, η are isomorphisms.

2.2. PROPOSITION. *Let A be a G-Azumaya R-algebra, Z its center,*

$$K = \{\sigma \in G | \sigma|_Z = 1\}.$$

(a) *If $\sigma a = a$ for all a in A, then $Z_\sigma \subseteq R$. Hence $Z^G = R = Z_1$.*

(b) *A is a separable R-algebra, and an Azumaya Z-algebra.*

(c) *Assume R is connected. Then Z is a Galois extension of R with group G/K. In particular, Z is R-projective.*

Proof. (a) This follows by noting that $\mu(1 \# \bar{z}) = \mu(z \# \bar{1})$ (or $\eta(\bar{1} \# z) = \eta(\bar{z} \# 1)$) implies z is in R (see [**13**, Theorem 1.9]).

(b) Let t in $\text{End}_R(A)$ satisfy $t(A_\sigma) = 0$ for $\sigma \neq 1$, $t(1) = 1$ [**15**, Corollary 1.4]. Let $e = \sum \bar{a}_i \# b_i$ in $\bar{A} \# A$ be such that $\eta(e) = t^{op}$. Then $t(x) = \sum {}^x a_i x b_i$. For a in A let f^{op}, g^{op} in $\text{End}_R(A)$ be defined by $f^{op} = \eta(\sum \bar{a}\bar{a}_i \# b_i)$, $g^{op} = \eta(\sum \bar{a}_i \# b_i a)$. Then $g(x) = t(x)a$ and

$$f(x) = \sum {}^x a {}^x a_i x b_i = \sum {}^x a t(x).$$

But ${}^x a t(x) = t(x) a$ since $t(x) = 0$ for x homogeneous of grade $\neq 1$. Thus $f = g$. Also, $\sum a_i b_i = 1$. Since the map $\bar{A} \# A \to A \otimes A^{op}$ given by $\bar{a} \otimes b \to a \otimes b^{op}$ is an R-module isomorphism, $\sum a_i \otimes b_i$ is a separability idempotent for A. Since A is R-separable it is also Z-separable.

(c) Since A is R-separable, so is its center Z. Now A is an R-projective Z-module, hence A is Z-projective by separability of Z. Hence Z is a direct Z-summand of A, hence is R-projective. Moreover, $Z^G = R$ by (a). It follows from Lemma 2.1 that Z is a Galois extension of R with group G/K.

2.3. *Remark.* It follows by looking at the map t used above that the "dimodule centers" of A are both R, i.e.,

$$\{x \in A | {}^a x a = a x \text{ for all } a \text{ in } A\} = R,$$
$$\{x \in A | {}^x a x = x a \text{ for all } a \text{ in } A\} = R.$$

2.4. PROPOSITION. *Let A be a G-Azumaya R-algebra. Suppose I is a two-sided ideal which is either a G-submodule of A or a homogeneous ideal. Then $I = I_o A$ for I_o an ideal of R.*

Proof. If I is a G-submodule (respectively, G-homogeneous) it is easy to see that $I \# \bar{A}$ (respectively, $\bar{A} \# I$) is a two-sided ideal of $A \# \bar{A}$ (respectively, $\bar{A} \# A$). Since $\text{End}_R(A)$ and $\text{End}_R(A)^{op}$ are Azumaya R-algebras, all their two-sided ideals are extensions of ideals of R. Thus, in the G-module case, there is an ideal I_o of R such that $I \# \bar{A} = I_o A \# \bar{A}$ and $I_o = (I \# \bar{A}) \cap R$ [**15**, Corollary 2.11]; the latter equality implies that $I_o A \subseteq I$, and since $\bar{A}$ is faithfully flat, $I_o A = I$. The other case is done similarly.

2.5. COROLLARY. *Let R be connected and A a G-Azumaya R-algebra with center Z.*

(a) *If $Z_\sigma \neq 0$, then $\text{ann}_A(Z_\sigma)$, the annihilator of Z_σ in A, is 0.*

(b) *$H = \{\sigma | Z_\sigma \neq 0\}$ is a subgroup of G.*

(c) *Suppose R is a domain and z is a nonzero element of Z_σ. Then $\text{ann}_A(z) = 0$.*

Proof. Let $I = \{a \in A | Z_\sigma a = 0\}$. Then $I = I_o A$ for some ideal I_o of R, by Proposition 2.4. Hence $Z_\sigma I_o = 0$. But Z_σ is R-projective by Proposition 2.2, and because R is connected, Z_σ is R-faithful [**3**, Proposition 4.6, p. 70]; hence $I_o = 0$ if $Z_\sigma \neq 0$. This proves (a), from which (b) follows easily. If R is a domain each Z_σ is torsion-free. Then (c) is proved by redefining $I = \text{ann}_A(z)$ and using the argument above. This completes the proof.

Let $f : G \times G \to U(R)$ be a 2-cocycle of G in the units of R, with G acting trivially on $U(R)$. We shall say f is abelian if $f(\sigma, \tau) = f(\tau, \sigma)$ for σ, τ in G. The crossed product RG_f is defined to be the R-algebra which as an R-module is freely generated by elements x_σ, σ in G, the multiplication being determined by the requirement that $x_\sigma x_\tau = f(\sigma, \tau)x_{\sigma\tau}$. The algebra RG_f has a natural G-grading relative to the Rx_σ.

If RG_f is a G-dimodule algebra then the G-action must be given by $\sigma x_\tau = \phi(\sigma, \tau)x_\tau$. It is easily checked that ϕ is a bilinear map from $G \times G$ to $U(R)$. We shall write $RG_f{}^\phi$ for this G-dimodule algebra.

2.6. Corollary. *Let R be a field, A a G-Azumaya R-algebra with center Z. Let $H = \{\sigma | Z_\sigma \neq 0\}$.*

(a) *There exist an abelian cocycle $f : H \times H \to U(R)$ and a bilinear map $\phi : G \times H \to U(R)$ for which $\phi(G, \tau) = 1$ implies $\tau = 1$, such that $Z \cong RH_f{}^\phi$.*

(b) char $R \nmid [H : 1]$.

Proof. This is proved in [**12**, Theorem 3.1], but we shall repeat this proof for later reference. For σ in H let $x_\sigma \neq 0$ be in Z_σ and let $o(\sigma)$ be the order of σ. By Corollary 2.5 $x_\sigma{}^{o(\sigma)}$ is nonzero, and is in R by Proposition 2.2. Thus $Z_\sigma = Z_\sigma x_\sigma{}^{o(\sigma)} \subseteq Rx_\sigma$ and Z_σ is one-dimensional. Since $x_\sigma x_\tau \neq 0$ we have that $x_\sigma x_\tau = f(\sigma, \tau)x_{\sigma\tau}$. The remarks preceding this discussion, together with Proposition 2.2, complete the proof of (a). It is shown in [**9**, Lemma 4] that if RH_f is R-separable then $[H : 1]$ is a unit in R. Then (b) follows from separability of A (Proposition 2.2), which implies that of Z.

2.7. Corollary. *Let G be cyclic of prime order p. Let R be connected and p not a unit in R. Then any G-Azumaya R-algebra is R-Azumaya.*

Proof. Let A be G-Azumaya, with center Z. Then A is R-separable by Proposition 2.2. Let m be a maximal ideal in R containing p. Then A/mA is a G-Azumaya R/m-algebra with center Z/mZ [**13**, Theorem 1.7; **15**, Proposition 2.3]. Because of (b) of Corollary 2.6 and the fact that G has order p we have that $Z/mZ = R/m$. By Proposition 2.2 Z is R-projective and R is a direct summand of Z with complementary summand $S = \oplus_{\sigma\neq 1} Z_\sigma$. Then $mS = S$, so S is annihilated by some element $1 - r$, r in m [**15**, Lemma 1.2]. By Corollary 2.5 S must be 0, thus $Z = R$. This generalizes [**14**, Proposition 5.2].

