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Divisors computing minimal log discrepancies on lc surfaces

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Abstract

Let $(X \ni x, B)$ be an lc surface germ. If $X \ni x$ is klt, we show that there exists a divisor computing the minimal log discrepancy of $(X \ni x, B)$ that is a Kollár component of $X \ni x$. If $B \neq 0$ or $X \ni x$ is not Du Val, we show that any divisor computing the minimal log discrepancy of $(X \ni x, B)$ is a potential lc place of $X \ni x$. This extends a result of Blum and Kawakita who independently showed that any divisor computing the minimal log discrepancy on a smooth surface is a potential lc place.

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1. *Introduction*

The *minimal log discrepancy (mld)* is an invariant that provides a sophisticated measure of the singularities of an algebraic variety. It not only plays an important role in the study of singularities but is also a central object in the minimal model program. Shokurov proved that the ascending chain condition (ACC) conjecture for mlds and the lower-semicontinuity (LSC) conjecture for mlds imply the termination of flips [**40**]. For papers related to these conjectures, we refer the readers to [**2**, **8**, **9**, **11**, **12**, **14**, **15**, **18**, **19**, **20**, **21**, **23**, **29**, **30**–**34**, **38**–**40**].

Very recently, there has been studies on the mld from the perspective of K-stability theory. In particular, in [**13**, **14**], *normalised volumes* [**27**] and *Kollár components* (in some references, called *reduced components*) have played essential roles to prove some important cases of the ACC conjecture for mlds. Since the structure of the Kollár components are very well-studied [**26**, **28**, **35**, **38**, **41**], we may propose the following natural folklore question:

Question 1·1. Let $(X \ni x, B)$ be an lc germ of dimension ≥ 2 such that $X \ni x$ is klt. Under what conditions will there exist a divisor E over $X \ni x$ such that $a(E, X, B) = \mathrm{mld}(X \ni x, B)$ and E is a Kollár component of $X \ni x$?

In the paper, we show that Question 1·1 always has a positive answer in dimension 2:

THEOREM 1·2. *Let* $(X \ni x, B)$ *be an lc surface germ such that* $X \ni x$ *is klt. Then there exists a divisor* E *over* $X \ni x$ *such that* $a(E, X, B) = \mathrm{mld}(X \ni x, B)$ *and* E *is a Kollár component of* $X \ni x$.

Regrettably, Question 1·1 does not always have a positive answer in dimension ≥ 3 even when $B = 0$ due to Example 6·1.

For smooth surfaces, a modified version of Question 1·1 was proved by Blum [5, theorem 1·2] and Kawakita [22, remark 3], who show that any divisor computing the mld is a potential lc place (see Definition 2·4 below) of the ambient variety, while Kawakita additionally shows that any such divisor is achieved by a weighted blow-up [22, theorem 1]. With this in mind, we may ask the following folklore question:

Question 1·3. Let $(X \ni x, B)$ be an lc germ. Under what conditions will every divisor that computes the mld be a potential lc place?

In this paper, we also answer Question 1·3 for surfaces:

THEOREM 1·4. *Let* $(X \ni x, B)$ *be an lc surface germ. Then every divisor* E *over* $X \ni x$ *such that* $a(E, X, B) = \mathrm{mld}(X \ni x, B)$ *is a potential lc place of* $X \ni x$ *if and only if* $(X \ni x, B)$ *is* ***not*** *of the following types*:

(i) $B = 0$ *and* $X \ni x$ *is a* D_m*-type Du Val singularity for some integer* $m \geq 5$, *or*

(ii) $B = 0$ *and* $X \ni x$ *is an* E_m*-type Du Val singularity for some integer* $m \in \{6, 7, 8\}$.

We say a few words about the intuition of Questions 1·1 and 1·3. Roughly speaking, a Kollár component always admits a log Fano structure that is compatible with the local singularity, and a potential lc place always admits a log Calabi–Yau structure that is compatible with the local singularity. These structures allow us to use results in global birational geometry to study the behaviour of the divisor and the local geometry of the singularity. On the other hand, we usually do not know whether a divisor calculating the mld is admits those good structures or not, and therefore, many powerful tools in global geometry are difficult to apply to the study on the mlds of a singularity.

Therefore, getting a satisfactory answer for either Question 1·1 or Question 1·3 could provide us with possibilities to apply global geometry results to tackle the ACC conjecture or the LSC conjecture for mlds. In particular, since Kollár components are well-studied in K-stability theory, with a satisfactory answer for Question 1·1, there is a strong potential for K-stability theory results to be applied to the study on mlds.

THEOREM 1·2 *and Theorem 1·4 follow from the following classification result on divisors computing mlds on lc surfaces:*

THEOREM 1·5. *Let* $(X \ni x, B)$ *be an lc surface germ and* $\mathcal{C}$ *the set of all prime divisors over* $X \ni x$ *which compute* $\mathrm{mld}(X \ni x, B)$.

(i) if $(X \ni x, B)$ *is dlt, then*:

(a) if $X \ni x$ *is smooth or an A-type singularity, then any element of* $\mathcal{C}$ *is a Kollár component of* $X \ni x$;

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(b) if $X \ni x$ is a D_m-type singularity for some integer $m \geq 4$ or an E_m-type singularity for some integer $m \in \{6,7,8\}$, let $f: Y \to X$ be the minimal resolution of $X \ni x$ and $\mathcal{D}(f)$ the dual graph of f. Then;

(1) there exists a unique element $E \in \mathcal{C}$ that is a Kollár component of $X \ni x$, and E is the unique fork of $\mathcal{D}(f)$;

(2) $\mathcal{C} \subset \operatorname{Exc}(f)$;

(3) if $B \neq 0$ or $X \ni x$ is not Du Val, then:

(A) *any element of $\mathcal{C}$ is a potential lc place of $X \ni x$*;

(B) *if $X \ni x$ is an E_m-type singularity, then $\mathcal{C}$ only contains the unique fork of $\mathcal{D}(f)$.*

(4) if $B = 0$ and $X \ni x$ is Du Val, then:

(A) $\mathcal{C} = \operatorname{Exc}(f)$;

(B) if $X \ni x$ is a D_m-type singularity, then an element $F \in \mathcal{C}$ is a potential lc place of $X \ni x$ if and only if either F is the fork of $\mathcal{D}(f)$, or the two branches which do contain F both have length 1;

(C) if $X \ni x$ is an E_m-type singularity, then an element $F \in \mathcal{C}$ is a potential lc place of $X \ni x$ if and only if F is the fork of $\mathcal{D}(f)$.

(ii) If $(X \ni x, B)$ is not dlt but $X \ni x$ is klt, then:

(a) Any element of $\mathcal{C}$ is a potential lc place of $X \ni x$;

(b) There exists an element of $\mathcal{C}$ that is a Kollár component of $X \ni x$.

(c) If $X \ni x$ is smooth, then any element of $\mathcal{C}$ is a Kollár component of $\operatorname{mld}(X \ni x, B)$

(iii) If $X \ni x$ is not klt, then any element of $\mathcal{C}$ is a potential lc place of $X \ni x$.

We hope that our results could inspire people to tackle Questions 1·1 and 1·3.

Remark 1·6. Some complementary examples of our main theorems are given in Section 6.

Remark 1·7. Although the study on minimal log discrepancies was traditionally considered over $\mathbb{C}$, recently there has been some studies on the structure of minimal log discrepancies over fields of arbitrary characteristics (cf. [7, 17, 36]). In this paper, the results hold over fields of arbitrary characteristics. This is because we only work on surfaces and we only care about the local behavior of surfaces. In this case, the concepts of minimal resolution, dual graphs, intersection numbers etc. will work in arbitrary characteristics. We emphasize that the key references we cite [25, section 4] and [15] also work for arbitrary characteristics.

2. *Preliminaries*

We adopt the standard notation and definitions in [25].

Definition 2·1. A pair (X,B) consists of a normal quasi-projective variety X and an $\mathbb{R}$-divisor $B \geq 0$ such that $K_X + B$ is $\mathbb{R}$-Cartier. If $B \in [0, 1]$, then B is called a boundary.

Let E be a prime divisor on X and D an $\mathbb{R}$-divisor on X. We define $\operatorname{mult}_E D$ *to be the multiplicity of E along D. Let $\phi: W \to X$ be any log resolution of (X,B) and let*

$$K_W + B_W := \phi^*(K_X + B).$$

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The log discrepancy *of a prime divisor D on W with respect to (X,B) is* $1-\mathrm{mult}_D B_W$ *and it is denoted by* $a(D,X,B)$*. We say that (X,B) is lc (resp.* klt*) if* $a(D,X,B)\geq 0$ *(resp.* >0*) for every log resolution* $\phi : W\to X$ *as above and every prime divisor D on W. We say that (X,B) is dlt if* $a(D,X,B)>0$ *for some log resolution* $\phi : W\to X$ *as above and every prime divisor D on W. We say that (X,B) is plt if* $a(D,X,B)>0$ *for any exceptional prime divisor D over X.*

A germ $(X\ni x,B)$ *consists of a pair (X,B) and a closed point* $x\in X$*. If* $B=0$*, the germ* $(X\ni x,B)$ *is usually represented by* $X\ni x$*. We say that* $(X\ni x,B)$ *is lc (resp.* klt, dlt, plt*) if (X,B) is lc (resp.* klt, dlt, plt*) near x. We say that* $(X\ni x,B)$ *is* smooth *if X is smooth near x. We say that* $(X\ni x,B)$ *is log* smooth *if X is log smooth near x. A divisor E over X is called over* $X\ni x$ *if* $\mathrm{center}_X E=\bar{x}$*.*

Definition 2·2. The *minimal log discrepancy (mld)* of an lc germ $(X\ni x,B)$ is

$$\mathrm{mld}(X\ni x,B):=\min\{a(E,X,B)\mid E\text{ is a prime divisor over }X\ni x\}.$$

Definition 2·3 (Plt blow-ups). Let $(X\ni x,B)$ be a klt germ. A *plt blow-up* of $(X\ni x,B)$ is a blow-up $f:Y\to X$ with the exceptional divisor E over $X\ni x$, such that (Y,f^{-1}_*B+E) is plt near E, and $-E$ is ample over X. The divisor E is called a *Kollár component* (in some references, *reduced component*) of $(X\ni x,B)$.

