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Nori Motives of Curves With Modulus and Laumon 1-motives

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Abstract. Let k be a number field. We describe the category of Laumon 1-isomotives over k as the universal category in the sense of M. Nori associated with a quiver representation built out of smooth proper k-curves with two disjoint effective divisors and a notion of H^1_{dR} for such "curves with modulus". This result extends and relies on a theorem of J. Ayoub and L. Barbieri-Viale that describes Deligne's category of 1-isomotives in terms of Nori's Abelian category of motives.

1 Introduction

Let k be a field of characteristic zero with an embedding $k \hookrightarrow \mathbb{C}$ into the field of complex numbers.

1.1 Let R be a field or a Dedekind ring and $T: \mathscr{D} \to \mathrm{mod}(R)$ a representation of a quiver $\mathscr{D}$ with values in the category $\mathrm{mod}(R)$ of finitely generated projective R-modules. In the unpublished work [9] (see also [11, 16] for surveys), M. Nori constructed an R-coalgebra $\mathcal{C}_T$ such that the representation T has a universal factorization (see Theorem 2.1)

$$\mathscr{D} \xrightarrow{\overline{T}} \mathrm{comod}(\mathcal{C}_T) \xrightarrow{F_T} \mathrm{mod}(R),$$

where $\mathrm{comod}(\mathcal{C}_T)$ is the category of left $\mathcal{C}_T$-comodules that are finitely generated over R, $\overline{T}$ is a representation, and F_T is the forgetful functor.

Then Nori applied this formalism to Betti homology to obtain the Abelian category EHM of effective homological motives over k (see [9, 11, 16]). By construction, given a k-variety X, a closed (reduced) subscheme $Y \subseteq X$, and an integer $i \in \mathbb{Z}$, there is a motive $\overline{H}_i(X, Y)$ in EHM that realizes to the usual Betti homology.

1.2 J. Ayoub and L. Barbieri-Viale showed [1, Theorem 5.2, Theorem 6.1] that the thick Abelian subcategory of Nori's category of effective homological motives generated by the $\overline{H}_0$ and $\overline{H}_1$ of pairs is equivalent to: (a) the Abelian category EHM_1 associated with the representation

$$\mathrm{Crv}_k^{\mathrm{op}} \longrightarrow \mathrm{mod}(\mathbb{Z}), \quad (C, Y) \longmapsto H_1(C, Y)$$

where Crv_k is the category of pairs (C, Y) where C is a smooth affine k-curve, $Y \subseteq C$ is a closed subset consisting of finitely many closed points, and $H_1(C, Y)$ is the first

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Betti homology group of the pair (C, Y); (b) the Abelian category ${}^t\mathcal{M}_1$ of Deligne's 1-motives with torsion [3, 8] .

Note that by [17, Théorème 3.4.1], the derived category of Deligne's Abelian category of 1-isomotives $\mathcal{M}_{1,\mathbb{Q}}$ is known to be equivalent to the thick triangulated subcategory of Voevodsky's category of geometrical effective motives with rational coefficients generated by motives of smooth k-curves.

1.3 Such a description is not possible integrally for the extension of the theory of 1-motives introduced by G. Laumon [14] and studied in [2,4,15,20]. Indeed, the category of Laumon 1-motives with torsion ${}^t\mathcal{M}_1^a$ of [4] contains the category of infinitesimal formal k-groups (equivalent via the Lie algebra to the category of finite-dimensional k-vector spaces) as a full subcategory. In particular not all Hom groups in ${}^t\mathcal{M}_1^a$ are finitely generated Abelian groups and therefore there cannot exist a quiver $\mathscr{D}$ and a representation $T\colon \mathscr{D} \to \mathsf{mod}(\mathbb{Z})$ such that ${}^t\mathcal{M}_1^a$ is equivalent to $\mathsf{comod}(\mathcal{C}_T)$.

If the field k is not a number field, the same obstruction applies with rational coefficients. The Abelian category $\mathcal{M}_{1,\mathbb{Q}}^a$ of Laumon 1-isomotives still contains the category of infinitesimal formal k-groups as a full subcategory and therefore not all its Hom groups are finite-dimensional $\mathbb{Q}$-vector spaces. Again this prevents the existence of a quiver $\mathscr{D}$ and a representation $T\colon \mathscr{D} \to \mathsf{mod}(\mathbb{Q})$ such that $\mathcal{M}_{1,\mathbb{Q}}^a$ is equivalent to $\mathsf{comod}(\mathcal{C}_T)$.

1.4 If k is a number field, one may still hope to describe the Abelian category $\mathcal{M}_{1,\mathbb{Q}}^a$ of Laumon 1-isomotives over k via Nori's tannakian formalism. The main result of this work is such a description in that case.

More precisely, let a k-curve with modulus be a triplet (X, Y, Z) where X is a smooth proper k-curve and Y, Z are effective divisors on X with disjoint supports. Define the de Rham cohomology of a such a k-curve with modulus as the finite-dimensional k-vector space

$$\mathbf{H}^1_{\mathrm{dR}}(X, Y, Z) := \mathbf{H}^1(X, [\mathcal{I}_Y \to \mathcal{I}_Z^{-1}\Omega^1_X]),$$

where $\mathcal{I}_Y$ and $\mathcal{I}_Z$ are the ideals in $\mathcal{O}_X$ that define Y and Z. The k-curves with modulus define a category $\overline{\mathrm{M}}\mathrm{Crv}_k$ for which a morphism $(X, Y, Z) \to (X', Y', Z')$ is a morphism $f\colon X \to X'$ of k-varieties such that $Y \leqslant f^*Y'$, $Z - Z_{\mathrm{red}} \geqslant f^*(Z' - Z'_{\mathrm{red}})$, and $Z_{\mathrm{red}} \geqslant (f^*Z')_{\mathrm{red}}$. If k is a number field, by forgetting the k-linear structure, the de Rham cohomology of curves with modulus define a functor

$$\mathbf{H}^1_{\mathrm{dR}}\colon \overline{\mathrm{M}}\mathrm{Crv}_k^{\mathrm{op}} \to \mathsf{mod}(\mathbb{Q})$$

with values in the category of finite-dimensional $\mathbb{Q}$-vector spaces. Our main theorem is the following (see Theorem 5.9).

Theorem 1.1 *Let k be a number field. The $\mathbb{Q}$-linear Abelian category associated with the representation of quiver*

$$\begin{aligned} \mathbf{H}^1_{\mathrm{dR}}\colon \overline{\mathrm{M}}\mathrm{Crv}_k^{\mathrm{op}} &\longrightarrow \mathsf{mod}(\mathbb{Q}) \\ (X, Y, Z) &\longmapsto \mathbf{H}^1_{\mathrm{dR}}(X, Y, Z) := \mathbf{H}^1(X, [\mathcal{I}_Y \to \mathcal{I}_Z^{-1}\Omega^1_X]) \end{aligned}$$

is equivalent to the category $\mathcal{M}_{1,\mathbb{Q}}^a$ of Laumon 1-isomotives over k.

Theorem 1.1 generalizes the equivalence between (a) and (b) recalled in §1.2 and proved by J. Ayoub and L. Barbieri-Viale [1, Theorem 5.2]. Note that we do not provide any definition for a non-homotopy invariant analog of the full category of Nori's motives of varieties (of arbitrary dimension) with modulus. Moreover in [1] the main theorems are valid over any field of characteristic zero embedded into the complex numbers, and they also admit integral coefficient variants. Here we are not able to provide such generality.[1] We leave this issue for future study.

Conventions. Throughout the paper we work over a base field k with a fixed embedding $k \hookrightarrow \mathbb{C}$. In §3.4, §3.6, and from §5.6 onward, we further assume that k is a number field. For a k-scheme X, we denote by Ω^1_X the sheaf of Kähler differentials on X relative to k. If Z is a closed subscheme of X, we write $\mathcal{I}_Z \subset \mathcal{O}_X$ for the ideal sheaf of Z. For a vector space V over k, we write V^* for the k-linear dual of V. Let R be a ring and let R' be an R-algebra. For an R-linear Abelian category $\mathscr{A}$, we denote by $\mathscr{A} \otimes_R R'$ its scalar extension. This is an R'-linear Abelian category having the same objects as $\mathscr{A}$ and such that

$$\operatorname{Hom}_{\mathscr{A}\otimes_R R'}(A, B) = \operatorname{Hom}_{\mathscr{A}}(A, B) \otimes_R R'. \tag{1.1}$$

2 Reminders on Nori's Tannakian Formalism

2.1 Let K be a field. Following [10, Chapitre II, §4], recall that a K-linear Abelian category $\mathscr{P}$ is said to be finite if it is Noetherian and Artinian, *i.e.*, $\mathscr{P}$ is essentially small and any object in $\mathscr{P}$ has finite length. We shall say that $\mathscr{P}$ is Hom finite if for any objects P, Q in $\mathscr{P}$ the K-vector space $\mathscr{P}(P, Q)$ is finite-dimensional. By [12, Theorem 2.1], we have the following theorem.