Apropos of the last result, we may inquire when an algebra of the form

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RG_f^{ϕ} is G-Azumaya. A rather precise criterion may be given, allowing some explicit constructions of G-Azumaya R-algebras.

Let $A = RG_f^{\phi}$, f a cocycle (not necessarily abelian) in $Z^2(G, U(R))$, $\phi : G \times G \to U(R)$ a bilinear map. To verify that A is G-Azumaya it is sufficient in this situation to check that

$$\mu : A \# \bar{A} \to \operatorname{End}_R(A) \quad \text{and} \quad \eta : \bar{A} \# A \to \operatorname{End}_R(A)^{op}$$

are isomorphisms, or even epimorphisms since we are dealing with free R-modules. A further simplification is possible. The left A-structure on A induces left and right A-structures on $\operatorname{End}_R(A)$. There are also such structures on $A \# \bar{A}$ making μ an (A, A)-bimodule map, viz.

$$a(x \# \bar{y}) = ax \# \bar{y}; \quad (x \# \bar{y})b = x^y b \# \bar{y}.$$

Using the right A-structure on A, a similar statement may be made for η.

Now to show μ, η are onto, it suffices to show that for each σ, τ in G there exists $a_{\sigma,\tau}$ in $A \# \bar{A}$ (respectively, $b_{\sigma,\tau}$ in $\bar{A} \# A$) such that

$$\mu(a_{\sigma,\tau})(x_\gamma) = \delta_{\gamma,\sigma} x_\tau = \eta(b_{\sigma,\tau})(x_\gamma).$$

By applying the remarks immediately above, it is easy to check that μ, for example, is onto if and only if there exists an a in $A \# \bar{A}$ such that $\mu(a)(x_\gamma) = \delta_{1,\gamma}$. These considerations may be applied to yield:

2.8. Proposition. *Let $A = RG_f^{\phi}$. Then A is G-Azumaya if and only if each of the following two matrices is invertible:*

$$(\phi(\alpha, \beta)c_{\alpha,\beta}), \quad (\phi(\beta, \alpha^{-1})c_{\alpha,\beta}), \quad c_{\alpha,\beta} = f(\alpha^{-1}, \beta)f(\alpha^{-1}\beta, \alpha).$$

If f is abelian, then A is G-Azumaya if and only if the matrix $(\phi(\alpha, \beta))$ is invertible. This implies ϕ is non-degenerate, and is equivalent to it in case R is connected and $[G : 1]$ is a unit in R.

Proof. The first statement is a straightforward consequence of the discussion above — μ, η are onto if there exist $(r_{\alpha,\beta})$, $(s_{\alpha,\beta})$ such that

$$\mu\left(\sum_\alpha r_{\alpha,\beta} x_{\alpha^{-1}} \# \bar{x}_\alpha\right)(x_\gamma) = \delta_{1,\beta} = \eta\left(\sum_\alpha s_{\alpha,\beta} \bar{x}_{\alpha^{-1}} \# \bar{x}_\alpha\right)^{op}(x_\gamma).$$

If f is abelian then

$$c_{\alpha,\beta} = f(\beta, \alpha^{-1})f(\beta\alpha^{-1}, \alpha) = f(\beta, 1)f(\alpha^{-1}, \alpha).$$

Since the diagonal matrices $\operatorname{diag}(f(\beta, 1))$, $\operatorname{diag}(f(\alpha^{-1}, \alpha))$ are invertible, the matrix $(\phi(\alpha, \beta))$ being invertible is equivalent to the two matrices above being invertible.

By saying ϕ is non-degenerate we mean that the two induced maps $G \to \operatorname{Hom}(G, U(R))$ are one-one. It is clear that if $\phi(\alpha, G) = 1 = \phi(1, G)$, with $\alpha \neq 1$, then $(\phi(\alpha, \beta))$ is not invertible. Now if R is connected and $n = [G : 1]$ is a unit in R, the number of n-th roots of unity in R is at most

n [**11**, Corollary 2.5]. Then the classical orthogonality relations hold for homomorphisms $\chi, \psi : G \to U(R)$:

$$\sum_{\sigma \in G} \chi(\sigma)\psi(\sigma^{-1}) = n\delta_{\chi,\psi};$$

(see, for example [**18**, § 126]). By applying this formula to $\phi(1, \ )$, $\phi(\alpha, \ )$, we obtain the last statement.

2.9. *Remarks.* (a) Note that if $[G : 1]$ is a unit and R is connected, then $G \to \mathrm{Hom}(G, U(R))$ being one-one implies it is onto as well. Thus if $\phi(\sigma, G) = 1$ implies $\sigma = 1$, it follows that $\phi(G, \tau) = 1$ implies $\tau = 1$. Thus, non-degeneracy of ϕ need be checked in only one variable.

(b) Let $A = RG_f^{\phi}$ and suppose ϕ, viewed as a cocycle in $Z^2(G, U(R))$, is a coboundary. Let c_σ be chosen so that $c_\sigma c_\tau = c_{\sigma\tau}\phi(\sigma, \tau)$ for σ in G. Then the correspondence $\bar{x}_\sigma \to c_\sigma x_\sigma$ establishes an isomorphism $\bar{A} \cong A$ of G-dimodule algebras.

2.10. *Examples.* Let $G = C_2 \times C_2$, the Klein four-group; write $G = \{1, \sigma, \tau, \sigma\tau\}$. Suppose 2 is a unit in R, and define bilinear maps ϕ, ψ by the tables below:

	1	σ	τ	$\sigma\tau$
1	1	1	1	1
σ	1	−1	−1	1
τ	1	1	1	1
$\sigma\tau$	1	−1	−1	1

ϕ

	1	σ	τ	$\sigma\tau$
1	1	1	1	1
σ	1	−1	1	−1
τ	1	−1	−1	1
$\sigma\tau$	1	1	−1	−1

ψ

The matrix $(\psi(\alpha^{-1}, \beta)\psi(\alpha^{-1}\beta, \alpha))$ is easily seen to be invertible. Hence, by Proposition 2.8, RG_ψ^1 is G-Azumaya. This shows that if f is not abelian, RG_f^{ϕ} may be G-Azumaya even if ϕ is degenerate.

The algebra RG_ϕ^{ψ} may be given another explicit interpretation. Identify $1, x_\sigma, x_\tau, x_{\sigma\tau}$ with the following matrices

$$1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad x_\sigma = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad x_\tau = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad x_{\sigma\tau} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then $RG_\phi^{\psi} = M_2(R)$, the ring of 2×2 matrices over R; the action of G is seen to be determined by requiring that

$$\sigma\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & -b \\ -c & d \end{bmatrix}, \quad \tau\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} d & c \\ b & a \end{bmatrix}.$$

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It is easily checked that $\overline{RG_f{}^\psi} = RG_{\psi f^o}{}^\psi$, where $f^o(\alpha, \beta) = f(\beta, \alpha)$. From this it is easy to see that $\overline{RG_\phi{}^\psi}$ is commutative for ϕ, ψ as above. Thus we have a situation where a G-Azumaya R-algebra A is central, but $\bar{A}$ is commutative. This makes concrete the observation [**13**, p. 250] that if G is not cyclic, then $B\mathrm{Az}(R, G)$, the set of R-central G-Azumaya R-algebras, need not yield a subgroup of $BD(R, G)$.