Definition 2·4 (Potential lc place). Let $(X\ni x,B)$ be an lc germ. A *potential lc place* of $(X\ni x,B)$ is a divisor E over $X\ni x$, such that there exists $G\geq 0$ on X such that $(X\ni x,B+G)$ is lc and $a(E,X,B+G)=0$.

Definition 2·5. A surface is a normal quasi-projective variety of dimension 2. A surface germ $X\ni x$ is called *Du Val* if $\mathrm{mld}(X\ni x,0)=1$.

Let $X\ni x$ be a klt surface germ of type A (resp. D,E) and $m\geq 1$ (resp. $m\geq 4, m\in\{6,7,8\}$) an integer. Let f be the minimal resolution of $X\ni x$. We say that $X\ni x$ is an A_m (resp. D_m, E_m)*-type singularity* if $\mathrm{Exc}(f)$ contains exactly m prime divisors.

For surfaces, to check that an extraction is a plt blow-up, we only need to control the singularity as the anti-ample requirement is automatically satisfied. The following lemma is well-known and we will use it many times:

LEMMA 2·6. *Let* $(X\ni x,B)$ *be a klt surface germ,* $f:Y\to X$ *an extraction of a prime divisor E, and* $B_Y:=f^{-1}_*B$*. Then:*

(i) *if* (Y,B_Y+E) *is plt near E, then E is a Kollár component of* $(X\ni x,B)$;

(ii) *if* (Y,B_Y+E) *is lc near E, then E is a potential lc place of* $(X\ni x,B)$.

In particular, any Kollár component of $(X\ni x,B)$ *is a potential lc place of* $(X\ni x,B)$.

Proof. Since $(X\ni x,B)$ is a klt surface germ, X is $\mathbb{Q}$-factorial, so there exists an f-exceptional divisor $F\geq 0$ such that $-F$ is ample over X [4, lemma 3·6·2(3)]. Since f only extracts E, $-E$ is ample over X. This implies (ii).

Since $-E$ is ample over X, $-(K_Y+B_Y+E)$ is ample over X. We may pick a general $G_Y\sim_{\mathbb{R},X}-(K_Y+B_Y+E)$ such that (Y,B_Y+E+G_Y) is lc near E and $K_Y+B_Y+E+G_Y\sim_{\mathbb{R},X}0$. Let $G:=f_*G_Y$, then $(X\ni x,B+G)$ is lc and E is a potential lc place of $(X\ni x,B)$.

Definition 2·7 (Dual graph). Let n be a non-negative integer, and $C = \cup_{i=1}^{n} C_i$ a collection of irreducible curves on a smooth surface U. We define the *dual graph* $\mathcal{D}(C)$ of C as follows.

(i) The vertices $v_i = v_i(C_i)$ of $\mathcal{D}(C)$ correspond to the curves C_i.

(ii) For $i \neq j$,the vertices v_i and v_j are connected by $C_i \cdot C_j$ edges.

In addition,

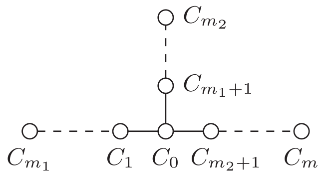
(iii) if we label each v_i by the integer $e_i := -C_i^2$, then $\mathcal{D}(C)$ is called the *weighted dual graph* of C.

A fork of a dual graph is a curve C_i such that $C_i \cdot C_j \geq 1$ for exactly three different $j \neq i$. A tail of a dual graph is a curve C_i such that $C_i \cdot C_j \geq 1$ for at most one $j \neq i$.

For any birational morphism $f:Y \to X$ between surfaces, let $E = \cup_{i=1}^{n} E_i$ be the reduced exceptional divisor for some non-negative integer n. We define $\mathcal{D}(f) := \mathcal{D}(E)$.

When we have a dual graph, we sometimes label C_i near the vertex v_i. We sometimes use black dots in the dual graph to emphasise the corresponding curves that are not exceptional.

Definition 2·8. Let $\mathcal{D}$ be a dual graph. If $\mathcal{D}$ looks like the following



for some integers $m > m_2 > m_1$, then $\cup_{i=1}^{m_1} C_i$, $\cup_{i=m_1+1}^{m_2} C_i$, and $\cup_{i=m_2+1}^{m} C_i$ will be called the *branches* of $\mathcal{D}$. The *length* of a branch is the number of irreducible curves in this branch.

We will use the following lemmas many times in this paper:

LEMMA 2·9. *Let $X \ni x$ be a smooth germ, $f:Y \to X$ the smooth blow-up of $X \ni x$ with exceptional divisor E, and C and D two $\mathbb{R}$-divisors on X without common irreducible components. Then*

$$(C \cdot D)_x = \sum_{y \in f^{-1}(x)} (f_*^{-1}C \cdot f_*^{-1}D)_y + (f_*^{-1}C \cdot E)(f_*^{-1}D \cdot E)$$
$$= \sum_{y \in f^{-1}(x)} (f_*^{-1}C \cdot f_*^{-1}D)_y + \mathrm{mult}_x C \cdot \mathrm{mult}_x D.$$

Proof. Possibly shrinking X to a neighbourhood of x and shrinking Y to a neighbourhood over x, we may assume that $(C \cdot D)_x = C \cdot D$ and $\sum_{y \in f^{-1}(x)} (f_*^{-1}C \cdot f_*^{-1}D)_y = f_*^{-1}C \cdot f_*^{-1}D$. Thus

$$0 = f^*C \cdot E = (f_*^{-1}C + (\mathrm{mult}_x C)E) \cdot E = f_*^{-1}C \cdot E - \mathrm{mult}_x C$$

and

$$0 = f^*D \cdot E = (f_*^{-1}D + (\mathrm{mult}_x C)E) \cdot E = f_*^{-1}D \cdot E - \mathrm{mult}_x D.$$

By the projection formula,

$$C \cdot D = f^*C \cdot f_*^{-1}D = f_*^{-1}C \cdot f_*^{-1}D + (\mathrm{mult}_x C)E \cdot f_*^{-1}D = f_*^{-1}C \cdot f_*^{-1}D + \mathrm{mult}_x C \cdot \mathrm{mult}_x D.$$

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LEMMA 2·10 *(cf. [25, lemma 3·41, corollary 4·2]). Let U be a smooth surface and $C = \cup_{i=1}^{m} C_i$ a connected proper curve on U. Assume that the intersection matrix $\{(C_i \cdot C_j)\}_{1\le i,j\le m}$ is negative definite. Let $A = \sum_{i=1}^{m} a_i C_i$ and $H = \sum_{i=1}^{m} b_i C_i$ be $\mathbb{R}$-linear combinations of the curves C_i. Assume that $H \cdot C_i \le A \cdot C_i$ for every i, then either $a_i = b_i$ for each i or $a_i < b_i$ for each i.*

3. *Divisors computing mlds over smooth surface germs*

In this section, we study the behavior of divisors computing mlds over a smooth surface germ. The following Definition-Lemma greatly simplify the notation in the rest of the paper and we will use it many times.

3·1. Definitions and lemmas

Definition-LEMMA 3·1. Let $(X \ni x, B)$ be a smooth lc surface germ and E a divisor over $X \ni x$. By [25, lemma 2·45], there exists a unique positive integer $n = n(E)$ and a unique sequence of smooth blow-ups

$$X_E := X_{n,E} \xrightarrow{f_{n,E}} X_{n-1,E} \xrightarrow{f_{n-1,E}} \dots \xrightarrow{f_{1,E}} X_{0,E} := X$$

such that E is on X_E and each $f_{i,E}$ is the smooth blow-up at center$_{X_{i-1,E}}E$. We define $f_E := f_{1,E} \circ f_{2,E} \cdots \circ f_{n,E}$, E^i the exceptional divisor of $f_{i,E}$ for each i, and E^i_j the strict transform of E^i on $X_{j,E}$ for any $i \le j$. In particular, $E^i_i = E^i$ for each i and $E = E^n = E^n_n$.

Remark 3·2. We need the following facts many times, which are elementary and we omit the proof. Let $(X \ni x, B)$ be a smooth lc surface germ, E a divisor over $X \ni x$, and $n := n(E)$. Then for any $i \le j$ such that $i, j \in \{1, 2, \dots, n\}$,

(i) E^i_j is a smooth rational curve,

(ii) $\cup_{k=1}^{i} E^k_i$ is simple normal crossing,

(iii) $(E^i)^2 = -1$, and

(iv) $(E^i_j)^2 \le -2$ when $i < j$.

LEMMA 3·3. *Let $(X \ni x, B)$ be a smooth surface germ and $f: Y \to X$ the smooth blow-up at x with exceptional divisor E. Then $a(E, X, B) = 2 - \mathrm{mult}_x B$. In particular, $\mathrm{mld}(X \ni x, B) \le 2 - \mathrm{mult}_x B$.*

Proof. This immediately follows from [25, lemma 2·29].

LEMMA 3·4. *Let $X \ni x$ be a smooth surface germ, $B \ge 0$ an $\mathbb{R}$-divisor on X and C a prime divisor on X. Then $(B \cdot C)_x \ge \mathrm{mult}_x B \cdot \mathrm{mult}_x C$.*

Proof. It immediately follows from [16, excercise 5·4(a)].