Theorem 2.1 *Let $\mathscr{P}$ be a K-linear Abelian category which is finite and* Hom *finite, $\mathscr{D}$ a quiver (*i.e.*, directed graph), and $T\colon \mathscr{D} \to \mathscr{P}$ a representation of the quiver $\mathscr{D}$ with values in $\mathscr{P}$. Then there exist a K-linear Abelian category $\mathscr{A}$, a representation $R\colon \mathscr{D} \to \mathscr{A}$, a K-linear faithful exact functor $F\colon \mathscr{A} \to \mathscr{P}$, and an invertible 2-morphism $\alpha\colon F \circ R \to T$ such that for every K-linear Abelian category $\mathscr{B}$, every representation $S\colon \mathscr{D} \to \mathscr{B}$, every K-linear exact faithful functor $G\colon \mathscr{B} \to \mathscr{P}$, and every invertible 2-morphism $\beta\colon G \circ S \to T$ the following conditions are satisfied.*

(i) *There exist a K-linear functor $H\colon \mathscr{A} \to \mathscr{B}$ and two invertible 2-morphisms*

$$\gamma\colon H \circ R \xrightarrow{\simeq} S \qquad \delta\colon G \circ H \xrightarrow{\simeq} F,$$

such that

$$\begin{array}{ccc} G \circ H \circ R & \xrightarrow{G\star\gamma} & G \circ S \\ \Big\downarrow{\scriptstyle \delta\star R} & & \Big\downarrow{\scriptstyle \beta} \\ F \circ R & \xrightarrow{\alpha} & T \end{array}$$

is commutative.

[1] Recent papers [6,7], introduced a new construction of the universal category without finite dimensionality assumption, which would enable us to define ECMM_1 for an arbitrary subfield of $\mathbb{C}$. Unfortunately, we would then lose a description of the category as $\operatorname{comod}(\mathcal{C}_T)$, which is essential in the proof of our main result (see Proposition 2.3).

(ii) *If* $H'\colon \mathscr{A} \to \mathscr{B}$ *is a* K*-linear functor and*

$$\gamma'\colon H' \circ R \xrightarrow{\simeq} S \quad \delta'\colon G \circ H' \xrightarrow{\simeq} F$$

are two invertible 2*-morphisms such that the square*

$$\begin{array}{ccc} G \circ H' \circ R & \xrightarrow{G \star \gamma'} & G \circ S \\ \downarrow{\scriptstyle \delta' \star R} & & \downarrow{\scriptstyle \beta} \\ F \circ R & \xrightarrow{\alpha} & T \end{array}$$

is commutative, then there exists a unique 2*-morphism* $\theta\colon H \to H'$ *such that* $\gamma' \circ (\theta \star R) = \gamma$ *and* $\delta' \circ (G \star \theta) = \delta$*.*

It will be useful to keep in mind the following remark.

Remark 2.2. When $\mathscr{P} = \mathsf{mod}(K)$, the previous theorem is due to M. Nori. More precisely, let $\mathscr{E}$ be a full subquiver of $\mathscr{D}$ with finitely many objects and $\mathrm{End}_K(T|_{\mathscr{E}})$ the subring of $\prod_{q\in\mathscr{E}} \mathrm{End}_K(T(q))$ formed by the elements $e = (e_q)_{q\in\mathscr{E}}$ such that $e_q \circ T(m) = T(m) \circ e_p$ for every object $p \in \mathscr{E}$ and every morphism $m\colon p \to q$ in $\mathscr{D}$. Then its linear dual $\mathcal{C}_{T|_{\mathscr{E}}} := \mathrm{End}_K(T|_{\mathscr{E}})^*$ is a coassociative, counitary K-coalgebra that is finite-dimensional over K. We may then consider the K-linear Abelian category $\mathsf{comod}(\mathcal{C}_T)$ of finite-dimensional left comodules over the coassociative and counitary K-coalgebra

$$\mathcal{C}_T := \operatorname*{colim}_{\mathscr{E}\subseteq\mathscr{D}} \mathcal{C}_{T|_{\mathscr{E}}},$$

where the colimit is taken over full subquivers of $\mathscr{D}$ with finitely many objects.

For every object $p \in \mathscr{D}$ the finite-dimensional K-vector space $T(p)$ inherits a structure of left $\mathcal{C}_T$-comodule. This provides a representation $\overline{T}\colon \mathscr{D} \to \mathsf{comod}(\mathcal{C}_T)$ such that $T = F_T \circ \overline{T}$ where $F_T\colon \mathsf{comod}(\mathcal{C}_T) \to \mathsf{mod}(K)$ is the forgetful functor. The main result proved by Nori is that the tuple $(\mathsf{comod}(\mathcal{C}_T), \overline{T}, F_T, \mathrm{id})$ satisfies the universal property of Theorem 2.1 when $\mathscr{P} = \mathsf{mod}(K)$.

The general case is deduced from Nori's result. Indeed, let $\mathscr{P}$ be a finite and Hom finite K-linear Abelian category and $T\colon \mathscr{D} \to \mathscr{P}$ a representation. A result [12, Corollary 4.3] that can be easily deduced from [23, 5.1 Theorem, 5.8] assures the existence of a K-linear exact faithful functor $\omega\colon \mathscr{P} \to \mathsf{mod}(K)$. Let $\mathscr{A} := \mathsf{comod}(\mathcal{C}_{\omega\circ T})$ and consider the associated representation

$$R := \overline{\omega \circ T}\colon \mathscr{D} \to \mathsf{comod}(\mathcal{C}_{\omega\circ T}) =: \mathscr{A}.$$

The universal property of $(\mathscr{A}, R, F_{\omega\circ T}, \mathrm{Id})$ applied to the tuple $(\mathscr{P}, T, \omega, \mathrm{Id})$ provides a K-linear exact faithful functor $F\colon \mathscr{A} \to \mathscr{P}$ and an invertible natural transformation $\alpha\colon F \circ R \to T$. One checks then that the tuple $(\mathscr{A}, R, F, \alpha)$ satisfies the universal property stated in Theorem 2.1 (see [12] for details).

2.2 Let $\mathscr{D}$ be a quiver and $T\colon \mathscr{D} \to \mathscr{P}$ a representation. Let $(\mathscr{B}, G, R, \beta)$ be an tuple where $\mathscr{B}$ is a K-linear Abelian category, $S\colon \mathscr{D} \to \mathscr{B}$ is a representation, $G\colon \mathscr{B} \to \mathscr{P}$ is a K-linear exact faithful functor, and $\beta\colon G \circ S \to T$ is an invertible natural transformation. By the universal property of Theorem 2.1, there exist a K-linear functor

$H\colon \mathrm{comod}(\mathcal{C}_T) \to \mathscr{B}$ and two invertible natural transformations

$$\gamma\colon H \circ \overline{T} \xrightarrow{\simeq} S, \qquad \delta\colon G \circ H \xrightarrow{\simeq} F_T$$

such that the square

$$\begin{array}{ccc} G \circ H \circ \overline{T} & \xrightarrow{G \star \gamma} & G \circ S \\ \downarrow{\scriptstyle \delta \star R} & & \downarrow{\scriptstyle \beta} \\ F_T \circ \overline{T} & = & T \end{array}$$

is commutative (here we use the notations from Remark 2.2). J. Ayoub and L. Barbieri-Viale gave a criterion for the functor H to be an equivalence [1, Proposition 2.1]. The proof of our main result relies on this criterion.

Proposition 2.3 (Ayoub and Barbieri-Viale [1]) *Assume the following conditions.*

(i) *For all vertices $p, q \in \mathscr{D}$, there exist $p \sqcup q$ in $\mathscr{D}$ and edges $i\colon p \to p \sqcup q$, $j\colon q \to p \sqcup q$ such that the map $S(i) + S(j)\colon S(p) \oplus S(q) \to S(p \sqcup q)$ is an isomorphism in $\mathscr{B}$.*

(ii) *Every object in $\mathscr{B}$ is a quotient of an object of the form $S(p)$ for some vertex $p \in \mathscr{D}$.*

(iii) *For every map $S(p) \to B$ in $\mathscr{B}$, there exists a finite sub-quiver $\mathscr{E} \subseteq \mathscr{D}$ containing p such that $\mathrm{Ker}\{T(p) = G \circ S(p) \to G(B)\}$ is a sub-$\mathrm{End}(T|_{\mathscr{E}})$-module of $T(p)$.*

Then the functor $H\colon \mathrm{comod}(\mathcal{C}_T) \to \mathscr{B}$ is an equivalence of categories.