2.11. PROPOSITION. *Let R be connected, A a G-Azumaya R-algebra with center Z. Let*

$$K = \{\sigma \in G | \sigma z = z \text{ for all } z \text{ in } Z\}, \quad H = \{\sigma \in G | Z_\sigma \neq 0\}.$$

Then

(a) *Z_σ has rank one and as an element of* $\mathrm{Pic}(R)$ *is annihilated by the order of σ, hence by* $\exp(H)$.

(b) *$[H : 1]$ is a unit in R.*

(c) *$[G : K] = [H : 1]$; if $H \cap K = \{1\}$ then $G = H \times K$.*

(d) *The multiplication map $Z_\sigma \otimes Z_\tau \to Z_{\sigma\tau}$ is an isomorphism for σ, τ in H.*

(e) *Let $m = \exp(H)$ and suppose $\mathrm{Pic}_m(R)$, the m-torsion subgroup of $\mathrm{Pic}(R)$, is zero. Then $Z \cong RH_f{}^\phi$ with f abelian. If $H \cap K = \{1\}$ then ϕ is non-degenerate and Z is G-Azumaya.*

Proof. Because R is connected, Z has a well-defined rank over R, say n. Let p be a maximal ideal of R; we have $R/p \cong R_p/pR_p$. Then $R_p \otimes_R Z \cong R_p{}^n$, hence Z/pZ has dimension n over R/p. By Proposition 2.2 A is separable over R, hence by [**15**, Proposition 2.5] and [**13**, Theorem 1.7], A/pA is a G-Azumaya R/p-algebra with center Z/pZ. Also $Z_\sigma/pZ_\sigma \neq 0$ for σ in H; for otherwise there would exist an element $1 - a$, a in p, annihilating Z_σ [**15**, Lemma 1.2], contradicting Corollary 2.5. We conclude from Corollary 2.6 that Z_σ/pZ_σ has dimension one over R/p, hence each Z_σ has rank one. Moreover, $[H : 1]$ is a unit in R/p for each p, hence is a unit in R. This proves (b) and part of (a).

We showed in Proposition 2.2 that Z is a Galois extension of R with group G/K. Thus Z has rank $[G/K : 1]$, so that $[G : K] = [H : 1]$. Thus (c) follows.

To show that $m_{\sigma,\tau} : Z_\sigma \otimes Z_\tau \to Z_{\sigma\tau}$ is an isomorphism it suffices to show it is onto, since all modules involved are projective of rank one. This can be done by showing that $m_{\sigma,\tau}$ is onto modulo each maximal ideal p of R. Details may be found in [**7**, Proof of Theorem 4.3, p. 319]. This proves (d) and incidentally shows that Z_σ is annihilated by the order of σ as an element of $\mathrm{Pic}(R)$, since $Z_1 = R$ by Proposition 2.2. Hence (a) is also proved.

If $\mathrm{Pic}_m(R) = 0$ for $m = \exp(H)$ then by (a) Z_σ is free of rank one for each σ in H. The proof of Corollary 2.6, aided by (d), shows that $Z \cong RH_f{}^\phi$. Then Proposition 2.8 and (c) above prove the rest of (e).

2.12. *Remarks.* The facts deduced above may be applied to strengthen a result of Knus, who in [**12**, Theorem 3.1] obtains a structure theorem for

G-graded central simple algebras A over a field R, where char $R \nmid \dim_K A$. These algebras may be given a G-action by defining $\sigma a = \sum \phi(\sigma, \tau)a_\tau$, where $\phi : G \times G \to U(R)$ is a given bilinear form relative to which all constructions in [**12**] are carried out. Then a G-graded Azumaya R-algebra becomes a G-Azumaya algebra. Knus obtains that $H \times H' \subseteq G$, assuming that ϕ is symmetric, and non-degenerate on every subgroup of G; H is as in 2.11 and $H' = \{\sigma | \phi(\sigma, H) = 1\}$. It is easily seen that $H' = K$ and $H \cap K = \{1\}$ by non-degeneracy of ϕ. Then (c) implies that $H \times H'$ is actually equal to G.

3. Structure of special G-Azumaya algebras. We begin with a description of how a G-Azumaya R-algebra decomposes when its center is particularly nice.

3.1. PROPOSITION. *Let R be connected and A a G-Azumaya R-algebra with center Z. Assume G acts faithfully on Z, so that Z is a Galois extension of R with group G (see Proposition* 2.2).

(a) *Suppose Z is the trivial Galois extension, i.e. $Z = \prod_{\sigma \in G} Re_\sigma$ with the e_σ pairwise orthogonal idempotents of sum* 1 *and $\sigma e_\tau = e_{\sigma\tau}$. Then $A \cong A^G \# Z$ as G-dimodule algebras. Moreover $A^G \cong Ae_1$ as R-algebras and each is an Azumaya R-algebra.*

(b) *Suppose $Z_\sigma = Ru_\sigma$ with $u_\sigma u_{\sigma^{-1}} = 1$ for σ in G. Then $A \cong Z \# A_1$ as G-dimodule algebras and A_1 is an Azumaya R-algebra.*

(c) *If the hypotheses of* (a) *and* (b) *hold then $[A^G] = [A_1]$ in $B(R)$, the Brauer group of R.*

Proof. Given an element a in A let $t(a) = \sum (\sigma a)e_\sigma$. Then $tA = A^G$. Any element a in A can be expressed as $\sum a(\sigma)e_\sigma$ with $a(\sigma)$ in A^G by taking $a(\sigma) = t(\sigma^{-1}a)$; such an expression is unique, for if $z = \sum x(\sigma)e_\sigma = \sum y(\sigma)e_\sigma$ then $\sum_\tau (\tau z)e_{\gamma\tau} = x(\gamma)$ for γ in G, hence $x(\gamma) = y(\gamma)$. Define $h : A^G \# Z \to A$ by $h(x \# z) = xz$. Then h is an isomorphism of R-algebras which preserves the action and grading by G.

R is embedded in Ae_1 via $r \to re_1$, since $re_1 = 0$ implies $re_\sigma = 0$ and $r = 0$. Define $j : Ae_1 \to A^G$ by $j(ae_1) = t(a)$. If $ae_1 = 0$, then $(\sigma a)e_\sigma = 0$, so j is well-defined. It is clear that j is an R-algebra isomorphism. $A^G \# Z$ and $A^G \otimes Z$ are isomorphic since G acts trivially on A^G. By Proposition 2.2 and [**2**, Proposition 2.18, p. 98] we know that A and Z are R-separable (with center Z) hence A^G is R-separable with center R.

(b) Define $s : A \to A_1$ by $sa = \sum a_\sigma u_{\sigma^{-1}}$. Then $sA = A_1$. By taking $x(\sigma) = s(a_\sigma)$ one can show easily that each element x of A can be expressed as $\sum x(\sigma)u_\sigma$ with $x(\sigma)$ in A_1. Suppose $z = \sum x(\sigma)u_\sigma = \sum y(\sigma)u_\sigma$. Then $z_\tau = x(\tau)u_\tau = y(\tau)u_\tau$ and $x(\tau) = y(\tau)$. Thus the $x(\sigma)$ are unique. Define $h : Z \# A_1 \to A$ by $h(z \# x) = zx$. This is an isomorphism of dimodule algebras. That A_1 is R-Azumaya follows by the same kind of argument used in (a) to show that A^G is R-Azumaya.