LEMMA 3·5. *Let $a \in [0, 1]$ be a real number and $(X \ni x, \Delta := B + aC)$ a smooth surface germ, where $B \ge 0$ is an $\mathbb{R}$-divisor and C is a prime divisor such that $C \not\subset \mathrm{Supp}\, B$ and C is smooth at x. Assume that $(B \cdot C)_x < 1$, then $\mathrm{mld}(X \ni x, \Delta) > 1 - a$.*

Proof. We only need to show that for any positive integer n and any sequence of smooth blow-ups

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$$X_n \xrightarrow{f_n} X_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} X_0 := X$$

over $X \ni x$ with exceptional divisors $E_k := \mathrm{Exc}(f_k)$ for each k, we have $a(E_k, X, \Delta) > 1 - a$.

In the following, we show that $a(E_k, X, \Delta) > 1 - a$ for each k by applying induction on n. When $n = 1$, by Lemmas 3·3 and 3·4,

$$a(E_1, X, \Delta) = 2 - \mathrm{mult}_x(B + aC) = 2 - a - \mathrm{mult}_x B \geq 2 - a - (B \cdot C)_x > 1 - a.$$

Therefore, we may assume that $n \geq 2$, and when we blow-up at most $n - 1$ times, each divisor we have extracted has log discrepancy $> 1 - a$. In particular, $a(E_k, X, \Delta) > 1 - a$ for any $k \in \{1, 2, \ldots, n - 1\}$.

Let $x_1 := \mathrm{center}_X E_n$, and B_1, C_1, Δ_1 the strict transforms of B,C and Δ on X_1 respectively. Then $x_1 \in E_1$. There are three cases:

Case 1. $a + \mathrm{mult}_x B - 1 < 0$. By Lemma 2·9,

$$(B_1 \cdot C_1)_{x_1} \leq (B \cdot C)_x - B_1 \cdot E_1 = (B \cdot C)_x - \mathrm{mult}_x B < 1 - \mathrm{mult}_x B \leq 1.$$

By induction hypothesis for the germ $(X_1 \ni x_1, \Delta_1)$ and blowing-up at most $n - 1$ times, we have

$$a(E_n, X, \Delta) = a(E_n, X_1, \Delta_1 + (a + \mathrm{mult}_x B - 1)E_1) \geq a(E_n, X_1, \Delta_1) > 1 - a$$

and finish the proof for Case 1.

Case 2. $a + \mathrm{mult}_x B - 1 \geq 0$ and $x_1 \in E_1 \cap C_1$. Let $\tilde{B}_1 := B_1 + (a + \mathrm{mult}_x B - 1)E_1$. Then

$$K_{X_1} + \tilde{B}_1 + aC_1 = f_1^*(K_X + \Delta).$$

We have

$$(\tilde{B}_1 \cdot C_1)_{x_1} = (B_1 \cdot C_1)_{x_1} + (a + \mathrm{mult}_x B - 1)(E_1 \cdot C_1)_{x_1}.$$

Since C is smooth at x, $E_1 \cup C_1$ is snc at x_1, so $(E_1 \cdot C_1)_{x_1} = 1$. By Lemma 2·9,

$$(B_1 \cdot C_1)_{x_1} \leq (B \cdot C)_x - B_1 \cdot E_1 = (B \cdot C)_x - \mathrm{mult}_x B < 1 - \mathrm{mult}_x B.$$

Thus

$$(\tilde{B}_1 \cdot C_1)_{x_1} < 1 - \mathrm{mult}_x B + (a + \mathrm{mult}_x B - 1) = a \leq 1.$$

By induction hypothesis for the germ $(X_1 \ni x_1, \tilde{B}_1 + aC_1)$ and blowing-up at most $n - 1$ times,

$$a(E_n, X, \Delta) = a(E_n, X_1, \tilde{B}_1 + aC_1) > 1 - a,$$

and we finish the proof for Case 2.

Case 3. $a + \mathrm{mult}_x B - 1 \geq 0$, $x_1 \in E_1$, but $x_1 \notin C_1$. By Lemma 2·9,

$$(B_1 \cdot E_1)_{x_1} \leq B_1 \cdot E_1 = \mathrm{mult}_x B \leq (B \cdot C)_x < 1.$$

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By induction hypothesis for the germ $(X_1 \ni x_1, B_1 + (a + \mathrm{mult}_x B - 1)E_1)$ and blowing-up at most $n-1$ times and apply Lemma 3·4,

$$a(E_n, X, \Delta) = a(E_n, X_1, B_1 + (a + \mathrm{mult}_x B - 1)E_1) \geq 1 - (a + \mathrm{mult}_x B - 1) > 1 - a,$$

and we finish the proof for Case 3.

LEMMA 3·6. *Let $X \ni x$ be a smooth germ, $B \geq 0$ an $\mathbb{R}$-divisor on X and C a prime divisor on X such that $C \not\subset \mathrm{Supp}\, B$ and C is smooth at x. Let $g_1{:}X_1 \to X$ be the smooth blow-up of x with exceptional divisor E_1, $C_1 := (g_1^{-1})_* C$, and $B_1 := (g_1^{-1})_* B$. Let $g_2{:}X_2 \to X_1$ be the smooth blow-up at $x_1 := C_1 \cap E_1$ with exceptional divisor E_2 and $B_2 := (g_2^{-1})_* B_1$. Then*

$$(B \cdot C)_x \geq 2(B_2 \cdot E_2).$$

Proof. Let $C_2 := (g_2^{-1})_* C_1$ and $E_1{}' := (g_2^{-1})_* E_1$. By Lemma 2·9,

$$B_1 \cdot E_1 = B_2 \cdot E'_1 + (E'_1 \cdot E_2)(B_2 \cdot E_2) \geq B_2 \cdot E_2$$

and

$$(B_1 \cdot C_1)_{x_1} = \sum_{y \in g_2^{-1}(x_1)} (B_2 \cdot C_2)_y + (B_2 \cdot E_2)(C_2 \cdot E_2) \geq (B_2 \cdot E_2)(C_2 \cdot E_2) = B_2 \cdot E_2.$$

Thus

$$(B \cdot C)_x = \sum_{y \in g_1^{-1}(x)} (B_1 \cdot C_1)_y + B_1 \cdot E_1 \geq (B_1 \cdot C_1)_{x_1} + B_1 \cdot E_1 \geq 2(B_2 \cdot E_2).$$

3·2. Dual graph of f_E

Roughly speaking, this subsection shows that the dual graph of f_E is almost always a chain when E is a divisor which computes the mld. We remark that [**15**, lemma 3·18] shows that there exists such an E such that the dual graph of f_E is a chain, while we show that for any such E, the dual graph of f_E is a chain.

We need the following result of Kawakita. Notice that the "in particular" part of the following theorem is immediate from the construction in [**22**, remark 3].

THEOREM 3·7 ([**22**, theorem 1, remark 3]). *Let $(X \ni x, B)$ be a smooth lc surface germ and E a divisor over $X \ni x$ such that $a(E, X, B) = \mathrm{mld}(X \ni x, B)$. Then there exists a weighted blow-up of $X \ni x$ which extracts E. In particular, E is a Kollár component of $X \ni x$.*

LEMMA 3·8. *Let $(X \ni x, B)$ be a smooth lc surface germ and E a divisor over $X \ni x$ such that $a(E, X, B) = \mathrm{mld}(X \ni x, B)$. Then $\mathcal{D}(f_E)$ is a chain.*

Proof. By Theorem 3·7, there exists a weighted blow-up $f{:}Y \to X$ which extracts E. E contains at most two singular points of Y which are cyclic quotient singularities, and locally analytically, E is one coordinate line of each cyclic quotient singularity. Let $g{:}W \to Y$ be the minimal resolution of Y near E, then $f \circ g = f_E$ and $\mathcal{D}(f_E)$ is a chain.

LEMMA 3·9. *Let $a \in [0, 1]$ be a real number. Assume that:*

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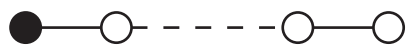
(i) $(X \ni x, \Delta := B + aC)$ *a smooth lc surface germ, where* $B \geq 0$ *is an* $\mathbb{R}$*-divisor and* C *is a prime divisor*;

(ii) $C \not\subset \operatorname{Supp} B$ *and* C *is smooth at* x*; and*

(iii) $(B \cdot C)_x < 2$.

Then any divisor E *over* $X \ni x$ *such that* $a(E, X, \Delta) = \operatorname{mld}(X \ni x, \Delta)$ *satisfies the following. Let* $C_E := (f_E^{-1})_* C$*, then* $C_E \cup \operatorname{Exc}(f_E)$ *is a chain and* C_E *is one tail of* $C_E \cup \operatorname{Exc}(f_E)$*. In particular, the dual graph of* $C_E \cup \operatorname{Exc}(f_E)$ *is the following*:



Here C_E *is denoted by the black circle.*

Proof. By the construction of f_E, if $C_E \cup \operatorname{Exc}(f_E)$ is a chain, then C_E is a tail of $C_E \cup \operatorname{Exc}(f_E)$. Let $n := n(E)$ and let C_i be the strict transform of C on $X_{i,E}$ for each $i \in \{0, 1, \ldots, n\}$. Since C is smooth at x, by the construction of f_E, $C_i \cup_{k=1}^{i} E_i^k$ is simple normal crossing over a neighbourhood of x and its dual graph does not contain a circle for each $i \in \{0, 1, \ldots, n\}$. Since $a(E, X, \Delta) = \operatorname{mld}(X \ni x, \Delta)$, $a(E^i, X, \Delta) \geq a(E, X, \Delta)$ for every $i \in \{1, 2, \ldots, n\}$.

Suppose that the lemma does not hold, then there exists a positive integer $m \in \{1, 2, \ldots, n\}$, such that $C_i \cup_{k=1}^{i} E_i^k$ is a chain, C_i is a tail of $C_i \cup_{k=1}^{i} E_i^k$ for any $i \in \{0, 1, \ldots, m-1\}$, and $C_m \cup_{k=1}^{m} E_m^k$ is not a chain. Since any graph without a circle that is not a chain contains at least 4 vertices, $m \geq 3$.