2.3 Let $\mathscr{P}_1$ and $\mathscr{P}_2$ be two finite and Hom finite K-linear Abelian categories. Let $\mathscr{D}_1$, $\mathscr{D}_2$ be quivers, $D\colon \mathscr{D}_1 \to \mathscr{D}_2$ a morphism of quivers, and $T_1\colon \mathscr{D}_1 \to \mathscr{P}_1$ and $T_2\colon \mathscr{D}_2 \to \mathscr{P}_2$ two representations. Let $(\mathscr{A}_1, F_1, R_1, \alpha_1)$ and $(\mathscr{A}_2, F_2, R_2, \alpha_2)$ be tuples obtained by applying Theorem 2.1 to the representations T_1 and T_2, respectively.

The next proposition shows that certain exact functors can be lifted to universal categories (for a proof, see [12, Proposition 6.6]).

Proposition 2.4 *Let (Φ, ϕ) be a pair where $\Phi\colon \mathscr{P}_1 \to \mathscr{P}_2$ is an* exact *K-linear functor and $\phi\colon \Phi \circ T_1 \to T_2 \circ D$ is an isomorphism of representations. There exist an exact functor $\Psi\colon \mathscr{A}_1 \to \mathscr{A}_2$, an invertible natural transformation $\rho\colon \Phi \circ F_1 \to F_2 \circ \Psi$, and an isomorphism of representations $\rho\colon \Psi \circ R_1 \to R_2 \circ D$ such that*

(2.1)
$$\begin{array}{ccccc} \Phi \circ F_1 \circ R_1 & \xrightarrow{\Phi \star \alpha_1} & \Phi \circ T_1 & & \\ & & & \searrow{\scriptstyle \phi} & \\ \downarrow{\scriptstyle \rho \star R_1} & & & & T_2 \circ D \\ & & & \nearrow{\scriptstyle \alpha_2 \star D} & \\ F_2 \circ \Psi \circ R_1 & \xrightarrow[F_2 \star \rho]{} & F_2 \circ R_2 \circ D & & \end{array}$$

is commutative.

2.4 In this work we will need to lift natural transformations as well. Let $D_1, D_2\colon \mathscr{D}_1 \to \mathscr{D}_2$ be a morphism of quivers. Let (Φ_1, ϕ_1), (Φ_2, ϕ_2) be a pairs, where $\Phi_1, \Phi_2\colon \mathscr{P}_1 \to \mathscr{P}_2$ are *exact* K-linear functor and $\phi_1\colon \Phi_1 \circ T_1 \to T_2 \circ D_1$, $\phi_2\colon \Phi_2 \circ T_1 \to T_2 \circ D_2$ are isomorphisms of representations.

By Proposition 2.4, there exist exact functors $\Psi_1, \Psi_2\colon \mathscr{A}_1 \to \mathscr{A}_2$, invertible natural transformations $\rho_1\colon \Phi_1 \circ F_1 \to F_2 \circ \Psi_1$, $\rho_2\colon \Phi_2 \circ F_1 \to F_2 \circ \Psi_2$, and isomorphisms of representations $\rho_1\colon \Psi_1 \circ R_1 \to R_2 \circ D_1$, $\rho_2\colon \Psi_2 \circ R_1 \to R_2 \circ D_2$ such that the corresponding diagrams as in (2.1) are commutative.

Proposition 2.5 *Let (θ, θ_D) be a pair where $\theta\colon \Phi_1 \to \Phi_2$ and $\theta_D\colon D_1 \to D_2$ are natural transformations such that the square*

$$\begin{array}{ccc} \Phi_1 \circ T_1 & \xrightarrow{\phi_1} & T_2 \circ D_1 \\ \downarrow{\scriptstyle \theta \star T_1} & & \downarrow{\scriptstyle T_2 \star \theta_D} \\ \Phi_2 \circ T_1 & \xrightarrow{\phi_2} & T_2 \circ D_2 \end{array}$$

is commutative. Then there exists one and only one natural transformation $\overline{\theta}\colon \Psi_1 \to \Psi_2$ that makes the squares

$$\begin{array}{ccc} \Psi_1 \circ R_1 & \xrightarrow{\rho_1} & R_2 \circ D_1 \\ \downarrow{\scriptstyle \overline{\theta} \star R_1} & & \downarrow{\scriptstyle R_2 \star \theta_D} \\ \Psi_2 \circ R_1 & \xrightarrow{\rho_2} & R_2 \circ D_2 \end{array} \qquad \begin{array}{ccc} \Phi_1 \circ F_1 & \xrightarrow{\rho_1} & F_2 \circ \Psi_1 \\ \downarrow{\scriptstyle \theta \star F_1} & & \downarrow{\scriptstyle F_2 \star \overline{\theta}} \\ \Phi_2 \circ F_1 & \xrightarrow{\rho_2} & F_2 \circ \Psi_2 \end{array}$$

commutative.

Proof Let X be an object in $\mathscr{A}_1$. Let us sketch the construction of a morphism $\overline{\theta}_X\colon \Psi_1(X) \to \Psi_2(X)$ in $\mathscr{A}_2$ which makes the square

$$\begin{array}{ccc} \Phi_1(F_1(X)) & \xrightarrow{\rho_{1,X}} & F_2(\Psi_1(X)) \\ \downarrow{\scriptstyle \theta_{F_1(X)}} & & \downarrow{\scriptstyle F_2(\overline{\theta}_X)} \\ \Phi_2(F_1(X)) & \xrightarrow{\rho_{2,X}} & F_2(\Psi_2(X)) \end{array}$$

commutative. Since F_2 is faithful, such a morphism is necessarily unique. When $X = R_1(p)$ for $p \in \mathscr{D}_1$, we define $\overline{\theta}_X$ to be the unique morphism that makes the square

$$\begin{array}{ccc} \Psi_1(X) & \xrightarrow{\rho_{1,p}} & R_2(D_1(p)) \\ \downarrow{\scriptstyle \overline{\theta}_X} & & \downarrow{\scriptstyle R_2(\theta_{D,p})} \\ \Psi_2(X) & \xrightarrow{\rho_{2,p}} & R_2(D_2(p)) \end{array}$$

commutative. This defines also $\overline{\theta}_X$ when X is a finite direct sum of such objects. Assume now the existence of an epimorphism $s\colon Y \to X$ in $\mathscr{A}_1$ where Y is an object for which $\overline{\theta}_Y$ has been constructed. It is then enough to check the existence of a factorization

$$\begin{array}{ccccc} \Psi_1(Y) & \xrightarrow{\Psi_1(s)} & \Psi_1(X) & \longrightarrow & 0 \\ \downarrow{\scriptstyle \overline{\theta}_Y} & & \vdots & & \\ \Psi_2(Y) & \xrightarrow{\Psi_2(s)} & \Psi_2(X) & \longrightarrow & 0. \end{array}$$

As the rows are exact, this amounts to checking that $\Psi_2(s)\circ\overline{\theta}_Y$ vanishes on the kernel of $\Psi_1(s)$. But this is true since it is after applying F_2, and F_2 is faithful.

Similarly, one shows the existence of $\overline{\theta}_X$ when X is any subobject of an object Y in $\mathscr{A}_1$ for which $\overline{\theta}_Y$ has already been constructed.

This concludes the proof since by [11, Proposition 7.1.16] every object in $\mathscr{A}_1$ is a subquotient of a finite direct sum of objects of the form $X = R_1(p)$ for $p \in \mathscr{D}_1$. ■

Remark 2.6. Note that since F_2 is a K-linear exact and faithful functor, if θ is a monomorphism (resp. epimorphism), then $\overline{\theta}$ is a monomorphism (resp. epimorphism).

3 Nori Motives of Curves With Modulus

3.1 In this subsection, we collect some preliminary results on cohomology of curves.

Proposition 3.1 *Let $f\colon C \to C'$ be a finite k-morphism of smooth, proper connected k-curves. Let D and D' be effective divisors on C and C', respectively.*

(i) *Suppose $D \leqslant f^*D'$. Then the canonical map $\mathscr{O}_{C'} \to f_*\mathscr{O}_C$ induces $\mathfrak{I}_{D'} \to f_*\mathfrak{I}_D$ and the trace map $f_*\Omega^1_C \to \Omega^1_{C'}$ induces $f_*(\mathfrak{I}_D^{-1}\Omega^1_C) \to \mathfrak{I}_{D'}^{-1}\Omega^1_{C'}$.*

(ii) *Suppose $D - D_{\mathrm{red}} \geqslant f^*(D' - D'_{\mathrm{red}})$ and $D_{\mathrm{red}} \geqslant (f^*D')_{\mathrm{red}}$. (The latter condition is equivalent to $f(C\smallsetminus|D|) \subset f(C'\smallsetminus|D'|)$). Then the canonical map $\Omega^1_{C'} \to f_*\Omega^1_C$ induces $\mathfrak{I}_{D'}^{-1}\Omega_{C'} \to f_*(\mathfrak{I}_D^{-1}\Omega_C)$ and the trace map $f_*\mathscr{O}_C \to \mathscr{O}_{C'}$ induces $f_*\mathfrak{I}_D \to \mathfrak{I}_{D'}$.*

(Recall that by our convention k is a subfield of $\mathbb{C}$, that Ω^1_C is the sheaf of Kähler differentials on C relative to k, and that $\mathfrak{I}_D$ is the ideal sheaf defining D.)