(c) If the hypotheses of (a) and (b) hold we have an isomorphism of R-

algebras $Z \otimes A_1 \cong A^G \otimes Z$ which is the identity on Z. Thus $[Z \otimes A_1] = [Z \otimes A^G]$ in $B(Z)$. But the map $R \to Z$ splits, so $B(R) \to B(Z)$ is a monomorphism, hence $[A_1] = [A^G]$.

The Brauer group $BD(R, G)$ of G-Azumaya R-algebras is defined as follows. Let P be a projective R-module of finite type having a G-grading $P = \oplus P_\sigma$ and a G-action satisfying $\sigma P_\tau = P_\tau$. Then $\text{End}_R(P)$ inherits a G-grading and a G-action ($[\sigma f](x) = \sigma f(\sigma^{-1}x)$), and is G-Azumaya R-algebra; such a G-Azumaya R-algebra will be said to be trivial. We say that the G-Azumaya R-algebras A and B are equivalent if $A \# \text{End}_R(P) \cong B \# \text{End}_R(Q)$ with $\text{End}_R(P)$ and $\text{End}_R(Q)$ trivial. This is an equivalence relation and the equivalence classes form a group $BD(R, G)$. The multiplication in $BD(R, G)$ is via $[A][B] = [A \# B]$; $[\bar{A}]$ is the inverse of $[A]$. Details may be found in [**13**].

Let $BD_0(R, G)$ denote the subset of $BD(R, G)$ consisting of equivalence classes of G-Azumaya R-algebras among whose representatives is an R-Azumaya algebra, i.e. a central R-algebra (since separability follows from Proposition 2.2). Long uses the notation $B\text{Az}(R, G)$ for this set; he also states the following result for R a separably closed field of characteristic $\neq p$, G a cyclic group of prime order p and $A = \text{End}_R(M)$ [**13**, Lemma 2.2]. The proof below is a rewording of that in [**13**] in our context.

3.2. Lemma. *Assume $H^2(G, U(R)) = 0$, where G acts trivially on R. Let A, B be G-Azumaya R-algebras. Assume A has center R and that G acts as inner automorphisms of A. Then $A \# B \cong A \otimes B$ as G-module algebras.*

Proof. Let u_σ be such that $u_\sigma a u_\sigma^{-1} = \sigma a$ for a in A, with $u_1 = 1$. Then $f(\sigma, \tau) = u_\sigma u_\tau u_{\sigma\tau}^{-1}$ is in the center of A, hence defines a 2-cocycle $f : G \times G \to U(R)$. Then f is a coboundary δg with g normalized ($g(1) = 1$). Thus u_σ may be replaced by $u_\sigma g(\sigma)^{-1}$ so we may assume $u_\sigma u_\tau = u_{\sigma\tau}$. Then

$$j : A \# B \to A \otimes B,$$

defined by $j(a \# b) = au_\sigma \otimes b$ for b homogeneous of grade σ, is an isomorphism of R-algebras and of G-modules.

3.3. Corollary. *Assume $H^2(G, U(R)) = 0$. Then $BD_0(R, G)$ is the subset of $BD(R, G)$ consisting of those classes of G-Azumaya R-algebras every representative of which is R-Azumaya.*

Proof. This follows from the definition of $BD(R, G)$, using the facts that $\text{End}_R(P)$ is R-Azumaya and that if $A \otimes B$ is R-Azumaya, so is A [**15**, Exercise 2.15].

3.4. Proposition. *Suppose one of the following sets of conditions holds:*

(i) $\text{Pic}_m(R) = 0$ *where* $m = \exp(G)$, *and* $H^2(G, U(R)) = 0$.

(ii) *G is a cyclic group.*

Then $BD_0(R, G)$ is a subgroup of $BD(R, G)$.

Proof. First assume G is cyclic. Let A, B be G-Azumaya R-algebras which are R-Azumaya as well. We shall show that if $C = A \# B$ or $C = \bar{A}$ then C is also R-Azumaya. To do this it suffices to prove that C/pC is R/p-Azumaya for each maximal ideal p of R [**2**, Theorem 4.1, p. 104]. Since $C/pC \cong A/pA \#_{R/p} B/mB$ and $\bar{A}/m\bar{A} \cong \overline{A/mA}$, we may assume R is a field, which we now do. Let K be the algebraic closure of R. If $K \otimes_R C$ is K-Azumaya then C is R-Azumaya [**15**, Lemma 4.6]. We may thus assume R is algebraically closed. Then $H^2(G, U(R)) = 0$ since for a cyclic group G of order n acting trivially on R, $H^2(G, U(R)) = U(R)/U(R)^n$ [**4**, p. 251]. By Lemma 3.2 we conclude that $A \# B$ is R-Azumaya. Now $A \# \bar{A} \cong A \otimes \bar{A}$, again by Lemma 3.2. Hence $A \otimes \bar{A} \cong \operatorname{End}_R(A)$, which is R-Azumaya, from which it follows that $\bar{A}$ is R-Azumaya [**15**, Exercise 2.15]. This completes the proof for the case where (ii) holds. If (i) holds we have that every element of G must act as an inner automorphism on A, since $\operatorname{Pic}_m(R) = 0$ [**12**, Corollary 4.6, p. 108]. The hypotheses of Lemma 3.2 hold, and the conclusion we desire now follows.

3.5. *Remark*. The example given in 2.10, where A is R-central but $\bar{A}$ is commutative shows that some hypothesis is needed for $BD_o(R, G)$ to be a subgroup of $BD(R, G)$.

4. The isomorphism $BD_o(R, G) \cong B(R) \times \operatorname{Aut}(G)$. Proposition 4.2 below is a key result in computations we shall carry out generalizing and unifying two examples of Long [**13**, Theorem 2.5; **14**, Theorem 5.8]. We shall mention its connection with work of Long and Sweedler following the proof, but first we require some preliminary remarks which will be useful in avoiding digressions within the proof.

4.1. *Remarks*. (a) Let A be a G-graded R-algebra containing R and for which A_1 is an R-module of finite type. Let u be in A. To check that u is homogeneous it suffices to check that $u + pA$ is homogeneous in A/pA for each maximal ideal p of R: for suppose this is the case and assume $u_\sigma \neq 0$, $\sigma \neq \tau$. Since u_σ and u_τ define multiplication maps from A_1 to A_σ and A_τ, and since 1 is in A_1, $u_\sigma + pA$ is nonzero for all p. Thus $u_\tau + pA$ is zero for all p. Let I be the image of the map $u_\tau : A_1 \to A_\tau$. We have $I = pI$, hence $(1 - a_p)I = 0$ for some a_p in p [**15**, Lemma 1.2]. Thus the annihilator of I is not contained in any maximal ideal of R, and $I = 0$.