By Lemma 2·8, $\cup_{k=1}^{n} E_n^k$ is a chain, hence $\cup_{k=1}^{m} E_m^k$ is a chain. Since $C_m \cup_{k=1}^{m} E_m^k$ is not a chain, $x_{m-1} := \operatorname{center}_{X_{m-1,E}} E^m \in E^{m-1} \backslash C_{m-1}$, and for any integer $i \in \{1, 2, \ldots, m-1\}$, $f_{i,E}$ is the smooth blow-up of a point $x_{i-1} \in C_{i-1}$, and if $i \in \{2, 3, \ldots, m-1\}$, then $f_{i,E}$ is the smooth blow-up of $C_{i-1} \cap E^{i-1}$. In particular, since $m \geq 3$, $x_{m-3} \in C_{m-3}$ and $x_{m-2} \in C_{m-2} \cap E^{m-2}$.

Claim 3·10. $(B_{m-1} \cdot E^{m-1})_{x_{m-1}} \geq 1$.

Proof. Let $a_{m-1} := \max\{0, 1 - a(E^{m-1}, X, \Delta)\}$. Since $(X \ni x, \Delta)$ is lc, $a_{m-1} \in [0, 1]$. If $(B_{m-1} \cdot E^{m-1})_{x_{m-1}} < 1$, then by Lemma 3·5,

$$\begin{aligned}
\operatorname{mld}(X \ni x, \Delta) = a(E^n, X, \Delta) &= a(E^n, X^{m-1}, B_{m-1} + (1 - a(E^{m-1}, X, \Delta))E^{m-1}) \\
&\geq a(E^n, X^{m-1}, B_{m-1} + a_{m-1}E^{m-1}) \\
&\geq \operatorname{mld}(X_{m-1} \ni x_{m-1}, B_{m-1} + a_{m-1}E^{m-1}) \\
&> 1 - a_{m-1} = a(E^{m-1}, X, \Delta),
\end{aligned}$$

a contradiction.

Proof of Lemma 2·9 continued. Since $m \geq 3$, by Lemmas 2·9, 3·6 and Claim 3·10,

$$(B \cdot C)_x \geq (B_{m-3} \cdot C_{m-3})_{x_{m-3}} \geq 2(B_{m-1} \cdot E^{m-1}) \geq 2(B_{m-1} \cdot E^{m-1})_{x_{m-1}} \geq 2,$$

which contradicts our assumptions.

LEMMA 3·11. *Let* $l, r \in [0, 1]$ *be two real numbers. Assume that*:

(i) $(X \ni x, \Delta := B + lL + rR)$ *is a smooth lc surface germ, where* $B \geq 0$ *is an* $\mathbb{R}$*-divisor and* L,R *are two different prime divisors*;

(ii) $L \not\subset \operatorname{Supp} B$, $R \not\subset \operatorname{Supp} B$*, and* $(X \ni x, L + R)$ *is log smooth; and*

(iii) *either* $(B \cdot L)_x < 1$ *or* $(B \cdot R)_x < 1$.

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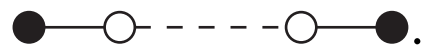
Then any divisor E over $X \ni x$ such that $a(E, X, \Delta) = \mathrm{mld}(X \ni x, \Delta)$ satisfies the following. Let $L_E := (f_E^{-1})_*L$ and $R_E := (f_E^{-1})_*R$, then $L_E \cup R_E \cup \mathrm{Exc}(f_E)$ is a chain and L_E, R_E are the tails of $L_E \cup R_E \cup \mathrm{Exc}(f_E)$. In particular, the dual graph of $L_E \cup R_E \cup \mathrm{Exc}(f_E)$ is the following:



Here L_E and R_E are denoted by the left black circle and the right black circle respectively.

Proof. By the construction of f_E, if $L_E \cup R_E \cup \mathrm{Exc}(f_E)$ is a chain, then L_E, R_E are the tails of $L_E \cup R_E \cup \mathrm{Exc}(f_E)$. Let $n := n(E)$ and let L_i, R_i be the strict transforms of L_i, R_i on $X_{i,E}$ for each $i \in \{0, 1, \ldots, n\}$. Since $(X \ni x, L + R)$ is log smooth and by the construction of f_E, $L_i \cup R_i \cup_{k=1}^{i} E_i^k$ is simple normal crossing over a neighbourhood of x and its dual graph does not contain a circle for each $i \in \{0, 1, \ldots, n\}$. Since $a(E, X, \Delta) = \mathrm{mld}(X \ni x, \Delta)$, $a(E^i, X, \Delta) \geq a(E, X, \Delta)$ for every $i \in \{1, 2, \ldots, n\}$.

Suppose that the lemma does not hold, then there exists a positive integer $m \in \{1, 2, \ldots, n\}$, such that $L_i \cup R_i \cup_{k=1}^{i} E_i^k$ is a chain and L_i, R_i are the tails of $L_i \cup R_i \cup_{k=1}^{i} E_i^k$ for any $i \in \{0, 1, \ldots, m-1\}$, and $L_m \cup R_m \cup_{k=1}^{m} E_m^k$ is not a chain. Since any graph without a circle that is not a chain contains at least 4 points, $m \geq 2$.

By Theorem 3·7, $\cup_{k=1}^{n} E_n^k$ is a chain, hence $\cup_{k=1}^{m} E_m^k$ is a chain. By the construction of f_E and the choice of m, $x_{m-1} := \mathrm{center}_{X_{m-1,E}} E^m \in E^{m-1} \backslash (L_{m-1} \cup R_{m-1})$.

Claim 3·12. $(B_{m-1} \cdot E^{m-1})_{x_{m-1}} \geq 1$.

Proof. Let $a_{m-1} := \max\{0, 1 - a(E^{m-1}, X, \Delta)\}$. Since $(X \ni x, \Delta)$ is lc, $a_{m-1} \in [0, 1]$. If $(B_{m-1} \cdot E^{m-1})_{x_{m-1}} < 1$, then by Lemma 3·5,

$$
\begin{aligned}
\mathrm{mld}(X \ni x, \Delta) = a(E^n, X, \Delta) &= a(E^n, X^{m-1}, B_{m-1} + (1 - a(E^{m-1}, X, \Delta))E^{m-1}) \\
&\geq a(E^n, X^{m-1}, B_{m-1} + a_{m-1}E^{m-1}) \\
&\geq \mathrm{mld}(X_{m-1} \ni x_{m-1}, B_{m-1} + a_{m-1}E^{m-1}) \\
&> 1 - a_{m-1} = a(E^{m-1}, X, \Delta),
\end{aligned}
$$

a contradiction.

Proof of Lemma 3·11 continued. Since $m \geq 2$, by Lemma 2·9 and Claim 3·12,

$$(B \cdot L)_x \geq (B_{m-2} \cdot L_{m-2})_{x_{m-2}} \geq (B_{m-1} \cdot E^{m-1})_{x_{m-1}} \geq 1$$

and

$$(B \cdot R)_x \geq (B_{m-2} \cdot R_{m-2})_{x_{m-2}} \geq (B_{m-1} \cdot E^{m-1})_{x_{m-1}} \geq 1$$

which contradict our assumptions.

4. *Classification of divisors computing mlds*

4·1. A key lemma

The following lemma is similar to [**25**, theorem 4·15] and plays an important role in the proof of our main theorems.

LEMMA 4·1. *Let m be a non-negative integer, $(X \ni x, B)$ a plt surface germ, $f: Y \to X$ the minimal resolution of $X \ni x$, and $B_Y := f_*^{-1}B$. Then*:

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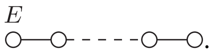
(i) *for any prime divisor* $F \subset \text{Exc}(f)$, $B_Y \cdot F < 2$;

(ii) *there exists at most one prime divisor* $F \subset \text{Exc}(f)$ *such that* $B_Y \cdot F \geq 1$; *and*

(iii) *if* $E \subset \text{Exc}(f)$ *is a prime divisor such that* $B_Y \cdot E \geq 1$, *then* $X \ni x$ *is an A-type singularity and* E *is a tail of* $\mathcal{D}(f)$.



Proof. Let $F_1, \ldots, F_m$ be the prime exceptional divisors of h for some positive integer m, and let $v_1, \ldots, v_m$ be the vertices corresponding to $F_1, \ldots, F_m$ in $\mathcal{D}(f)$ respectively. We construct an extended graph $\bar{\mathcal{D}}(f)$ in the following way:

(i) the vertices of $\bar{\mathcal{D}}(f)$ are $v_0, v_1, \ldots, v_m$;

(ii) for any $i, j \in \{1, 2, \ldots, m\}$, v_i and v_j are connected by a line if and only if v_i and v_j are connected by a line in $\mathcal{D}(f)$;

(iii) for any $i \in \{1, 2, \ldots, m\}$, v_0 and v_i are connected by $\lfloor B_Y \cdot F_i \rfloor$ lines.

Moreover, we may write

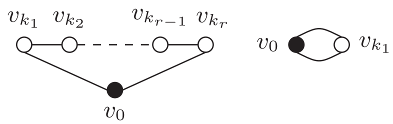
$$K_Y + B_Y - \sum_{i=1}^{m} a_i F_i = f^*(K_X + B),$$

where $a_i := a(F_i, X, B) - 1$. Since $(X \ni x, B)$ is plt and f is the minimal resolution of $X \ni x$, $0 \geq a_i > -1$ for each i. Let $A := \sum_{i=1}^{m} a_i F_i$.

If $\bar{\mathcal{D}}(f)$ is not connected, then $B_Y \cdot F_i < 1$ for each i and there is nothing left to prove. Therefore, we may assume that $\bar{\mathcal{D}}(f)$ is connected.