Proof This follows from the following elementary lemma. ■

Lemma 3.2 *Let K be a function field of one variable over k, and let $R \subset K$ be a discrete valuation ring containing k. Let L be a finite extension of K and let S be the integral closure of R in L. Denote by $\mathfrak{m}$ the maximal ideal of R, and by $\mathfrak{n}_1,\dots,\mathfrak{n}_r$ the maximal ideals of S. Let $e_i \in \mathbb{Z}_{>0}$ be the ramification index of $\mathfrak{n}_i$. Let $m, n_1,\dots,n_r \geqslant 1$ be integers and put $\mathfrak{n}^n := \mathfrak{n}_1^{n_1}\cdots\mathfrak{n}_r^{n_r}$, $\mathfrak{n}^{-n} := \mathfrak{n}_1^{-n_1}\cdots\mathfrak{n}_r^{-n_r}$.*

(i) *Suppose $n_i \leqslant e_i m$ for all i. Then the canonical map $K \to L$ sends $\mathfrak{m}^m$ to $\mathfrak{n}^n$, and the trace map $\Omega^1_{L/k} \to \Omega^1_{K/k}$ sends $\mathfrak{n}^{-n}\Omega^1_{S/k}$ to $\mathfrak{m}^{-m}\Omega^1_{R/k}$.*

(ii) *Suppose $n_i - 1 \geqslant e_i(m-1)$ for all i. Then the canonical map $\Omega^1_{K/k} \to \Omega^1_{L/k}$ sends $\mathfrak{m}^{-m}\Omega_{R/k}$ to $\mathfrak{n}^{-n}\Omega_{S/k}$, and the trace map $L \to K$ sends $\mathfrak{n}^n$ to $\mathfrak{m}^m$.*

Proof The last statement of (ii) follows from [22, Chapter III, Propositions 7, 13]. All other statements are elementary. ■

Proposition 3.3 *Let C be a smooth proper curve over k and let D be an effective divisor on C. We set*

$$U(C,D) := H^0(C, \mathfrak{I}_{D_{\mathrm{red}}}/\mathfrak{I}_D) \quad V(C,D) := H^0(C, \mathfrak{I}_{D_{\mathrm{red}}}\mathfrak{I}_D^{-1}/\mathscr{O}_C).$$

Then the differential map induces isomorphisms

$$d\colon U(C,D) \xrightarrow{\cong} H^0(C, (\mathscr{O}_C/\mathfrak{I}_D\mathfrak{I}_{D_{\mathrm{red}}}^{-1}) \otimes \Omega_C^1),$$
$$d\colon V(C,D) \xrightarrow{\cong} H^0(C, (\mathfrak{I}_D^{-1}/\mathfrak{I}_{D_{\mathrm{red}}}^{-1}) \otimes \Omega_C^1).$$

Proof Write $D = \sum_{P\in|C|} n_P P$. Then we have

$$U(C,D) \cong \bigoplus_{P\in|D|} \mathfrak{m}_P/\mathfrak{m}_P^{n_P},$$
$$H^0(C, (\mathscr{O}/\mathfrak{I}_D\mathfrak{I}_{D_{\mathrm{red}}}^{-1}) \otimes \Omega_C^1) \cong \bigoplus_{P\in|D|} \Omega_{C,P}^1/\mathfrak{m}_P^{n_P-1}\Omega_{C,P}^1,$$

where $\mathfrak{m}_P$ denotes the maximal ideal of the local ring $\mathscr{O}_{C,P}$ of C at P. Thus the first statement follows from the bijectivity of

$$d\colon \mathfrak{m}_P/\mathfrak{m}_P^{n_P} \longrightarrow \Omega_{C,P}^1/\mathfrak{m}_P^{n_P-1}\Omega_{C,P}^1,$$

which is readily seen. Similarly, we have

$$V(C,D) \cong \bigoplus_{P\in|D|} \mathfrak{m}_P^{1-n_P}/\mathscr{O}_{C,P},$$
$$H^0(C, (\mathfrak{I}_D^{-1}/\mathfrak{I}_{D_{\mathrm{red}}}^{-1}) \otimes \Omega_C^1) \cong \bigoplus_{P\in|D|} \mathfrak{m}_P^{-n_P}\Omega_{C,P}^1/\mathfrak{m}_P^{-1}\Omega_{C,P}^1.$$

Thus the second statement follows from the bijectivity of

$$d\colon \mathfrak{m}_P^{1-n_P}/\mathscr{O}_{C,P} \longrightarrow \mathfrak{m}_P^{-n_P}\Omega_{C,P}^1/\mathfrak{m}_P^{-1}\Omega_{C,P}^1,$$

which is readily seen. ■

Corollary 3.4 *The two k-vector spaces $U(C,D)$ and $V(C,D)$ are canonically dual to each other.*

Proof We may suppose D is (effective and) non-trivial. Then we get

$$U(C,D) = \ker[H^1(C,\mathfrak{I}_D) \to H^1(C,\mathfrak{I}_{D_{\mathrm{red}}})]$$

from an exact sequence $0 \longrightarrow \mathfrak{I}_D \longrightarrow \mathfrak{I}_{D_{\mathrm{red}}} \longrightarrow \mathfrak{I}_{D_{\mathrm{red}}}/\mathfrak{I}_D \longrightarrow 0$. On the other hand, another exact sequence $0 \longrightarrow \mathfrak{I}_{D_{\mathrm{red}}}^{-1} \otimes \Omega_C^1 \longrightarrow \mathfrak{I}_D^{-1} \otimes \Omega_C^1 \longrightarrow (\mathfrak{I}_D^{-1}/\mathfrak{I}_{D_{\mathrm{red}}}^{-1}) \otimes \Omega_C^1 \longrightarrow 0$ and the above proposition yield

$$V(C,D) = \mathrm{Coker}[H^0(C,\mathfrak{I}_{D_{\mathrm{red}}}^{-1}\Omega_C^1) \to H^0(C,\mathfrak{I}_D^{-1}\Omega_C^1)].$$

Now the corollary follows from the Serre duality. ■

Corollary 3.5 *Let (C,D) and (C',D') be pairs consisting of a smooth proper k-curve and an effective divisor. Let $f\colon C \to C'$ be a finite k-morphism. The canonical map $\mathscr{O}_{C'} \to f_*\mathscr{O}_C$ and the trace map $f_*\Omega_C^1 \to \Omega_{C'}^1$ induce the following functoriality.*

(i) *If* $D \leqslant f^*D'$, *then we have*

$$f^*: U(C', D') \longrightarrow U(C, D) \quad f_*: V(C, D) \longrightarrow V(C', D').$$

(ii) *If* $D - D_{\mathrm{red}} \geqslant f^*(D' - D'_{\mathrm{red}})$ *and* $D_{\mathrm{red}} \geq (f^*D')_{\mathrm{red}}$, *then we have*

$$f^*: V(C', D') \longrightarrow V(C, D) \quad f_*: U(C, D) \longrightarrow U(C', D').$$

Proof Since $D \leqslant f^*D'$ implies $D_{\mathrm{red}} \leqslant (f^*D')_{\mathrm{red}} \leqslant f^*(D'_{\mathrm{red}})$, this follows from Propositions 3.1 and 3.3. ∎

3.2 Let us denote by $\overline{\mathrm{M}}\mathrm{Crv}$ the following category. An object in $\overline{\mathrm{M}}\mathrm{Crv}$ is a triplet (X, Y, Z) where X is a smooth proper k-curve and Y, Z are effective divisors on X such that $|Y| \cap |Z| = \varnothing$. A morphism $(X, Y, Z) \to (X', Y', Z')$ in $\overline{\mathrm{M}}\mathrm{Crv}$ is a morphism $f: X \to X'$ of k-varieties such that $Y \leqslant f^*Y'$, $Z - Z_{\mathrm{red}} \geqslant f^*(Z' - Z'_{\mathrm{red}})$, and $Z_{\mathrm{red}} \geqslant (f^*Z')_{\mathrm{red}}$ (equivalently, $f(X \smallsetminus |Z|) \subset f(X' \smallsetminus |Z'|)$). It then follows from Proposition 3.1 that the canonical map $\mathscr{O}_{X'} \to f_*\mathscr{O}_X$ induces morphisms of sheaves

$$\mathcal{I}_{Y'} \longrightarrow f_*\mathcal{I}_Y \quad \text{and} \quad \mathcal{I}_{Z'}^{-1}\Omega^1_{X'} \longrightarrow f_*(\mathcal{I}_Z^{-1}\Omega^1_X). \tag{3.1}$$