(b) Let G be a finite abelian group. Let GR denote $(RG)^*$, the R-dual of the group algebra RG. We may identify GR with the set of functions from G to R. For σ in G let e_σ be the function in GR given by $e_\sigma(\tau) = \delta_{\sigma,\tau}$ for all τ in G. The formula $(\sigma v)(\tau) = v(\sigma^{-1}\tau)$ defines a G-action on GR; then $\sigma e_\tau = e_{\sigma\tau}$. The multiplication $m : RG \otimes RG$ induces a comultiplication

$$\Delta : GR \to GR \otimes GR$$

since $(RG \otimes RG)^*$ and $GR \otimes GR$ are isomorphic. It is straightforward to verify that Δ satisfies

$$(*)\quad \Delta(v) = \textstyle\sum_\sigma e_\sigma \otimes \sigma^{-1}v = \sum_\sigma \sigma^{-1}v \otimes e_\sigma.$$

Now let A be a G-graded R-algebra, with $A = \oplus A_\sigma$. We define an action of GR on A by constructing it on homogeneous elements of A and extending linearly:

$$va = \textstyle\sum_\sigma v(\sigma)a_\sigma, \quad \text{where } a = \sum_\sigma a_\sigma \text{ and } a_\sigma \in A_\sigma.$$

Using the relation $(ab)_\sigma = \sum_\tau a_\tau b_{\tau^{-1}\sigma}$ and manipulating sums, it is straightforward to verify that:

$$(**)\quad v(ab) = \textstyle\sum_\sigma a_\sigma(\sigma^{-1}v)(b) = \sum_\sigma (\sigma^{-1}v)(a)b_\sigma.$$

4.2. Proposition. *Let G be a finite abelian group, A a G-graded R-algebra with center R. Assume A_1 is a finitely generated R-module. Let u be a unit in A such that the inner automorphism $u(\ )u^{-1}$ preserves the grading on G. Then u is homogeneous.*

Proof. By the remark of 4.1(a) we may assume R is a field. Let the G-grading on A induce a GR-action in the manner described in (b) of 4.1, so that for v in GR and x in A_σ, $vx = v(\sigma)x$. It is straightforward to verify that

$$A_\sigma = \{x \in A | vx = v(\sigma)x \text{ for } v \text{ in } GR\}.$$

We shall make use of this characterization to show u is homogeneous. We continue to assume R is a field.

Suppose x is a homogeneous element of A whose grade is τ. By assumption, uxu^{-1} is also in A_τ. Let w be in GR and apply the equalities (**) of Remark 4.1(b), the first with $a = uxu^{-1}$, $b = u$, the second with $a = u$, $b = x$; for v take τw. Then

$$v(ux) = uxu^{-1}w(u) = w(u)x.$$

Since x is an arbitrary homogeneous element, it follows that $u^{-1}w(u)$ is in the center of A, i.e. in R, for any w in GR. Let $w(u) = ru$, with r in R. Write $u = \sum u_\sigma$, with u_σ in A_σ. Then

$$w(u) = \textstyle\sum_\sigma w(\sigma)u_\sigma = \sum_\sigma ru_\sigma,$$

and $w(\sigma) = r$ whenever $u_\sigma \neq 0$, since R is a field. In particular, $w(u) = w(\sigma)u$ for σ chosen so that $u_\sigma \neq 0$. The characterization of A_σ displayed above shows that u is homogeneous.

4.3. *Remarks.* (a) The proposition above generalizes special cases proved by Long, viz. for A a central simple algebra over an algebraically closed field K and G a group of prime order p; $\text{char}(K) \neq p$ is done in [**13**, § 2, pp. 244–245], $\text{char}(K) = p$ in [**14**, pp. 589–593]. Our formulation avoids use of the necessity of every R-algebra automorphism (respectively, R-derivation) being inner, a fact used in [**13**] (respectively [**14**]).

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(b) The discussion in 4.1(b) may be given a more general framework. A grading of the R-module amounts to a co-action of the Hopf algebra $H = RG$ on A, i.e. an R-module map α,

$$\alpha : A \rightarrow A \otimes H,$$

making certain diagrams commutative [**14**, Section 2]. The condition $A_\sigma A_\tau \subseteq A_{\sigma\tau}$ is reflected in α being an algebra map (or equivalently, in the multiplication map $A \otimes A \rightarrow A$ being an H-comodule map. The H-comodule action on A induces an H^*-module action on A,

$$\beta : H^* \otimes A \rightarrow A,$$

defined by $\beta(v \otimes a) = (1 \otimes v)(\alpha a)$. That A is an H^*-module follows from A being an H-comodule. But the condition that α is an algebra map also has its effect. It implies that (β, H^*) is a measuring from A to A in the sense of Sweedler [**17**, p. 137ff]. It is this condition of being a measuring that is summarized by our formula (**) in 4.1.

4.4. THEOREM. *Let G have exponent m and assume* $\mathrm{Pic}_m(R) = 0$ *and that R has a primitive m-th root of* 1. *There is then a map*

$$\beta : BD_o(R, G) \rightarrow \mathrm{Aut}(G).$$

If $H^2(G, U(R)) = 0$ then β is a homomorphism. The map

$$\gamma : BD_o(R, G) \rightarrow B(R) \times \mathrm{Aut}(G)$$

defined by $\gamma([A]) = ([A], \beta[A])$ is then an epimorphism. If in addition R is connected and either of the following sets of conditions holds, γ is an isomorphism:

(i) $[G : 1]$ *is a unit in R;*

(ii) *G is cyclic of prime order p and R is a separately closed ring of characteristic p.*

Proof. Let A be a G-Azumaya R-algebra with center R. Because $\mathrm{Pic}_m(R) = 0$, every R-algebra automorphism of A whose order divides m is inner [**2**, Corollary 4.6, p. 108]. For σ in G let u_σ be chosen so that $\sigma a = u_\sigma a u_\sigma^{-1}$. By Proposition 4.2, u_σ is homogeneous, and because A has center R the grade of u_σ depends only on σ and not on u_σ. Define $\alpha_A : G \rightarrow G$ by

$$\alpha_A(\sigma) = \text{grade of } u_\sigma.$$

It is clear that $u_\sigma u_\tau u_{\sigma\tau}^{-1}$ commutes with all elements of A, hence is in $U(R)$. It follows that α_A is a group homomorphism. We now define $\beta_A : G \rightarrow G$ by

$$\beta_A(\sigma) = \sigma\alpha_A(\sigma)^{-1}.$$

Then β_A is a group homomorphism and in fact an automorphism. For suppose $\beta_A(\tau) = 1$. Then $\alpha_A(\tau) = \tau$ so that μ_τ has grade 1. Recall from Section 1 that the map $\mu : A \# \bar{A} \rightarrow \mathrm{End}_R(A)$, given by $\mu(a \# \bar{b})(x) = a^b x b$, is an isomorphism. Because u_τ has grade τ, it follows that $\mu(1 \# \bar{u}_\tau) = \mu(u_\tau \# \bar{1})$.

Thus u_τ is in R and its grade, τ, must be 1. Thus β_A has trivial kernel and since G is finite β_A is onto as well.

We proceed to show that β_A depends on the equivalence class of A in $BD(R, G)$ rather than on A itself. Suppose $[A] = [B]$ in $BD(R, G)$, with A and B both R-central. Then

$$A \# \operatorname{End}_R(P) = B \# \operatorname{End}_R(Q)$$

as dimodule algebras, for G-dimodules P and Q which are faithfully projective R-modules. But

$$A \# \operatorname{End}_R(P) \cong A \otimes \operatorname{End}_R(P),$$

and similarly for B, $\operatorname{End}_R(Q)$ by [**13**, Theorem 1.3]. The action of G on $\operatorname{End}_R(P)$ is defined by $(\sigma f)(x) = \sigma f(\sigma^{-1}x)$, so that $\sigma f = \sigma f \sigma^{-1}$, elements of G being viewed as lying in $\operatorname{End}_R(P)$ by their given action on P. But G acts as grading-preserving maps on P, hence σ has grade 1 as an element of $\operatorname{End}_R(P)$. Now with u_σ inducing the action of G on A, $u_\sigma \otimes \sigma$ may be chosen to induce the G-action on $A \otimes \operatorname{End}_R(P)$. Since $u_\sigma \otimes \sigma$ has the same grade as u_σ, the desired conclusion follows and we have a well-defined map

$$\beta : BD_0(R, G) \rightarrow \operatorname{Aut}(G)$$

given by $\beta([A]) = \beta_A$.