Claim 4·2. *$\bar{\mathcal{D}}(f)$ does not contain a circle.*

Proof. Suppose that $\bar{\mathcal{D}}(f)$ contains a circle. We let $v_{k_0} := v_0, v_{k_1}, \ldots, v_{k_r}$ be the vertices of this circle for some positive r such that $\bar{\mathcal{D}}(f)$ contains one of the following sub-graphs:



Let $H := -\sum_{i=1}^{r} F_{k_i}$. Then

$$H \cdot F_{k_i} = -2 - F_{k_i}^2 = K_Y \cdot F_{k_i} \leq (K_Y + B_Y) \cdot F_{k_i} = A \cdot F_{k_i}$$

for each $i \in \{2, 3, \ldots, r-1\}$,

$$H \cdot F_{k_i} = -1 - F_{k_i}^2 = K_Y \cdot F_{k_i} + 1 \leq (K_Y + B_Y) \cdot F_{k_i} = A \cdot F_{k_i}$$

for each $i \in \{1, r\}$ when $r \geq 2$,

$$H \cdot F_{k_1} = -F_{k_1}^2 = K_Y \cdot F_{k_i} + 2 \leq (K_Y + B_Y) \cdot F_{k_i} = A \cdot F_{k_i}$$

when $r = 1$, and

$$H \cdot F_i \leq 0 \leq (K_Y + B_Y) \cdot F_i = A \cdot F_i$$

for each $i \notin \{k_1, \ldots, k_r\}$. By Lemma 2·10, $a_{k_i} \leq -1$ for each i, a contradiction.

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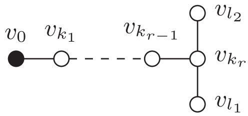
Claim 4·3. *$\bar{\mathcal{D}}(f)$ does not contain a fork.*

Proof. Assume that $\bar{\mathcal{D}}(f)$ contains a fork. By Claim 4·2, $\bar{\mathcal{D}}(f)$ does not contain a circle. Therefore, v_0 is not a fork of $\bar{\mathcal{D}}(f)$. In particular, there exist a positive integer r and vertices $v_{k_0} := v_0, v_{k_1}, \dots, v_{k_r}, v_{l_1}, v_{l_2}$ such that $\bar{\mathcal{D}}(f)$ contains the following sub-graph:



Let $H := -\sum_{i=1}^{r} F_{k_i} - \frac{1}{2}F_{l_1} - \frac{1}{2}F_{l_2}$. Then

$$H \cdot F_{l_i} = -1 - \frac{1}{2}F_{l_i}^2 \le -2 - F_{l_i}^2 = K_Y \cdot F_{l_i} \le (K_Y + B_Y) \cdot F_{l_i} = A \cdot F_{l_i}$$

for each $i \in \{1, 2\}$,

$$H \cdot F_{k_i} = -2 - F_{k_i}^2 = K_Y \cdot F_{k_i} \le (K_Y + B_Y) \cdot F_{k_i} = A \cdot F_{k_i}$$

for each $i \in \{2, 3, \dots, r-1\}$,

$$H \cdot F_{k_i} = -1 - F_{k_i}^2 = K_Y \cdot F_{k_i} + 1 \le (K_Y + B_Y) \cdot F_{k_i} = A \cdot F_{k_i}$$

for $i = 1$, and

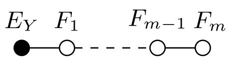
$$H \cdot F_i \le 0 \le (K_Y + B_Y) \cdot F_i = A \cdot F_i$$

for each $i \notin \{k_1, \dots, k_r, l_1, l_2\}$. By Lemma 2·10, $a_{k_r} \le -1$, a contradiction.

Proof of Lemma 4·1 continued. By Claims 4·2 and 4·3, $\bar{\mathcal{D}}(f)$ does not contain a circle or a fork. Therefore, $\bar{\mathcal{D}}(f)$ is a chain. Since $\mathcal{D}(f)$ is connected and $\bar{\mathcal{D}}(f)$ has $\mathcal{D}(f)$ as a sub-graph, $\mathcal{D}(f)$ is a chain and v_0 is a tail of $\bar{\mathcal{D}}(f)$. The lemma immediately follows from the structure of $\bar{\mathcal{D}}(f)$ and $\mathcal{D}(f)$.

4·2. A-type singularities

LEMMA 4·4. *Let $X \ni x$ be a surface germ of A-type, E a prime divisor on X, $f : Y \to X$ the minimal resolution of $X \ni x$, and $E_Y := f_*^{-1}E$. Let $F_1, \dots, F_m$ be the prime exceptional divisors over $X \ni x$. Assume that $E_Y \cup \cup_{i=1}^m F_i$ is simple normal crossing over a neighbourhood of x and the dual graph of $E_Y \cup \cup_{i=1}^m F_i$ is the following:*



Then $(X \ni x, E)$ is plt.

Proof. We may write

$$K_Y + E_Y - \sum_{i=1}^{m} a_i F_i = f^*(K_X + E),$$

where $a_i := a(F_i, X, E) - 1$. Let $H := -\sum_{i=1}^{m} F_i$ and $A := \sum_{i=1}^{m} a_i F_i$. Then

$$H \cdot F_1 = -F_1^2 = 2 + K_Y \cdot F_1 = 1 + (K_Y + E_Y) \cdot F_1 = 1 + A \cdot F_1 > A \cdot F_1$$

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if $m=1$,

$$H\cdot F_1=-1-F_1^2=1+K_Y\cdot F_1=(K_Y+E_Y)\cdot F_1=A\cdot F_1$$

if $m\geq 2$,

$$H\cdot F_i=-2-F_i^2=K_Y\cdot F_i=(K_Y+E_Y)\cdot F_i=A\cdot F_i$$

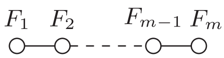
if $m\geq 2$ and $2\leq i\leq m-1$, and

$$H\cdot F_m=-1-F_m^2=1+K_Y\cdot F_m=1+(K_Y+E_Y)\cdot F_m=1+A\cdot F_i>A\cdot F_m$$

if $m\geq 2$. By Lemma 2·10, $a_i>-1$ for each i, hence $(X\ni x,E)$ is plt.

THEOREM 4·5. *Let $(X\ni x,B)$ be a plt surface germ such that $X\ni x$ is an A-type singularity. Then for any divisor E over $X\ni x$ such that $a(E,X,B)=\mathrm{mld}(X\ni x,B)$, E is a Kollár component of $X\ni x$.*

Proof. For any model X' of X such that $\mathrm{center}_{X'}E$ is a divisor, we let $E_{X'}$ be the center of E on X'. Let $h:Y\to X$ be the minimal resolution of X and $F_1,\ldots,F_m$ the prime exceptional divisors of h with the following dual graph:



Let $B_Y:=f_*^{-1}B$ and $a_i:=1-a(F_i,X,B)$ for each i. Then $a_i\in[0,1)$ for each i. There are three cases:

Case 1. E is on Y. In this case, we let $W:=Y$ and $g:=h$. Then $\mathcal{D}(g)$ is a chain. Moreover, by the construction of g, for any prime divisor $F\neq E_W$ in $\mathrm{Exc}(g)$, $F^2\leq -2$.

Case 2. E is not on Y, $\mathrm{center}_Y E:=y\in F_i$ for some i, and $\mathrm{center}_Y E\notin F_j$ for any $j\neq i$. In this case, since $a(E,X,B)=\mathrm{mld}(X\ni x,B)$, $a(E,X,B)\leq 1-a_i$. By Lemma 3·5, $B_Y\cdot F_i\geq 1$. By Lemma 4·1(iii), $i=1$ or m. We let $f_E:W\to Y$ the sequence of smooth blow-ups as in Definition-Lemma 3·1, and $g:=h\circ f_E$. By Lemma 4·1(i), $B_Y\cdot F_i<2$, hence $(B_Y\cdot F_i)_y<2$. Since

$$K_Y+B_Y+\sum a_iF_i=h^*(K_X+B),$$

by Lemma 3·9, $\mathcal{D}(g)$ is a chain. Moreover, by the construction of g, for any prime divisor $F\neq E_W$ in $\mathrm{Exc}(g)$, $F^2\leq -2$.

Case 3. E is not on Y and $\mathrm{center}_Y E:=y\in F_i\cap F_{i+1}$ for some i. In this case, we let $f_E:W\to Y$ the sequence of smooth blow-ups as in Definition-Lemma 3·1, and $g:=h\circ f_E$. By Lemma 4·1(ii), either $B_Y\cdot F_i<1$ or $B_Y\cdot F_{i+1}<1$, hence either $(B_Y\cdot F_i)_y<1$ or $(B_Y\cdot F_{i+1})_y<1$. Since

$$K_Y+B_Y+\sum a_iF_i=h^*(K_X+B),$$

by Lemma 3·11, $\mathcal{D}(g)$ is a chain. Moreover, by the construction of g, for any prime divisor $F\neq E_W$ in $\mathrm{Exc}(g)$, $F^2\leq -2$.

By [**25**, remark 4·9(2)], there exists a contraction $\phi:W\to Z$ over X of $\mathrm{Supp}\,\mathrm{Exc}(g)\backslash E_W$. Since $\mathcal{D}(g)$ is a chain in all three cases, $\mathrm{Exc}(g)$ is snc. By Lemma 4·4, $(Z\ni z,E_Z)$ is plt for

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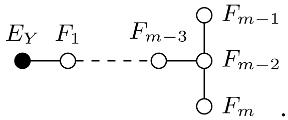
any singular point z of Z in E_Z. Thus (Z, E_Z) is plt near E_Z, hence E is a Kollár component of $X \ni x$.

4·3. D-type and E-type singularities

LEMMA 4·6. *Let $X \ni x$ be a surface germ of D_m-type for some integer $m \geq 3$, E a prime divisor on X, $f: Y \to X$ the minimal resolution of $X \ni x$, and $E_Y := f_*^{-1}E$. Let $F_1, \dots, F_m$ be the prime exceptional divisors over $X \ni x$. Assume that $E_Y \cup \cup_{i=1}^m F_i$ is simple normal crossing over a neighbourhood of x, $F_{m-1}^2 = F_m^2 = -2$, and the dual graph of $E_Y \cup \cup_{i=1}^m F_i$ is the following:*



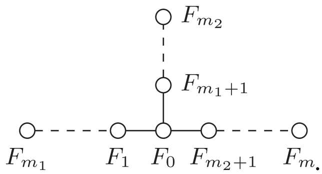
Then $(X \ni x, E)$ is lc but not plt.