It will be useful to consider also the following variant: $\underline{\mathrm{M}}\mathrm{Crv}$ is the category with the same objects as $\overline{\mathrm{M}}\mathrm{Crv}$, but this times a morphism $(X, Y, Z) \to (X', Y', Z')$ in $\underline{\mathrm{M}}\mathrm{Crv}_k$ is a morphism $f: X \to X'$ of k-varieties such that $Y - Y_{\mathrm{red}} \geqslant f^*(Y' - Y'_{\mathrm{red}})$, $Y_{\mathrm{red}} \geqslant (f^*Y')_{\mathrm{red}}$, and $Z \leqslant f^*Z'$. Again it then follows from Proposition 3.1 that the trace map $f_*\mathscr{O}_X \to \mathscr{O}_{X'}$ induces morphisms of sheaves

$$f_*\mathcal{I}_{Y'} \longrightarrow \mathcal{I}_Y \quad \text{and} \quad f_*(\mathcal{I}_{Z'}^{-1}\Omega^1_{X'}) \longrightarrow \mathcal{I}_Z^{-1}\Omega^1_X. \tag{3.2}$$

Definition 3.6 Let (X, Y, Z) be an object in the category $\overline{\mathrm{M}}\mathrm{Crv}$. We define

$$\mathbf{H}^1_{\mathrm{dR}}(X, Y, Z) := \mathbf{H}^1(X, [\mathcal{I}_Y \to \mathcal{I}_Z^{-1}\Omega^1_X])$$

to be the first hypercohomology group of the complex of $\mathscr{O}_X$-modules $[\mathcal{I}_Y \to \mathcal{I}_Z^{-1}\Omega^1_X]$, where $\mathcal{I}_Y$ is placed in degree zero. This is a finite-dimensional k-vector space. By (3.1), we obtain a functor $\mathbf{H}^1_{\mathrm{dR}}: \overline{\mathrm{M}}\mathrm{Crv}^{\mathrm{op}} \to \mathsf{mod}(k)$, where $\mathsf{mod}(k)$ is the category of finite-dimensional k-vector spaces. We also have a functor

$${}^t\mathbf{H}^1_{\mathrm{dR}}: \underline{\mathrm{M}}\mathrm{Crv} \longrightarrow \mathsf{mod}(k) \tag{3.3}$$

which takes the same value on objects as $\mathbf{H}^1_{\mathrm{dR}}$, but acts on morphisms via (3.2).

3.3 In the following, see Proposition 3.3 for the definition of $U(X, Y)$ and $V(X, Z)$.

Proposition 3.7 *For any* $(X, Y, Z) \in \overline{\mathrm{M}}\mathrm{Crv}$, *there is a canonical decomposition*

$$\mathbf{H}^1_{\mathrm{dR}}(X, Y, Z) \cong \mathbf{H}^1_{\mathrm{dR}}(X, Y_{\mathrm{red}}, Z_{\mathrm{red}}) \oplus U(X, Y) \oplus V(X, Z). \tag{3.4}$$

Moreover, the decomposition (3.4) *is functorial with respect to maps in* $\overline{\mathrm{M}}\mathrm{Crv}$.

Proof Since $U(C, D_{\mathrm{red}}) = V(C, D_{\mathrm{red}}) = 0$ for a smooth proper k-curve C and an effective divisor D, we are reduced to showing

$$\mathbf{H}^1_{\mathrm{dR}}(X, Y, Z) \cong \mathbf{H}^1_{\mathrm{dR}}(X, Y_{\mathrm{red}}, Z) \oplus U(X, Y) \cong \mathbf{H}^1_{\mathrm{dR}}(X, Y, Z_{\mathrm{red}}) \oplus V(X, Z).$$

To show the first isomorphism, we construct canonical maps

$$a\colon \mathbf{H}^1_{\mathrm{dR}}(X, Y_{\mathrm{red}}, Z) \longrightarrow \mathbf{H}^1_{\mathrm{dR}}(X, Y, Z), \quad b\colon \mathbf{H}^1_{\mathrm{dR}}(X, Y, Z) \longrightarrow \mathbf{H}^1_{\mathrm{dR}}(X, Y_{\mathrm{red}}, Z)$$

such that $b \circ a = \mathrm{id}$ and $\ker(b) \cong U(X, Y)$. For this we first note that the map

$$[\mathcal{I}_Y \to \mathcal{I}_Y \mathcal{I}_{Y_{\mathrm{red}}}^{-1} \mathcal{I}_Z^{-1} \Omega^1_X] \longrightarrow [\mathcal{I}_{Y_{\mathrm{red}}} \to \mathcal{I}_Z^{-1} \Omega^1_X]$$

(induced by the inclusions $\mathcal{I}_Y \subset \mathcal{I}_{Y_{\mathrm{red}}}$ and $\mathcal{I}_Y \mathcal{I}_{Y_{\mathrm{red}}}^{-1} \mathcal{I}_Z^{-1} \subset \mathcal{I}_Z^{-1}$) is a quasi-isomorphism by Proposition 3.3. Using this, we define a to be the composition

$$\begin{aligned} \mathbf{H}^1_{\mathrm{dR}}(X, Y_{\mathrm{red}}, Z) &= \mathbf{H}^1(X, [\mathcal{I}_{Y_{\mathrm{red}}} \to \mathcal{I}_Z^{-1}\Omega^1_X]) \\ &\xleftarrow{\cong} \mathbf{H}^1(X, [\mathcal{I}_Y \to \mathcal{I}_Y \mathcal{I}_{Y_{\mathrm{red}}}^{-1} \mathcal{I}_Z^{-1}\Omega^1_X]) \longrightarrow \\ &\qquad \mathbf{H}^1(X, [\mathcal{I}_Y \to \mathcal{I}_Z^{-1}\Omega^1_X]) = \mathbf{H}^1_{\mathrm{dR}}(X, Y, Z), \end{aligned}$$

where the second map is induced by the inclusion $\mathcal{I}_Y \mathcal{I}_{Y_{\mathrm{red}}}^{-1} \mathcal{I}_Z^{-1} \subset \mathcal{I}_Z^{-1}$. Next, b is given by

$$\begin{aligned} \mathbf{H}^1_{\mathrm{dR}}(X, Y, Z) &= \mathbf{H}^1(X, [\mathcal{I}_Y \to \mathcal{I}_Z^{-1}\Omega^1_X]) \\ &\longrightarrow \mathbf{H}^1(X, [\mathcal{I}_{Y_{\mathrm{red}}} \to \mathcal{I}_Z^{-1}\Omega^1_X]) = \mathbf{H}^1_{\mathrm{dR}}(X, Y_{\mathrm{red}}, Z), \end{aligned}$$

which is induced by the inclusion $\mathcal{I}_Y \subset \mathcal{I}_{Y_{\mathrm{red}}}$. It is obvious that the composition $b \circ a$ is the identity. It is also clear from this construction that $\ker(b) \cong U(X, Y)$. Note also that Proposition 3.3 tells us that $\mathrm{Coker}(a) \cong U(X, Y)$, as it should be.

The second isomorphism $\mathbf{H}^1_{\mathrm{dR}}(X, Y, Z) \cong \mathbf{H}^1_{\mathrm{dR}}(X, Y, Z_{\mathrm{red}}) \oplus V(X, Z)$ is constructed in a similar way. We omit it. ∎

Proposition 3.8 *For any $(X, Y, Z) \in \overline{\mathrm{M}}\mathrm{Crv}$, the two k-vector spaces $\mathbf{H}^1_{\mathrm{dR}}(X, Y, Z)$ and $\mathbf{H}^1_{\mathrm{dR}}(X, Z, Y)$ are canonically dual to each other.*

Proof Apply Lemma 3.9 with $C^* = [\mathcal{I}_Y \to \mathcal{I}_Z^{-1}\Omega^1_X]$ and $D^* = [\mathcal{I}_Z \to \mathcal{I}_Y^{-1}\Omega^1_X]$. ∎

Lemma 3.9 *Let C^* and D^* be two complexes of sheaves of k-vector spaces on X such that C^i and D^i are locally free $\mathcal{O}_X$-modules for all i and that $C^i = D^i = 0$ unless $i \notin \{0,1\}$. Let $\wedge\colon \mathrm{Tot}(C^* \otimes_k D^*) \to \Omega^\bullet_X$ be a map of complexes and suppose that it induces $C^0 \cong \underline{\mathrm{Hom}}_{\mathcal{O}_X}(D^1, \Omega^1_X)$ and $C^1 \cong \underline{\mathrm{Hom}}_{\mathcal{O}_X}(D^0, \Omega^1_X)$. Then $\wedge$ induces a perfect duality between $\mathbf{H}^i(X, C^*)$ and $\mathbf{H}^{2-i}(X, D^*)$ for all i.*

Proof This is reduced to the Serre duality by an exact sequence

$$\cdots \longrightarrow H^{i-1}(X, C^1) \longrightarrow \mathbf{H}^i(X, C^*) \longrightarrow H^i(X, C^0) \longrightarrow H^i(X, C^1) \longrightarrow \cdots$$

and a similar sequence for D^*. ∎

3.4 The following definition introduces our main object of studies.

Definition 3.10 Let k be a number field. The category ECMM_1 of effective cohomological isomotives of curves with modulus is the $\mathbb{Q}$-linear category associated with the representation $\mathbf{H}^1_{\mathrm{dR}}\colon \overline{\mathrm{M}}\mathrm{Crv}^{\mathrm{op}} \to \mathrm{mod}(\mathbb{Q})$.