Now assume $H^2(G, U(R)) = 0$. Under this additional hypothesis $BD_0(R, G)$ is a subgroup of $BD(R, G)$ (see Proposition 3.4). The u_σ which define the action of G on A may be chosen so that

$$u_\sigma u_\tau = u_{\sigma\tau}, \quad u_1 = 1, \quad \sigma u_\tau = u_\tau.$$

It is a straightforward matter to verify from these equations that if u_σ are so chosen in A, v_σ in B and $\rho = \beta_B(\sigma)$, then $u_\rho \# v_\sigma$ induces the action of σ on $C = A \# B$. But $u_\rho \# v_\sigma$ has degree $\alpha_A(\rho)\alpha_B(\sigma)$ and thus

$$\begin{aligned}\beta_C(\sigma) = \sigma\alpha_C(\sigma)^{-1} = \sigma\alpha_B(\sigma)^{-1}\alpha_A(\rho)^{-1} &= \beta_B(\sigma)\alpha_A(\beta_B(\sigma)) \\ &= \beta_A(\beta_B(\sigma)).\end{aligned}$$

This shows that $\beta_{A\#B} = \beta_A \circ \beta_B$.

Now define γ as in the statement of the current theorem. Under our hypotheses that $H^2(G, U(R)) = 0$ and $\operatorname{Pic}_m(R) = 0$ we know that $A \# B \cong A \otimes B$ (see Lemma 3.2) and it is clear that γ is a homomorphism. We shall show that γ maps onto $B(R) \times \operatorname{Aut}(G)$. Let $[A]$ in $B(R)$ and j in $\operatorname{Aut}(G)$ be given. Let $\alpha(\sigma) = \sigma j(\sigma)^{-1}$. Let P be a free R-module on the basis x_σ, σ in G, graded by $P_\sigma = Rx_\sigma$. Define a G-action on P by $\tau x_\sigma = x_{\alpha(\tau)\sigma}$. Let $B = \operatorname{End}_R(P)$, with induced grading and action. The induced G-action on B is such that acting by σ is just conjugation by the element σ viewed as lying in B. Let u_σ be σ viewed as an element of B. Then u_σ has grade $\alpha(\sigma)$. Let $C = A \otimes B$ with the G-action and grading induced by the ones constructed on B, relative to the trivial ones on A. If C is G-Azumaya it is clear that $\gamma([C]) = ([A], j)$.

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To prove that C is G-Azumaya one may use the following facts; the proof of the first of these is readily adapted from [**14**]:

(1) There is an algebra map $t : \overline{\text{End}_R(P)} \to \text{End}_R(P)^{op}$ given by $t(\bar{f})(x) = (\sigma f)(x)$ for x in P_σ. If $j(\sigma) = \sigma\alpha(\sigma)^{-1}$ defines an automorphism of G then t is an isomorphism [**14**, Lemma 5.6].

(2) To show $\mu : A \# \bar{A} \to \text{End}_R(A)$ is an isomorphism it suffices to do this for R/p as p ranges over the maximal ideals of R (similarly for

$$\eta : \bar{A} \# A \to \text{End}_R(A)^{op}).$$

After thus reducing to R being a field, the argument in [**14**, Proposition 5.7] shows that A is G-Azumaya. This completes the proof that γ is onto.

Suppose now that $\gamma([A]) = 0$. This implies that as an R-algebra $A \cong \text{End}_R(P)$ with P a faithfully projective R-module. If we can give P the structure of a G-dimodule such that the induced structure on A agrees with the one we started with, then $[A]$ will be the trivial element of $BD_o(R, G)$.

First we define the G-action on P. Choose u_σ in A such that $\sigma x = u_\sigma x u_\sigma^{-1}$ for x in A, $u_1 = 1$, $u_\sigma u_\tau = u_{\sigma\tau}$. Define $\sigma x = u_\sigma(x)$, giving a well-defined G-action on P (relative to the choice of the u_σ).

Let $H = RG$, $H^* = GR$, the dual Hopf algebra to RG. Having a G-grading on P is equivalent to making P into an RG-comodule, i.e. defining a co-action $P \to P \otimes H$ (cf. 4.3(b)). Because H is a projective R-module of finite type, $\text{Hom}(P, P \otimes H)$ is naturally isomorphic to $\text{Hom}(H^* \otimes P, P)$. Thus obtaining a G-grading on P is equivalent to making P into an H^*-module.

Consider case (i) first. Let $G = G_1 \times \ldots \times G_s$, with G_i cyclic. Let $n = [G : 1]$, $m = \exp(G)$. Because $H^2(G, U(R)) = 0$ it follows that $H^2(G_i, U(R)) = 0$ for $i = 1, \ldots, s$ [**20**, Theorem 2.1]. R contains a primitive m-th root of unity and since n is a unit in R the dual group $G^* = \text{Hom}(G, U(R))$ is isomorphic to G and $H^* \cong RG^*$. Let $\pi : G^* \to G$ be an isomorphism. The grading on $A = \text{End}_R(P)$ defines a G^*-action by $\chi(a) = \sum_{\sigma \in G} \chi(\sigma) a_\sigma$, and G^* acts as R-algebra automorphisms of A. Thus for each χ in G^* there exists u_χ in A such that $\chi(a) = u_\chi a u_\chi^{-1}$. Because $H^2(G^*, U(R)) \cong H^2(G, U(R)) = 0$ it follows that we may choose the u_χ with $u_1 = 1$, $u_\chi u_\psi = u_{\chi\psi}$ for χ, ψ in G^*. Now define an action of G^* on P by $\chi y = u_\chi(y)$ for y in P. This induces a G-grading as discussed above.

The setting of (ii) is essentially that considered by Long in [**14**]; the essential steps in Long's proof are his Propositions 5.1 and 5.3. The first of these (that the dual H^* of the group algebra $H = RG$ has basis $1^*, d, d^2, \ldots, d^{p-1}$ where d satisfies $d^p = d$ and $\Delta(d) = 1^* \otimes d + d \otimes 1^*$) remains valid because R contains $\boldsymbol{F}_p$. The second proposition hinges on the following two facts, whose validity in our setting holds by the indicated results: a derivation on an R-Azumaya algebra is inner [**15**, Proposition 4.11]; and $X^p - X + r$ is a separable polynomial in $R[X]$ for r in R (i.e. $R[X]/(X^p - X + r)$ is a separable extension of R) [**11**, Theorem 2.2]. We refer the reader to [**14**] for details.

5. The isomorphism $BD(R, C_p) \cong B(R) \times D_{2(p-1)}$. Let p be a prime and G a cyclic group of order p. Assume R is connected and contains a p-th root for each of its elements, i.e. $H^2(G, U(R)) = 0$.

If p is not a unit in R then every G-Azumaya R-algebra is R-Azumaya by Corollary 2.7. Thus $BD(R, G) = BD_o(R, G)$ and by Theorem 4.4 we conclude that $BD(R, G) \cong B(R) \times C_{p-1}$, where C_{p-1} denotes a cyclic group of order $p - 1$. We shall concern ourselves with the case where p is a unit in R.