Proof. By computing intersections numbers,

$$K_Y + E_Y + \sum_{i=1}^{m-2} F_i + \frac{1}{2}(F_{m-1} + F_m) = f^*(K_X + E).$$

Since $(Y, E_Y + \sum_{i=1}^{m-2} F_i + \frac{1}{2}(F_{m-1} + F_m))$ is log smooth over a neighbourhood of x, $(X \ni x, E)$ is lc but not plt.

LEMMA 4·7. *Let $0 < m_1 < m_2 < m_3 := m$ be integers, $(X \ni x, B)$ a plt surface germ, $f: Y \to X$ the minimal resolution of $X \ni x$ with prime exceptional divisors $F_0, \dots, F_m$ with the following dual graph $\mathcal{D}(f)$:*



Let $a_i := a(F_i, X, B) - 1$ for each i. Then:

(i) *$a(F_0, X, B) \leq a(F_i, X, B)$ for any $i \in \{1, 2, \dots, m_1\}$; and*

(ii) *if $a(F_0, X, B) = a(F_l, X, B)$ for some $l \in \{1, 2, \dots, m_1\}$, then*

(a) *$a_i = a_0$ for every $i \in \{0, 1, \dots, l\}$,*
(b) *$a_{m_1+1} = a_{m_2+1} = \frac{1}{2}a_0$,*
(c) *$F_i^2 = -2$ for any $i \in \{0, 1, \dots, l-1, m_1+1, m_2+1\}$, and*
(d) *either $m = m_2 + 1 = m_1 + 2$, or $B = 0$ and $X \ni x$ Du Val.*

Proof. Let $B_Y := f_*^{-1}B$. Then

$$K_Y + B_Y - \sum_{i=1}^{m} a_i F_i = f^*(K_X + B).$$

Since $(X \ni x, B)$ is plt and f is the minimal resolution of $X \ni x$, $-1 < a_i \leq 0$ for each i.

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If $a_0 < a_i$ for any $i \in \{1, 2, \ldots, m_1\}$ there is nothing to prove. Otherwise, there exists $k \in \{1, 2, \ldots, m_1\}$, such that $a_k = \min\{a_i \mid 0 \le i \le m_1\} \le a_0$. We define

$$A := \sum_{i=0}^{k} a_i F_i + a_{m_1+1} F_{m_1+1} + a_{m_2+1} F_{m_2+1}, \text{ and } H := a_k \left(\sum_{i=0}^{k} F_i + \frac{1}{2} F_{m_1+1} + \frac{1}{2} F_{m_2+1} \right).$$

Then

$$H \cdot F_i = a_k(F_i^2 + 2) = -a_k K_Y \cdot F_i \le K_Y \cdot F_i \le (K_Y + B_Y) \cdot F_i = A \cdot F_i,$$

when $0 \le i < k$ and if the equality holds then $K_Y \cdot F_i = 0$,

$$H \cdot F_k = a_k(F_k^2 + 1) \le a_k F_k^2 + a_{k-1} = A \cdot F_k$$

and if the equality holds then $a_k = a_{k-1}$, and

$$\begin{aligned} H \cdot F_{m_i+1} = a_k(1 + \frac{1}{2} F_{m_i+1}^2) = -\frac{a_k}{2} K_Y \cdot F_{m_i+1} \le K_Y \cdot F_{m_i+1} \\ \le (K_Y + B_Y) \cdot F_{m_i+1} \le A \cdot F_{m_i+1} \end{aligned}$$

for $i \in \{1, 2\}$, and if the equality holds, then:

(i) $K_Y \cdot F_{m_i+1} = 0$, and

(ii) either $m_{i+1} = m_i + 1$, or $m_{i+1} \ge m_i + 2$ and $a_{m_i+2} = 0$.

Thus $H \cdot F_i \le A \cdot F_i$ for any $i \in \{0, 1, \ldots, k, m_1 + 1, m_2 + 1\}$. By Lemma 2·10, $a_i \le a_k$ for any $i \in \{0, 1, \ldots, k\}$ and $a_{m_1+1} = a_{m_2+1} = 1/2a_k$. Since $a_k = \min\{a_i \mid 0 \le i \le m_1\} \le a_0$, $a_i = a_k = \min\{a_i \mid 0 \le i \le m_1\}$ for any $i \in \{0, 1, \ldots, k\}$. Thus for any $l \in \{1, 2, \ldots, m_1\}$ such that $a(F_0, X, B) = a(F_l, X, B)$, we may pick $k = l$, which shows (i) and (ii)(a). Moreover, we have that $H \cdot F_i = A \cdot F_i$ for any $i \in \{0, 1, \ldots, l, m_1 + 1, m_2 + 1\}$, which implies that:

(i) $a_{m_1+1} = a_{m_2+1} = 1/2a_l$, hence (ii)(b);

(ii) $F_i^2 = K_Y \cdot F_i = -2$ for every $i \in \{0, 1, \ldots, l-1, m_1 + 1, m_2 + 1\}$, hence (ii)(c); and

(iii) either $m = m_2 + 1 = m_1 + 2$, or there exists $i \in \{1, 2\}$ such that $m_{i+1} \ge m_i + 2$ and $a_{m_i+2} = 0$.

If $m = m_2 + 1 = m_1 + 2$ then we get (ii)(d) and the proof is completed. Otherwise, there exists $i \in \{1, 2\}$ such that $m_{i+1} \ge m_i + 2$ and $a_{m_i+2} = 0$. Thus

$$1 = a(F_{m_i+2}, X, B) \le a(F_{m_i+2}, X, 0) \le 1,$$

which implies that $a(F_i, X, B) = a(F_i, X, 0)$, hence $B = 0$. Moreover, since f is the minimal resolution of $X \ni x$, $\sum_{i=1}^{m} a_i F_i \sim_X K_Y$ is nef over X. By Lemma 2·10, $a_i = 0$ for every i, and $X \ni x$ is Du Val.

LEMMA 4·8. *Let $(X \ni x, B)$ be a plt surface germ such that $X \ni x$ is a D_m-type singularity for some integer $m \ge 4$, or an E_m-type singularity for some integer $m \in \{6, 7, 8\}$. Let $f: Y \to X$ be the minimal resolution of $X \ni x$, and E a divisor over $X \ni x$ such that $a(E, X, B) = \mathrm{mld}(X \ni x, B)$. Then $E \subset \mathrm{Exc}(f)$.*

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Proof. Let $F_1,\dots,F_m$ be the prime exceptional divisors of f, $a_i := 1 - a(F_i, X, B)$ for each i, and $B_Y := f_*^{-1}B$. Then

$$K_Y + B_Y + \sum_{j=1}^{m} a_j F_j = f^*(K_X + B).$$

Suppose that $E \not\subset \mathrm{Exc}(f)$. Let $y := \mathrm{center}_Y E$.

Claim 4·9. *$y = F_i \cap F_k$ for some $i \neq k$.*
Proof. Then there exists an integer $i \in \{1, 2, \dots, m\}$ such that $y \in F_i$. Thus

$$\mathrm{mld}(Y \ni y, B_Y + \sum_{j=1}^{m} a_j F_j) \leq a\left(E, Y, B_Y + \sum_{j=1}^{m} a_j F_j\right)$$
$$= a(E, X, B) = \mathrm{mld}(X \ni x, B) \leq 1 - a_i.$$

By Lemma 35,

$$(B + \sum_{j\neq i} a_j F_j) \cdot F_i \geq \left(\left(B + \sum_{j\neq i} a_j F_j\right) \cdot F_i\right)_y \geq 1.$$

By Lemma 4·1(iii), $B \cdot F_i < 1$, which implies that there exists $k \neq i$ such that $y \in F_k$. In particular, $y = F_i \cap F_k$.

Proof of Lemma 4·8 *continued*. By Claim 4·9, $y = F_i \cap F_k$ for some $i \neq k$. Possibly switching i and k, we may assume that F_i is closer to the fork of $\mathcal{D}(f)$ than F_k. Since $(X \ni x, B)$ is plt and f is the minimal resolution of $X \ni x$, $0 \leq a_k < 1$. Thus there exists an extraction $g{:}W \to X$ of F_k with induced morphism $h{:}Y \to W$. Let $\bar{F}_k$ be the center of F_k on W and $w := \mathrm{center}_W E$. Then h is the minimal resolution of $W \ni w$. Moreover, if F_i is not the fork of $\mathcal{D}(f)$, then $W \ni w$ is not an A-type singularity, and if F_i is the fork of $\mathcal{D}(f)$, then $W \ni w$ is an A-type singularity but F_i is not a tail of $\mathcal{D}(h)$. Since $(X \ni x, B)$ is plt, $(W \ni w, g_*^{-1}B + a_k F_k)$ is plt. By Lemma 4·1(iii),

$$\left(\left(B + \sum_{j\neq i} a_j F_j\right) \cdot F_i\right)_y = ((B + a_k F_k) \cdot F_i)_y \leq (B + a_k F_k) \cdot F_i < 1.$$

By Lemma 3·5,

$$\mathrm{mld}(X \ni x, B) = a(E, X, B) = a(E, Y, B_Y + \sum_{j=1}^{m} a_j F_j)$$
$$\geq \mathrm{mld}(Y \ni y, B_Y + \sum_{j=1}^{m} a_j F_j) > 1 - a_i \geq \mathrm{mld}(X \ni x, B),$$

a contradiction.