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By construction the representation $\mathbf{H}^1_{\mathrm{dR}}$ has a factorization

$$\overline{\mathrm{M}}\mathrm{Crv}^{\mathrm{op}} \xrightarrow{\overline{\mathbf{H}}^1_{\mathrm{dR}}} \mathrm{ECMM}_1 \xrightarrow{F^a_{\mathrm{dR}}} \mathsf{mod}(\mathbb{Q})$$

into a representation $\overline{\mathbf{H}}^1_{\mathrm{dR}}$ and a $\mathbb{Q}$-linear faithful exact functor F^a_{dR}.

3.5 Let Crv be the category defined as follows (see [1, §5.1]). An object is a pair (C, Y) where C is a smooth affine curve and $Y \subseteq C$ is a closed subset consisting of finitely many closed points. A morphism $(C, Y) \to (C', Y')$ is given by a k-morphism $f: C \to C'$ such that $f(Y) \subset Y'$.

Recall that by definition [1, §5.1] the $\mathbb{Q}$-linear Abelian category EHM_1 of effective homological isomotives of curves [2] is the universal category associated with the representation

$$H^{\mathrm{B}}_1: \mathrm{Crv}_k \longrightarrow \mathsf{mod}(\mathbb{Q}), \quad (C, Y) \longmapsto H^{\mathrm{B}}_1(C, Y) \otimes_{\mathbb{Z}} \mathbb{Q}, \tag{3.5}$$

where $H^{\mathrm{B}}_1(C, Y)$ is the Betti homology of the pair (C, Y) (with integral coefficients). Let us denote by ECM_1 the universal category associated with the representation

$$H^1_{\mathrm{B}}: \mathrm{Crv}^{\mathrm{op}}_k \longrightarrow \mathsf{mod}(\mathbb{Q}), \quad (C, Y) \longmapsto H^1_{\mathrm{B}}(C, Y) \otimes_{\mathbb{Z}} \mathbb{Q},$$

where $H^1_{\mathrm{B}}(C, Y)$ is the Betti cohomology of the pair (C, Y). The $\mathbb{Q}$-linear dual functor $\mathsf{mod}(\mathbb{Q})^{\mathrm{op}} \to \mathsf{mod}(\mathbb{Q})$ induces an equivalence

$$(\mathrm{EHM}_1)^{\mathrm{op}} \longrightarrow \mathrm{ECM}_1. \tag{3.6}$$

3.6 In this work, it will be convenient to define effective cohomological motives of curves using algebraic de Rham cohomology instead of Betti cohomology. For this we assume that k is a number field and consider the representation

$$\begin{aligned} H^1_{\mathrm{dR}}: \mathrm{Crv}^{\mathrm{op}} &\longrightarrow \mathsf{mod}(k) \\ (C, Y) &\longmapsto H^1_{\mathrm{dR}}(C, Y) := \mathbf{H}^1_{\mathrm{dR}}(C, [\mathcal{I}_Y \to \Omega^1_C])). \end{aligned} \tag{3.7}$$

If $\overline{C}$ is the smooth compactification of C and $C_\infty = \overline{C} \setminus C$ is the set of points at infinity, then we have $H^1_{\mathrm{dR}}(C, Y) \cong \mathbf{H}^1_{\mathrm{dR}}(\overline{C}, Y, C_\infty)$, where $\mathbf{H}^1_{\mathrm{dR}}(\overline{C}, Y, C_\infty)$ is defined as in Definition 3.6 with both Y, C_∞ viewed as closed reduced subschemes of $\overline{C}$. Let us denote by $\mathrm{ECM}^{\mathrm{dR}}_1$ the $\mathbb{Q}$-linear Abelian category associated with the representation H^1_{dR} in (3.7). By construction the representation H^1_{dR} has a factorization

$$\mathrm{Crv}^{\mathrm{op}} \xrightarrow{\overline{H}^1_{\mathrm{dR}}} \mathrm{ECM}^{\mathrm{dR}}_1 \xrightarrow{F_{\mathrm{dR}}} \mathsf{mod}(\mathbb{Q})$$

into a representation $\overline{H}^1_{\mathrm{dR}}$ and a $\mathbb{Q}$-linear faithful exact functor F_{dR}. Note that by the universal property the functor F_{dR} factorizes in $\mathsf{mod}(k)$ via the forgetful functor.

Lemma 3.11 *There is a canonical isomorphism of functors $H^1_{\mathrm{dR}} \otimes_k \mathbb{C} \xrightarrow{\sim} H^1_{\mathrm{B},\mathbb{C}}$ on the category* Crv.

---

[2] Note that in [1] the category EHM_1 is denoted by EHM''_1, while EHM_1 stands for the the thick Abelian subcategory of Nori's category of effective cohomological isomotives generated by the first cohomology motive of pairs. These categories are equivalent by [1, Theorem 5.2, Theorem 6.1].

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Proof For a k-variety V we write V^{an} for the complex analytic variety associated with V. Let (C, Y) in Crv and let $\mathcal{I}, \mathcal{J}$ be the ideals of Y^{an} and C_∞^{an} in $\mathcal{O}_{\overline{C}^{\mathrm{an}}}$.The canonical map

$$\mathbf{H}^1_{\mathrm{dR}}(\overline{C}, Y, C_\infty) \otimes_k \mathbb{C} \longrightarrow \mathbf{H}^1(\overline{C}^{\mathrm{an}}, [\mathcal{I} \to \mathcal{J}^{-1}\Omega^1_{\overline{C}^{\mathrm{an}}}])$$

is an isomorphism of $\mathbb{C}$-vector spaces by GAGA. On the other hand, we have canonical quasi-isomorphisms

$$j_*\mathbb{C}_{C^{\mathrm{an}}} \cong [\mathcal{O}_{\overline{C}^{\mathrm{an}}} \to \mathcal{J}^{-1}\Omega^1_{\overline{C}^{\mathrm{an}}}], \quad i_*\mathbb{C}_{Y^{\mathrm{an}}} \cong [\mathcal{O}_{\overline{C}^{\mathrm{an}}}/\mathcal{I} \to 0],$$

where $j\colon C^{\mathrm{an}} \to \overline{C}^{\mathrm{an}}$ and $i\colon Y^{\mathrm{an}} \to \overline{C}^{\mathrm{an}}$ are immersions and $\mathbb{C}_{C^{\mathrm{an}}}$ (resp. $\mathbb{C}_{Y^{\mathrm{an}}}$) denotes the constant sheaf on C^{an} (resp. Y^{an}). There is an exact sequence of complexes

$$0 \longrightarrow [\mathcal{I} \to \mathcal{J}^{-1}\Omega^1_{\overline{C}^{\mathrm{an}}}] \longrightarrow [\mathcal{O}_{\overline{C}^{\mathrm{an}}} \to \mathcal{J}^{-1}\Omega^1_{\overline{C}^{\mathrm{an}}}] \longrightarrow [\mathcal{O}_{\overline{C}^{\mathrm{an}}}/\mathcal{I} \to 0] \longrightarrow 0.$$

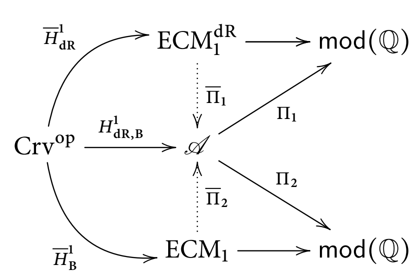
Hence the lemma follows from the fact that $H^i_{\mathrm{B}}(C, Y) \otimes_{\mathbb{Z}} \mathbb{C}$ is computed as the hypercohomology of the cone of $j_*\mathbb{C}_{C^{\mathrm{an}}} \to i_*\mathbb{C}_{Y^{\mathrm{an}}}$ with degree shifted by one. ■