5.1. THEOREM. *Let G by cyclic of prime order p. Let R be connected, with $H^2(G, U(R)) = 0$, $\text{Pic}_p(R) = 0$ and p a unit in R. Then $BD(R, G) \cong B(R) \times D_{2(p-1)}$, where $D_{2(p-1)}$ denotes a dihedral group with $2(p - 1)$ elements.*

Proof. The idea for this proof is taken from [**13**]. Because the setting there implies that $B(R) = 0$ (R is a separably closed field) some unpleasant technicalities are avoided which we shall find it necessary to deal with. We shall write

$$D_{2(p-1)} = \{1, 2, \ldots, p - 1, a_1, a_2, \ldots, a_{p-1}\}$$

where the group multiplication is given by the following rules, each interpreted mod p where necessary:

$$a_i j = a_{ij},$$
$$i a_j = a_{i^{-1}j},$$
$$a_i a_j = i^{-1}j.$$

For A a G-Azumaya algebra, its center Z must be either R or else a Galois extension of R with group G, by Proposition 2.2. Following the terminology in [**13**] the two cases will be labelled as A being of type (i) and type (ii) respectively.

Suppose A is of type (ii). By (e) of Proposition 2.11 we have that $Z \cong RG_f{}^\phi$ with f an abelian cocycle in $Z^2(G, U(R))$. By hypothesis on R, f is cohomologous to the trivial cocycle hence $Z \cong RG_1{}^\phi$ as G-dimodule algebras. Moreover ϕ is nondegenerate, again by (e) of Proposition 2.11. The center of A is an invariant of the class of A in $BD(R, G)$, since

$$A \# \text{End}_R(P) \cong A \otimes \text{End}_R(P)$$

[**13**, Theorem 1.3] and $Z(A \otimes B) \cong Z(A) \otimes Z(B)$ for A and B R-separable [**15**, Proposition 2.3]. It follows readily that ϕ is an invariant of the class of A in $BD(R, G)$, as the G-dimodule structure on Z determines ϕ.

The nondegeneracy of ϕ together with our hypotheses on R imply that Z is the trivial Galois extension of R. For let $Z = \oplus Rx_\sigma$ (as in Proposition 2.8) and define

$$e_\sigma = \frac{1}{[G:1]} \sum_{\tau \in G} \phi(\sigma, \tau) x_\tau.$$

The nondegeneracy of ϕ yields orthogonality relations

$$\textstyle\sum_{\sigma \in G} \phi(\sigma, \tau)\phi(\sigma^{-1}, \gamma) = [G:1]\delta_{\tau,\gamma}$$

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(cf. Proposition 2.8 and see [**18**, § 126]) which imply that the e_σ are pairwise orthogonal idempotents with sum 1. The relation $\sigma u_\tau = \phi(\sigma, \tau)x_\tau$ and the bilinearity of ϕ yield that $\sigma e_\tau = e_{\sigma\tau}$. Thus Z is the trivial Galois extension of R.

By Proposition 3.1 there are isomorphisms of G-dimodule algebras

$$A \cong A^G \# Z \cong Z \# A_1;$$

furthermore both A^G and A_1 are R-separable, and $[A^G] = [A_1]$ in $B(R)$. Write $[A_o]$ for the common value of these elements in $B(R)$.

We fix the following bits of notation: π is a generator of G, ω a primitive p-th root of unity in R. We identify $\text{Aut}(G)$ as a subgroup of $D_{2(p-1)}$ by sending β to i, where $\beta(\pi) = \pi^i$.

Now define $\psi : BD(R, G) \to B(R) \times D_{2(p-1)}$ by

$$\psi([A]) = \begin{cases} \gamma([A]) \text{ for } A \text{ of type (i)}, \\ ([A_o], a_i) \text{ for } A \text{ of type (ii)}, \end{cases}$$

where $\gamma : BD_o(R, G) \to B(R) \times \text{Aut}(G)$ is defined as in Theorem 4.4, $\text{Aut}(G)$ is embedded in $D_{2(p-1)}$ as mentioned above, A has center RG_1^ϕ with $\phi(\pi, \pi) = \omega^i$ and $[A_o]$ in $B(R)$ is as given above.

We know from Theorem 4.4 that ψ is well-defined on algebras of type (i). Suppose $B = A \# E$ with $E = \text{End}_R(P)$ trivial in $BD(R, G)$. Then $B = A \otimes E$ [**13**, Theorem 1.3]. We noted above that A and B have the same center Z, and we know that $Z = \prod Re_\sigma$ with $e_\sigma e_\tau = \delta_{\sigma,\tau} e_\sigma$, $\sum e_\sigma = 1$ and $\sigma e_\tau = e_{\sigma\tau}$. Then $Be_1 \cong Ae_1 \otimes E$ and by (a) of Proposition 3.1 we have isomorphisms of R-algebras $Ae_1 \cong A^G$, $Be_1 \cong B^G$. Thus $[A^G]$ in $B(R)$ depends only on the class of A in $BD(R, G)$ and not on A itself.

To show ψ is onto we first note that ψ is onto elements of the form $([A], i)$, $0 < i < p$, by Theorem 4.4. The element $([A], a_i)$ is also the image of something under ψ, namely of $A \# RG_1^\phi$, where $\phi(\pi, \pi) = \omega^i$.

We shall show below that ψ is a homomorphism. Assuming that we wish to show that ψ is one-one. We know from Theorem 4.4 that $\psi([A]) = 0$ with $[A]$ in $BD_o(R, G)$ implies $[A] = 0$. But for A of type (ii) the second component of $\psi([A])$ is a_i for some i with $0 < i < p$, so $\psi([A]) \neq 0$.

We know, also from Theorem 4.4, that $\psi(xy) = \psi(x)\psi(y)$ when x and y are in $BD_o(R, G)$. We shall check this next when x is in $BD_o(R, G)$ and y in $BD(R, G)$. Let $\psi(x) = ([A]), i)$ where A is G-Azumaya and R-Azumaya, $\sigma a = u_\sigma a u_\sigma^{-1}$ for a in A, u_σ has grade $\alpha_A(\sigma)$ and $\beta_A(\sigma) = \sigma\alpha_A(\sigma)^{-1}$ (see the proof of Theorem 4.4 for details). Saying that $\psi(x) = ([A], i)$ means that $\beta_A(\pi) = \pi^i$. Let $k = i^{-1}$ in $D_{2(p-1)}$, i.e. $ki \equiv 1 \pmod p$ and $\beta_A(\pi^k) = \pi$. Now let $y = [B]$ where B has center RG_1^ϕ and $\phi(\pi, \pi) = \omega^j$. Thus $B \cong B^G \# RG_1^\phi$ and $\psi(y) = ([B^G], a_j)$. By Lemma 3.2 there is an isomorphism of G-module algebras