THEOREM 4·10. *Let $(X \ni x, B)$ be a plt surface germ such that $X \ni x$ is a D_m-type singularity for some integer $m \geq 4$ or an E_m-type singularity for some integer $m \in \{6, 7, 8\}$. Let $f{:}Y \to X$ be the minimal resolution of $X \ni x$. Then there exists a unique divisor E over $X \ni x$,*

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such that $a(E,X,B)=\mathrm{mld}(X\ni x,B)$*, and* E *is a Kollár component of* $X\ni x$*. Moreover,* E *is the unique fork of* $\mathcal{D}(f)$*.*

Proof. By Lemma 4·8, we may only consider divisors in $\mathrm{Exc}(f)$. Let $F_1,\ldots,F_m$ be the prime exceptional divisors of f such that F_1 is the unique fork. Since $a(F_i,X,B)\le a(F_i,X,0)\le 1$ for each i, there exists an extraction $g{:}Y_i\to X$ of F_i for each i, and we let $\bar{F}_i$ be the strict transform of F_i on Y_i. By Lemmas 4·4 and 4·6, $(Y_i,\bar{F}_i)$ is not plt near $\bar{F}_i$ for any $i\in\{2,3,\ldots,m\}$ and $(Y_1,\bar{F}_1)$ is plt near F_1. By Lemma 4·7, $a(F_1,X,B)=\mathrm{mld}(X\ni x,B)$. So $E=F_1$ is the unique divisor we want.

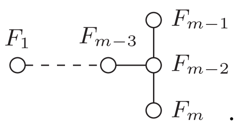
THEOREM 4·11. *Let* $(X\ni x,B)$ *be a plt surface germ such that* $X\ni x$ *is a* D_m*-type singularity for some integer* $m\ge 4$ *or an* E_m*-type singularity for some integer* $m\in\{6,7,8\}$*. Assume that either* $B\ne 0$ *or* $X\ni x$ *is not Du Val. Let* E *be a divisor over* $X\ni x$ *such that* $a(E,X,B)=\mathrm{mld}(X\ni x,B)$*, then* E *is a potential lc place of* $X\ni x$*.*

Proof. Let $f{:}Y\to X$ be the minimal resolution of $X\ni x$. By Lemma 4·8, $E\subset\mathrm{Exc}(f)$. By Theorem 4·10, we may assume that E is not the fork of $\mathcal{D}(f)$. Let L_1,L_2,L_3 be the three branches of $\mathcal{D}(f)$ and assume that E belongs to L_1. By Lemma 4·7(ii)(c),(ii)(d), $X\ni x$ is a D_m-type singularity, L_2 contains a unique curve F_2 and L_3 contains a unique curve F_3 respectively, such that $F_2^2=F_3^2=-2$. Since $(X\ni x,B)$ is plt and f is the minimal resolution of $X\ni x$, $0<a(E,X,B)\le 1$, so there exists an extraction $g{:}W\to X$ of E with the induced morphism $h{:}Y\to W$. Let E_W be the strict transform of E on W. By Lemmas 4·4 and 4·6, (W,E_W) is lc near E_W, so E is a potential lc place of $X\ni x$.

THEOREM 4·12. *Let* $X\ni x$ *be a Du Val singularity of* D_m*-type for some integer* $m\ge 4$ *or of* E_m*-type for some integer* $m\in\{6,7,8\}$*. Let* $f{:}Y\to X$ *be the minimal resolution of* $X\ni x$*, and* E *be a divisor over* $X\ni x$ *such that* $a(E,X,0)=\mathrm{mld}(X\ni x,0)$*, then* E *is a potential lc place of* $X\ni x$ *if and only if one of the following holds*:

(i) *E is the fork of the* $\mathcal{D}(f)$;

(ii) $X\ni x$ *is of* D_m*-type, the dual graph of* $X\ni x$ *looks like the following, and* $E\in\{F_1,\ldots,F_{m-2}\}$.



Proof. Since $a(E,X,0)=\mathrm{mld}(X\ni x,0)$, $E\subset\mathrm{Exc}(f)$. Since $X\ni x$ is Du Val, for any prime divisor $F\subset\mathrm{Exc}(f)$, $F^2=-2$. The if part follows from Lemmas 4·4 and 4·6.

To prove the only if part, we may assume that E is not the fork of $\mathcal{D}(f)$. Then E is a potential lc place of $X\ni x$ if and only if there exists an extraction $g{:}W\to X$ of E such that (W,E_W) is lc near E_W, where E_W is the strict transform of E on W. The theorem follows from [**25**, theorem 4·15].

4·4. Non-plt singularities

Definition-LEMMA 4·13 (25, theorem 4·7]). Let $X\ni x$ be an lc but not klt surface germ and $f{:}Y\to X$ the minimal resolution of $X\ni x$. Then exactly one of the following holds:

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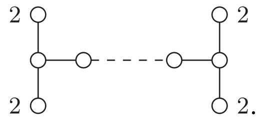
(i) (**B**-type) $\mathrm{Exc}(f) = F$ is a smooth elliptic curve.

(ii) (**C**-type) $\mathrm{Exc}(f) = F$ is a nodal cubic curve;

(iii) (**F**-type) $\mathrm{Exc}(f)$ is a circle of smooth rational curves;

(iv) (**H**-type) $\mathrm{Exc}(f)$ has ≥ 5 rational curves and $\mathcal{D}(f)$ has the following weighted dual graph:



Let $m \geq 5$ be an integer. For an lc singularity of **H**-*type as above with m exceptional divisors, we call it an* $\mathbf{H}_m$-*type singularity.*

Although the concept of Kollár component is not defined over an lc but not klt germ, the classification of surface lc singularities tells us when there exists a divisor which "looks like a Kollár component".

THEOREM 4·14. *Let $(X \ni x, B)$ be an lc surface germ such that $X \ni x$ is not klt, and E a prime divisor over X such that $a(E, X, B) = \mathrm{mld}(X \ni x, B)$. Then*:

(i) *E is a potential lc place of $(X \ni x, B)$. In particular, there exists a divisorial contraction $f : Y \to X$ which extracts E*;

(ii) *$K_Y + E$ is plt near E if and only if E is a* **B**-*type or an* $\mathbf{H}_5$-*type singularity.*

Proof. E is an lc place of $(X \ni x, B)$ which implies (i). Since $(X \ni x, B)$ is lc and $X \ni x$ is not klt, $B = 0$. By the connectedness of lc places, $K_Y + E$ is plt near E if and only if E is the only lc place over $X \ni x$, and (ii) follows from Definition-Lemma 4·13.

Now we deal with the case when $X \ni x$ is klt but $(X \ni x, B)$ is not plt:

THEOREM 4·15. *Let $(X \ni x, B)$ be an lc surface germ such that $X \ni x$ is klt but $(X \ni x, B)$ is not plt. Then any divisor E over $X \ni x$ such that $a(E, X, B) = \mathrm{mld}(X \ni x, B)$ is a potential lc place of $(X \ni x, B)$.*

Proof. Since $(X \ni x, B)$ is not plt, $a(E, X, B) = \mathrm{mld}(X \ni x, B) = 0$, so E is an lc place of $(X \ni x, B)$, hence a potential lc place of $(X \ni x, B)$.

THEOREM 4·16. *Let $(X \ni x, B)$ be an lc surface germ such that $X \ni x$ is klt but $(X \ni x, B)$ is not plt. Then there exists a divisor E over $X \ni x$ such that $a(E, X, B) = \mathrm{mld}(X \ni x, B) = 0$ and E is Kollár component of $X \ni x$.*

Proof. Let

$$\delta := \min\{a(E, X, B) \mid E \text{ is over } X \ni x, a(E, X, B) > 0\}.$$

Then there exists $\epsilon \in (0, 1)$ such that $\delta > \mathrm{mld}(X \ni x, (1 - \epsilon)B) > 0$. By Theorems 4·5 and 4·10, there exists a divisor E over $X \ni x$ such that $a(E, X, (1 - \epsilon)B) = \mathrm{mld}(X \ni x, (1 - \epsilon)B) < \delta$ and E is a Kollár component of $X \ni x$. Since

$$0 \leq a(E, X, B) < a(E, X, (1 - \epsilon)B) < \delta,$$

$a(E, X, B) = \mathrm{mld}(X \ni x, B) = 0$, so E satisfies our requirements.

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Finally, recall the following result:

THEOREM 4·17. *Let $(X \ni x, B)$ be a dlt surface germ that is not plt. Then $(X \ni x, B)$ is log smooth and $B = \lfloor B \rfloor$ has exactly two irreducible components near x.*

Proof. Since $(X \ni x, B)$ is dlt but not plt, $\lfloor B \rfloor$ contains at least 2 irreducible components near x. Since X is a surface, $B = \lfloor B \rfloor$ has exactly two irreducible components near x.

There exists a divisor E over $X \ni x$ such that $a(E, X, B) = 0$. Since $(X \ni x, B)$ is dlt, $x := \text{center}_X E$ belongs to the log smooth strata of (X,B), so $(X \ni x, B)$ is log smooth.

5. *Proof of the main theorems*

Proof of Theorem 1·5. (i)(a) follows from Theorems 3·7, 4·5 and 4·17. For (i)(b), by Theorem 4·17, $(X \ni x, B)$ is plt. (i)(b)(1) follows from Theorem 4·10. (i)(b)(2) follows from Lemma 4·8. (i)(b)(3)(A) follows from Theorem 4·11. (i)(b)(3)(B) follows from Lemma 4·7 and (i)(b)(2). (i)(b)(4)(A) is the classification of surface Du Val singularities. (i)(b)(4)(B) and (i)(b)(4)(C) follow from Theorem 4·12. (ii)(a) follows from Theorem 4·15. (ii)(b) follows from Theorem 4·16. (ii)(c) follows from Theorem 3·7. (iii) follows from Theorem 4·14.

Proof of Theorem 1·2. It follows from Theorem 1·5(i)(a) (i)(b)(1), (ii)(b).

Proof of Theorem 1·4. It follows from Theorem 1·5(i)(a), (i)(b)(3)(A), (i)(b)(4)(B), (i)(b)(4)(C), (ii)(a), (iii).

6. *Examples*

The following example is given by Zhuang which shows that Question 1·1 does not have a general positive answer in dimension ≥ 3 even when $B = 0$. We are grateful for him sharing the example with us.