Proposition 3.12 *Let k be a number field. The categories* ECM_1 *and* $\mathrm{ECM}_1^{\mathrm{dR}}$ *are equivalent.*

Proof Consider the 2-fiber product $\mathscr{A}$ of the categories $\mathsf{mod}(k)$ and $\mathsf{mod}(\mathbb{Q})$ over $\mathsf{mod}(\mathbb{C})$. An object of $\mathscr{A}$ is thus a triplet (V, W, α) where V is a finite-dimensional k-vector space, W is a finite-dimensional $\mathbb{Q}$-vector space, and $\alpha\colon V \otimes_k \mathbb{C} \to W \otimes_{\mathbb{Q}} \mathbb{C}$ is an isomorphism of $\mathbb{C}$-vector spaces. The category $\mathscr{A}$ is a $\mathbb{Q}$-linear Abelian category with two $\mathbb{Q}$-linear exact faithful functors $\Pi_1\colon \mathscr{A} \to \mathsf{mod}(\mathbb{Q})$, $\Pi_2\colon \mathscr{A} \to \mathsf{mod}(\mathbb{Q})$ given by the projection on the first factor composed with the forgetful functor and the projection on the second factor. We may then consider the representation

$$\begin{aligned} H^1_{\mathrm{dR,B}}\colon \mathrm{Crv}^{\mathrm{op}} &\longrightarrow \mathscr{A} \\ (C, Y) &\longmapsto H^1_{\mathrm{dR,B}}(C, Y) := (H^1_{\mathrm{dR}}(C, Y), H^1_{\mathrm{B}}(C, Y) \otimes_{\mathbb{Z}} \mathbb{Q}, \alpha), \end{aligned}$$

where the isomorphism $\alpha\colon H^1_{\mathrm{dR}}(C, Y) \otimes_k \mathbb{C} \to H^1_{\mathrm{B}}(C, Y) \otimes_{\mathbb{Z}} \mathbb{C}$ is the one of Lemma 3.11. We have the commutative diagram



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where $\overline{\Pi}_1$ and $\overline{\Pi}_2$ are the functors provided by the universal properties. The subdiagram

$$\begin{array}{ccccc} & & \mathrm{ECM}_1^{\mathrm{dR}} & & \\ & \overset{\overline{H}^1_{\mathrm{dR}}}{\nearrow} & \vdots\uparrow & \overset{\Pi_2\circ\overline{\Pi}_1}{\searrow} & \\ \mathrm{Crv}^{\mathrm{op}} & & \vdots & & \mathrm{mod}(\mathbb{Q}) \\ & \underset{\overline{H}^1_{\mathrm{B}}}{\searrow} & \vdots & \nearrow & \\ & & \mathrm{ECM}_1 & & \end{array}$$

then provides a $\mathbb{Q}$-linear functor $\mathrm{ECM}_1 \to \mathrm{ECM}_1^{\mathrm{dR}}$. Similarly we get a $\mathbb{Q}$-linear functor $\mathrm{ECM}_1^{\mathrm{dR}} \to \mathrm{ECM}_1$ and it is easy to check that they are quasi-inverse to one another. ∎

Let C be any smooth affine k-curve. We denote by $\overline{C}$, its smooth compactification and set $C_\infty = \overline{C} \smallsetminus C$ viewed as a reduced subscheme of $\overline{C}$. This induces a morphism of quivers

$$\overline{(-)}\colon \mathrm{Crv} \longrightarrow \overline{\mathrm{M}}\mathrm{Crv}, \quad (C, Y) \longmapsto (\overline{C}, Y, C_\infty).$$

Remark 3.13. Let $f\colon (C, Y) \to (C', Y')$ be a morphism in Crv. Then f extends to a morphism $\overline{f}\colon \overline{C} \to \overline{C}'$ between smooth compactifications. This morphism satisfies $f(\overline{C} \smallsetminus C_\infty) \subset \overline{C}' \smallsetminus C'_\infty$ and since $f(Y) \subset Y'$, we have

$$Y = Y_{\mathrm{red}} \leqslant (f^*(Y'_{\mathrm{red}}))_{\mathrm{red}} \leqslant f^*(Y'_{\mathrm{red}}) = f^*(Y').$$

Therefore, $\overline{f}$ defines a morphism between $(\overline{C}, Y, C_\infty)$ and $(\overline{C}', Y', C'_\infty)$ in $\overline{\mathrm{M}}\mathrm{Crv}$. Similarly, we have another morphism of quivers

$$\mathrm{Crv} \longrightarrow \underline{\mathrm{M}}\mathrm{Crv}, \quad (C, Y) \longmapsto (\overline{C}, C_\infty, Y). \tag{3.8}$$

Since, by definition $H^1_{\mathrm{dR}} = \mathbf{H}^1_{\mathrm{dR}} \circ \overline{(-)}$ as representations of the quiver $\mathrm{Crv}^{\mathrm{op}}$, the universal property of Nori's construction [12, Theorem 2] ensures the existence of a $\mathbb{Q}$-linear exact faithful functor $I_{\mathrm{ECM}}\colon \mathrm{ECM}_1^{\mathrm{dR}} \to \mathrm{ECMM}_1$ and isomorphisms of functors

$$I_{\mathrm{ECM}} \circ \overline{H}^1_{\mathrm{dR}} \longrightarrow \overline{\mathbf{H}}^1_{\mathrm{dR}} \circ \overline{(-)}, \quad F^a_{\mathrm{dR}} \circ I_{\mathrm{ECM}} \longrightarrow F_{\mathrm{dR}}$$

that makes the square

$$\begin{array}{ccc} F^a_{\mathrm{dR}} \circ I_{\mathrm{ECM}} \circ \overline{H}^1_{\mathrm{dR}} & \longrightarrow & F^a_{\mathrm{dR}} \circ \overline{\mathbf{H}}^1_{\mathrm{dR}} \circ \overline{(-)} \\ \downarrow & & \downarrow{\scriptstyle =} \\ F_{\mathrm{dR}} \circ \overline{H}^1_{\mathrm{dR}} & \xrightarrow{=} & H^1_{\mathrm{dR}} \end{array}$$

commutative.

Let us consider now the $\mathbb{Q}$-linear Abelian category $\mathscr{B}$ defined as follows. An object in $\mathscr{B}$ is a tuple (V, W, a, b) where V, W are finite-dimensional k-vector spaces and $a\colon V \to W$ and $b\colon W \to V$ are morphisms of k-vector spaces such that $b \circ a = \mathrm{Id}$. A morphism $(V, W, a, b) \to (V', W', a', b')$ in $\mathscr{B}$ is simply a pair of k-linear morphisms $(f\colon V \to V', g\colon W \to W')$ such that $a' \circ f = g \circ a$ and $b' \circ g = f \circ b$. Note that by construction, we have two $\mathbb{Q}$-linear exact functors obtained by projection on the first and second factor composed with the forgetful functor $\Pi_1: \mathscr{B} \to \mathrm{mod}(\mathbb{Q})$, $\Pi_2: \mathscr{B} \to \mathrm{mod}(\mathbb{Q})$ and that, moreover, Π_2 is faithful.

Let X be a smooth proper k-curve and Y, Z be closed subschemes of X. Recall from Proposition 3.7 that there are two morphisms

$$a: \mathbf{H}^1_{\mathrm{dR}}(X, Y_{\mathrm{red}}, Z_{\mathrm{red}}) \longrightarrow \mathbf{H}^1_{\mathrm{dR}}(X, Y, Z) \tag{3.9}$$

and

$$b: \mathbf{H}^1_{\mathrm{dR}}(X, Y, Z) \longrightarrow \mathbf{H}^1_{\mathrm{dR}}(X, Y_{\mathrm{red}}, Z_{\mathrm{red}}) \tag{3.10}$$

such that $b \circ a = \mathrm{id}$. We may therefore consider the representation

$$\begin{aligned} \mathbf{H}^1_{\mathrm{dR},\mathscr{B}}: \overline{\mathrm{M}}\mathrm{Crv}^{\mathrm{op}} &\longrightarrow \mathscr{B} \\ (X, Y, Z) &\longmapsto (\mathbf{H}^1_{\mathrm{dR}}(X, Y_{\mathrm{red}}, Z_{\mathrm{red}}), \mathbf{H}^1_{\mathrm{dR}}(X, Y, Z), a, b), \end{aligned}$$

where a and b are the morphisms (3.9) and (3.10). By construction $\Pi_2 \circ \mathbf{H}^1_{\mathrm{dR},\mathscr{B}} = \mathbf{H}^1_{\mathrm{dR}}$ and from (3.7) we have $\Pi_1 \circ \mathbf{H}^1_{\mathrm{dR},\mathscr{B}} = H^1_{\mathrm{dR}} \circ (-)_{\text{ét}}$, where $(-)_{\text{ét}}$ is the morphism of quivers

$$(-)_{\text{ét}}: \overline{\mathrm{M}}\mathrm{Crv} \longrightarrow \mathrm{Crv}, \quad (X, Y, Z) \longmapsto (X \smallsetminus Z_{\mathrm{red}}, Y_{\mathrm{red}}).$$