$$A \# B \cong A \otimes B$$

under which $a \# b$ corresponds to $au_\sigma \otimes b$ for b in B_σ. Thus $A \# B$ has for its center the image of $1 \otimes RG_1{}^\phi$ under the inverse isomorphism, i.e. as a G-module algebra the center of $A \# B$ is a free R-module on the elements $y_\sigma = u_\sigma{}^{-1} \otimes x_\sigma$, where $RG_1{}^\phi = \oplus Rx_\sigma$. Now y_σ has grade $\beta_A(\sigma)$. Write $\bar{\sigma}$ for $\beta_A{}^{-1}(\sigma)$, i.e. $\beta_A(\bar{\sigma}) = \sigma$. Then the element $z_\sigma = y_{\bar{\sigma}}$ has grade σ and the center of $A \# B$ is a free R-module on the z_σ. Because $\sigma u_\tau = u_\tau$ it follows that $\sigma y_\tau = \phi(\sigma, \tau) y_\tau$; then $\sigma z_\tau = \sigma y_{\bar{\tau}} = \phi(\sigma, \bar{\tau}) z_\tau$. The second component in $\psi([A \# B])$ is by definition determined as a_l, where $\pi z_\pi = \omega^l z_\pi$. But $\pi z_\pi = \phi(\pi, \bar{\pi})$ and $\bar{\pi} = \pi^k$ according to the definition above for k. Thus the second component of $\psi(xy)$ is $\phi(\pi, \pi)^k = \omega^{jk}$ where we had $\psi([B]) = ([B^G], a_j)$. Thus $\psi(xy)$ has $a_{i^{-1}j}$ for its second component and as seen by consulting the very beginning of this proof, $ia_j = a_{i^{-1}j}$, so that $\psi(x)\psi(y) = \psi(xy)$ is valid insofar as the second components are concerned. But by Lemma 3.2 we have that $A \# B^G$ and $A \otimes B$ are isomorphic as G-module algebras hence

$$A \# B^G \# RG_1{}^\phi \cong A \otimes B^G \# RG_1{}^\phi$$

as R-algebras and $\psi(xy)$ has $[A \otimes B^G]$ for its first component, and $\psi(xy) = \psi(x)\psi(y)$.

We remark next that if A is of type (ii) then $\psi([\bar{A}]) = \psi([A])^{-1}$. Let w_σ in A_1 be chosen so that $\sigma a = w_\sigma a w_\sigma{}^{-1}$ for a in A_1, $w_1 = 1$, $w_\sigma w_\tau = w_{\sigma\tau}$ (and hence $\sigma w_\tau = w_\tau$) for σ, τ in G; this is possible because $\mathrm{Pic}_p(R) = 0$ implies that G acts as inner automorphisms of A_1, and $H^2(G, U(R)) = 0$ does the rest (cf. Lemma 3.2). It is not difficult to compute that $\overline{x_\sigma w_\sigma}$ is in the center of $\bar{A}$: to do this one uses that an element of $\bar{A}$ is of the form $\overline{x_\tau a}$ with a in A_1 (by Proposition 3.1); that $\sigma x_\tau = \phi(\sigma, \tau) x_\tau$ and $\phi(\sigma, \tau) = \phi(\tau, \sigma)$ (by cyclicity of G); and that w_σ is in A_1. The element $y_\sigma = \overline{x_\sigma w_\sigma}$ has grade σ in $\bar{A}$ and $\sigma y_\tau = \phi(\sigma, \tau) y_\tau$. Thus if $\psi([A]) = ([A_o], a_i)$, $\psi([\bar{A}])$ has a_i for its second component. Since $\bar{A}^G \cong \overline{A^G}$ and the latter is easily seen to be isomorphic to $(A^G)^{op}$, we may conclude that $\psi([\bar{A}]) = \psi([A])^{-1}$.

It now follows that $\psi(yx) = \psi(y)\psi(x)$ for x of type (i) and y of type (ii), for we can write yx as $(x^{-1}y^{-1})^{-1}$ and apply the above results.

There remains to prove $\psi(xy) = \psi(x)\psi(y)$ for x and y of type (ii). Let $x = [A]$, $y = [B]$ with

$$A \cong A^G \# RG_1{}^\phi, \; B \cong RG_1{}^\theta \# B_1.$$

Because A^G (respectively, B_1) has trivial G-action (respectively G-grading) we have that

$$A \# B \cong A^G \otimes (RG_1{}^\phi \# RG_1{}^\theta) \otimes B_1$$

as G-dimodule algebras. As a G-graded R-algebra $RG_1{}^\theta$ is isomorphic to $RG_1{}^\phi$, hence $\overline{RG_1{}^\phi} \# RG_1{}^\theta$ is isomorphic to $RG_1{}^\phi \# RG_1{}^\phi$ as a G-graded R-algebra. But $RG_1{}^\phi = RG_1{}^\phi$ because $H^2(G, U(R)) = 0$ (Remark 2.9(b)) and we know $RG_1{}^\phi$ is G-Azumaya; hence $RG_1{}^\phi \# RG_1{}^\theta$ is isomorphic to $\mathrm{End}_R(RG)$ as a G-graded R-algebra. This allows us to conclude that the first component of $\psi(xy)$ is

$[A^G][B_1]$ in $B(R)$, which is $[A_o][B_o]$ or just the first component of $\psi(x)\psi(y)$. To show equality of the relevant second components we must show that there is an element v_π in $A \# B$ such that $v_\pi c v_\pi^{-1} = \pi c$ for c in $A \# B$ and $\beta_{A \# B}(\pi)$ (as defined in Theorem 4.4) is the appropriate power of π; if $\psi(x)$ (respectively $\psi(y)$) has second component a_i (respectively a_j) this power is $i^{-1}j$ (by the rule for computing $a_i a_j$). Write k for $i^{-1}j$ and C for $A \# B$. Because $\beta_C(\pi) = \pi\alpha_C(\pi)^{-1}$, where $\alpha_C(\pi)$ is the grade of v_π, we must find a v_π of grade π^{1-k}. Let l be chosen so that $\phi(\pi, \pi^{-l}) = \theta(\pi, \pi)$ (where $Z(A) = RG_1^\phi$, $Z(B) = RG_1^\theta$), and let w_σ in B_1 be chosen so that $w_\sigma b w_\sigma^{-1} = \sigma b$ for b in B_1. Write $RG_1^\phi = \oplus_\sigma Rx_\sigma$, $RG_1^\theta = \oplus_\sigma Ry_\sigma$. It is not hard to verify that with $v_\pi = x_{\pi^l} \# y_\pi w_\pi$ we have $v_\pi c v_\pi^{-1} = \pi c$ for c in B (write $c = a \# y_\tau b_1$ with b_1 in B_1). But this v_π has grade π^{1+l}. Finally, since $\phi(\pi, \pi) = \omega^i$ and $\theta(\pi, \pi) = \omega^j$ the choice of l implies that $\omega^{-il} = \omega^j$ or $l = -i^{-1}j = -k \pmod p$. This completes the proof of the theorem.

5.2. THEOREM. *Let $G = G_1 \times \ldots \times G_m$ where G_i is cyclic of prime order p_i and the p_i are distinct. Let R be connected, with $H^2(G, U(R)) = 0$. Let $n = p_1 \ldots p_m$ and assume* $\mathrm{Pic}_n(R) = 0$ *and that n is a unit in R. Then $BD(R, G) \cong B(R) \times D_1 \times \ldots \times D_m$ where D_i is a dihedral group of order* $2(p_i - 1)$.

Proof. The most natural proof of this result uses the previous theorem and results on the Morita theory for G-dimodule algebras referred to in the introduction. Besides this theory some of the crucial facts needed are: 1) For A a G-Azumaya R-algebra, its center Z is a tensor product $Z_1 \otimes \ldots \otimes Z_n$ where $Z_i = R$ or Z_i is a Galois extension of R with group G_i; 2) $BD(R, G_i)$ is embedded in $BD(R, G)$ and these subgroups of $BD(R, G)$ commute; 3) The subgroup of $BD(R, G)$ generated by the $BD(R, G_i)$ is naturally embedded in $B(R) \times D_1 \times \ldots \times D_n$ and this subgroup is in fact $BD(R, G)$. Details will appear elsewhere.

The theorem above can be deduced by applying the exact sequence derived in [**6**]. The methods employed here are completely different from those of [**6**].

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The Institute for Advanced Study,
Princeton, New Jersey