EXAMPLE 6·1 *(c.f. [24, excercise 41]). Consider the threefold singularity given by*

$$(x^3 + y^3 + z^3 + w^4 = 0) \subset (\mathbb{C}^4 \ni 0).$$

This is a canonical singularity, and the only divisor E which computes the mld is attained at the ordinary blow-up. However, E is a cone over an elliptic curve, so E is lc but not klt. In particular, E is not a Kollár component of the ambient variety.

The following example of Kawakita shows that there may not exist a divisor computing $\text{mld}(X \ni x, B)$ that is also a potential place of $(X \ni x, B)$ even when X is a smooth surface. We remark that Theorem 1·2 shows that there always exists a divisor computing $\text{mld}(X \ni x, B)$ that is a Kollár component of $(X \ni x, 0)$.

EXAMPLE 6·2 *(cf. [22, example 2]). Let $D := (x_1^2 + x_2^3 + rx_1x_2^2 = 0) \subset \mathbb{A}^2$ for some general real number r, and $B := 2/3D$. Then there exists a unique divisor E over $\mathbb{A}^2 \ni 0$ such that* $\text{mld}(\mathbb{A}^2 \ni 0, B) = a(E, X, B) = 2/3$. *However, E is not a potential place of $(\mathbb{A}^2 \ni 0, B)$.*

The following example is complementary to Theorem 4·5, which shows that the assumption "$(X \ni x, B)$ is plt" is necessary.

EXAMPLE 6·3. *Let $Z \ni z$ be a D_4-type Du Val singularity with the following dual graph:*

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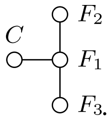


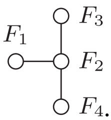
Let $g:X \to Z$ be the extraction of C. Then X has a unique singularity $x \in C$ such that $X \ni x$ is an A_3-type Du Val singularity. Let $f:Y \to X$ be the minimal resolution of $(X \ni x, C)$ and $C_Y := f_*^{-1}C$. Then

$$K_Y + C_Y + F_1 + \frac{1}{2}F_2 + \frac{1}{2}F_3 = f^*(K_X + C).$$

In particular, let $h:W \to Y$ be the smooth blow-up of $C_Y \cap F_1$ with the exceptional divisor E. Then $a(E, X, C) = \mathrm{mld}(X \ni x, C) = 0$, but by [25, Theorem 4·15(2)], E is not a Kollár component of $X \ni x$.

The last example is complementary to Theorem 4·10 and 4·12, which shows that even for non-Du Val singularities of D-type and $B = 0$, it is possible that some divisor which computes the minimal log discrepancy is not a Kollár component.

EXAMPLE 6·4. *Let $G := BD_{12}(5, 3)$ be a binary dihedral group in $\mathrm{GL}(2, \mathbb{C})$. By [10, table 3·2], the quotient singularity $X \ni x \cong \mathbb{C}^2/G \ni 0$ is a D_4-type singularity and its minimal resolution $f:Y \to X$ has the following dual graph:*



where $F_1^2 = -3$ and $F_i^2 = -2$ for $i \in \{2, 3, 4\}$. Thus $a(F_1, X, 0) = 1/2 = \mathrm{mld}(X \ni x, 0)$, but by Theorem 4·10, F_1 is not a Kollár component of $X \ni x$.

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REFERENCES

[1] V. ALEXEEV. Two two–dimensional terminations. *Duke Math. J.* **69** (1993), no. 3, 527–545.
[2] F. AMBRO. On minimal log discrepancies. *Math. Res. Lett.* 6 (1999), no. 5–6, 573–580.
[3] F. AMBRO. The set of toric minimal log discrepancies. *Cent. Eur. J. Math.* 4 (2006), no. 3, 358–370.
[4] C. BIRKAR, P. CASCINI, C. D. Hacon J. McKernan. Existence of minimal models for varieties of log general type. *J. Amer. Math. Soc.* **23** (2010), no. 2, 405–468.
[5] H. BLUM. On divisors computing MLD's and LCT's. *Bull. Korean Math. Soc.* **58** (2021), no. 1, 113–132.
[6] A. A. BORISOV. Minimal discrepancies of toric singularities. *Manuscripta Math.* **92** (1997), no. 1, 33–45.

https://doi.org/10.1017/S0305004123000051 Published online by Cambridge University Press

[7] B. CHEN. Upper bound of discrepancies of divisors computing minimal log discrepancies on surfaces. arXiv:2009.03613v1.
[8] G. CHEN J. HAN. Boundedness of inline923-complements for surfaces. *Adv. Math.* **383** (2021), 107703, 40pp.
[9] W. CHEN, G. DI CERBO, J. HAN, C. JIANG, R. SVALDI. Birational boundedness of rationally connected Calabi–Yau 3-folds. *Adv. Math.* **378** (2021), 107541, 32pp.
[10] A. NOLLA DE CELIS. Dihedral groups and G-Hilbert schemes. Ph.D. thesis. University of Warwick (2008).
[11] L. Ein M. Mustaţă. Inversion of adjunction for local complete intersection varieties. *Amer. J. Math.* **126** (2004), no. 6, 1355–1365.
[12] L. EIN, M. MUSTAŢĂ, T. YASUDA. Jet schemes, log discrepancies and inversion of adjunction. *Invent. Math.* **153** (2003), no. 3, 519–535.
[13] J. HAN, J. LIU, V. V. SHOKUROV. ACC for minimal log discrepancies for exceptional singularities. arXiv:1903.04338v2.
[14] J. HAN, Y. LIU, L. QI. ACC for local volumes and boundedness of singularities. arXiv:2011.06509v2.
[15] J. Han Y. Luo. On boundedness of divisors computing minimal log discrepancies for surfaces. arXiv:2005.09626v2.
[16] R. HARTSHORNE. *Algebraic Geometry*. Graduate Texts in Math. no. 52 (Springer-Verlag, New York-Heidelberg, 1977).
[17] S. ISHII. The minimal log discrepancies on a smooth surface in positive characteristic. *Math. Z.* **297** (2021) 389–397 (2021).
[18] C. JIANG. A gap theorem for minimal log discrepancies of non-canonical singularities in dimension three. *J. Algebraic Geom.* **30** (2021), 759–800.
[19] M. KAWAKITA. Towards boundedness of minimal log discrepancies by the Riemann–Roch theorem. *Amer. J. Math.* **133** (2011), no. 5, 1299–1311.
[20] M. KAWAKITA. Discreteness of log discrepancies over log canonical triples on a fixed pair. *J. Algebraic Geom.* **23** (2014), no. 4, 765–774.
[21] M. KAWAKITA. A connectedness theorem over the spectrum of a formal power series ring. *Internat. J. Math.* **26** (2015), no. 11, 1550088, 27pp.
[22] M. KAWAKITA. Divisors computing the minimal log discrepancy on a smooth surface. *Math. Proc. Camb. Phil. Soc.* **163** (2017), no. 1, 187–192.
[23] M. KAWAKITA. On equivalent conjectures for minimal log discrepancies on smooth threefolds. *J. Algebraic Geom.* **30** (2021), 97–149.
[24] J. KOLLÁR. Exercises in the birational geometry of algebraic varieties. arXiv:0809.02579v2.
[25] J. Kollár S. Mori. *Birational Geometry of Algebraic Varieties*. Cambridge Tracts in Math. no. 134 (Cambridge University Press, 1998).
[26] S. A. KUDRYAVTSEV. Pure log terminal blow-ups. *Math. Notes* **69** (2001), no. 5, 814–819.
[27] C. LI. Minimizing normalized volumes of valuations. *Math. Z.* **289** (2018), no. 1–2, 491–513.
[28] C. Li C. Xu. Stability of valuations and Kollár components. *J. Eur. Math. Soc.* **22** (2020), no. 8, 2573–2627.
[29] J. LIU. Toward the equivalence of the ACC for a-log canonical thresholds and the ACC for minimal log discrepancies. arXiv:1809.04839v3.
[30] J. LIU and L. XIAO. An optimal gap of minimal log discrepancies of threefold non-canonical singularities. *J. Pure Appl. Algebra* 225 (2021), no. 9, 106674, 23 pp.
[31] M. Mustaţă Y. Nakamura. A boundedness conjecture for minimal log discrepancies on a fixed germ. *Local and global methods in algebraic geometry. Contemp. Math*. no. 712 (2018), 287–306.
[32] Y. NAKAMURA. On semi-continuity problems for minimal log discrepancies. *J. Reine Angew. Math.* **711** (2016), 167–187.
[33] Y. NAKAMURA. On minimal log discrepancies on varieties with fixed Gorenstein index. *Michigan Math. J.* **65** (2016), no. 1, 165–187.
[34] Y. Nakamura K. Shibata. Inversion of adjunction for quotient singularities. arXiv:2011.07300v2.
[35] Y. G. PROKHOROV. Blow-ups of canonical singularities. *Algebra (Moscow, 1998)* (2000), 301–318.
[36] K. SHIBATA. Minimal log discrepancies in positive characteristic. *Comm. Algebra* **50** (2022), no. 2, 571–582.
[37] V. V. Shokurov. A.c.c. in codimension 2. (1994), preprint.

[38] V. V. SHOKUROV. 3–fold log models. *J. Math. Sciences* **81** (1996), 2677–2699.
[39] V. V. SHOKUROV. Complements on surfaces. *J. Math. Sci. (New York)* **102** (2000), no. 2, 3876–3932.
[40] V. V. SHOKUROV. Letters of a bi-rationalist, V. Minimal log discrepancies and termination of log flips (Russian). *Tr. Mat. Inst. Steklova* 246 (2004), Algebr. Geom. Metody, Svyazi i Prilozh., 328–351; translation in *Proc. Steklov Inst. Math.* **246** (2004), no. 3, 315–336.
[41] C. XU. Finiteness of algebraic fundamental groups. *Compositio Math.* **150** (2014), no. 3, 409–414.