By [12, Theorem 2], there exists a faithful exact $\mathbb{Q}$-linear functor $F^a_{\mathscr{B}}: \mathrm{ECMM}_1 \to \mathscr{B}$ and two isomorphisms of functors $\gamma: F^a_{\mathrm{dR}} \circ \overline{\mathbf{H}}^1_{\mathrm{dR}} \to \mathbf{H}^1_{\mathrm{dR},\mathscr{B}}$, $\delta: \Pi_2 \circ F^a_{\mathscr{B}} \to F^a_{\mathrm{dR}}$ such that

$$\begin{array}{ccc} \Pi_2 \circ F^a_{\mathscr{B}} \circ \overline{\mathbf{H}}^1_{\mathrm{dR}} & \xrightarrow{\Pi_2 \star \gamma} & \Pi_2 \circ \mathbf{H}^1_{\mathrm{dR},\mathscr{B}} \\ \Big\downarrow {\scriptstyle \delta \star \overline{\mathbf{H}}^1_{\mathrm{dR}}} & & \Big\downarrow {\scriptstyle =} \\ F^a_{\mathrm{dR}} \circ \overline{\mathbf{H}}^1_{\mathrm{dR}} & \xrightarrow{\quad = \quad} & \mathbf{H}^1_{\mathrm{dR}} \end{array}$$

is commutative.

We may apply [12, Proposition 6.6] to Π_1 to obtain the existence of a $\mathbb{Q}$-linear exact and faithful functor $\Pi_{\mathrm{ECM}}: \mathrm{ECMM}_1 \to \mathrm{ECM}_1^{\mathrm{dR}}$ and isomorphisms of functors

$$\Pi_1 \circ F^a_{\mathscr{B}} \longrightarrow F_{\mathrm{dR}} \circ \Pi_{\mathrm{ECM}}, \quad \Pi_{\mathrm{ECM}} \circ \overline{\mathbf{H}}^1_{\mathrm{dR}} \longrightarrow \overline{H}^1_{\mathrm{dR}} \circ (-)_{\text{ét}}$$

such that the diagram

$$\begin{array}{ccccc} \Pi_1 \circ F^a_{\mathscr{B}} \circ \overline{\mathbf{H}}^1_{\mathrm{dR}} & \longrightarrow & F_{\mathrm{dR}} \circ \Pi_{\mathrm{ECM}} \circ \overline{\mathbf{H}}^1_{\mathrm{dR}} & \longrightarrow & F_{\mathrm{dR}} \circ \overline{H}^1_{\mathrm{dR}} \circ (-)_{\text{ét}} \\ \Big\downarrow {\scriptstyle \Pi_1 \star \gamma} & & & \swarrow {\scriptstyle =} & \\ \Pi_1 \circ \mathbf{H}^1_{\mathrm{dR},\mathscr{B}} & \xrightarrow{\quad = \quad} & H^1_{\mathrm{dR}} \circ (-)_{\text{ét}} & & \end{array}$$

commutes. (See [12, Proposition 6.7] for uniqueness.)

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Remark 3.14. Let $I_{\mathscr{B}}\colon \mathsf{mod}(k) \to \mathscr{B}$ be the functor that maps V to $(V, V, \mathrm{Id}, \mathrm{Id})$. The diagram

$$\begin{array}{ccccc} \mathrm{ECM}_1^{\mathrm{dR}} & \xrightarrow{F_{\mathrm{dR}}} & \mathsf{mod}(k) & \searrow & \\ \downarrow{\scriptstyle I_{\mathrm{ECM}}} & & \downarrow{\scriptstyle I_{\mathscr{B}}} & & \\ \mathrm{ECMM}_1 & \xrightarrow{F^a_{\mathscr{B}}} & \mathscr{B} & \xrightarrow{\Pi_2} & \mathsf{mod}(\mathbb{Q}) \\ & & \underset{F^a_{\mathrm{dR}}}{\curvearrowright} & & \end{array}$$

is commutative up to isomorphisms of functors.

Proposition 3.15 *The composition $\Pi_{\mathrm{ECM}} \circ I_{\mathrm{ECM}}$ is isomorphic to the identity. Moreover, the functor I_{ECM} is fully faithful.*

Proof Since $(-)_{\text{ét}} \circ \overline{(-)}$ is the identity on the quiver Crv, the first assertion is an immediate consequence of the uniqueness statement [12, Proposition 6.7]. Let M, N be objects in $\mathrm{ECM}_1^{\mathrm{dR}}$ and $\alpha\colon I_{\mathrm{ECM}}(M) \to I_{\mathrm{ECM}}(N)$ be a morphism in ECMM_1. Note that for such an α, we have $\Pi_1 \circ F^a_{\mathscr{B}}(\alpha) = \Pi_2 \circ F^a_{\mathscr{B}}(\alpha) = F^a_{\mathrm{dR}}(\alpha)$. Let $\beta = \Pi_{\mathrm{ECM}}(\alpha)$. It is enough to show that $I_{\mathrm{ECM}}(\beta) = \alpha$ and since F^a_{dR} is faithful, it is enough to show this equality after applying F^a_{dR}. We have

$$F^a_{\mathrm{dR}}(I_{\mathrm{ECM}}(\beta)) = F_{\mathrm{dR}}(\beta) = F_{\mathrm{dR}}(\Pi_{\mathrm{ECM}}(\alpha)) = \Pi_1 \circ F^a_{\mathscr{B}}(\alpha) = \Pi_2 \circ F^a_{\mathscr{B}}(\alpha) = F^a_{\mathrm{dR}}(\alpha).$$

This concludes the proof. ■

4 Review of Laumon 1-motives and Their de Rham Realization

In this section, we recall necessary material introducing notations [4, 14].

4.1 Recall that we are working over a field k of characteristic zero. Let Aff be the category of affine schemes over k, and let $\mathscr{S}$ be the category of sheaves of Abelian groups on the fppf site on Aff. For $F \in \mathscr{S}$, we abbreviate $F(R) := F(\operatorname{Spec} R)$ for a k-algebra R, and we put $\mathrm{Lie}(F) := \ker[F(k[\epsilon]/(\epsilon^2)) \to F(k)]$.

4.2 We shall consider full subcategories of $\mathscr{S}$.

Let $\mathscr{S}_0$ be the full subcategory of $\mathscr{S}$ consisting of objects that are represented by connected commutative algebraic groups G over k [14, (4.1)]. We identify such a G with the object in $\mathscr{S}$ represented by G.

Let $\mathscr{S}_l$ be the full subcategory of $\mathscr{S}_0$ consisting of linear commutative algebraic groups over k. We write $\mathscr{S}_{\mathrm{uni}}$ (resp. $\mathscr{S}_{\mathrm{mul}}$) for the full subcategory of $\mathscr{S}_l$ consisting of unipotent (resp. multiplicative) groups. For any $L \in \mathscr{S}_l$, there is a canonical decomposition $L \cong L_{\mathrm{uni}} \times L_{\mathrm{mul}}$, where $L_{\mathrm{uni}} \in \mathscr{S}_{\mathrm{uni}}$ and $L_{\mathrm{mul}} \in \mathscr{S}_{\mathrm{mul}}$. The functor $\mathscr{S}_{\mathrm{uni}} \to \mathsf{mod}(k)$, $L \mapsto L(k)$ is an equivalence by which we often identify them.

Let $\mathscr{S}_a$ be the full subcategory of $\mathscr{S}_0$ consisting of Abelian varieties. Recall that any $G \in \mathscr{S}_0$ canonically fits in an extension $0 \to G_l \to G \to G_{\mathrm{ab}} \to 0$, where $G_{\mathrm{ab}} \in \mathscr{S}_a$ and $G_l \in \mathscr{S}_l$. We ease the notation by putting $G_{\mathrm{uni}} = (G_l)_{\mathrm{uni}}$ and $G_{\mathrm{mul}} = (G_l)_{\mathrm{mul}}$. We call $G_{\mathrm{sa}} := G/G_{\mathrm{uni}}$ the *semi-Abelian part* of G.

Let $\mathscr{S}_{-1}$ be the full subcategory of $\mathscr{S}$ consisting of formal groups over k without torsion [14, (4.2)]. We write $\mathscr{S}_{\mathrm{inf}}$ (resp. $\mathscr{S}_{\text{ét}}$) for the full subcategory of $\mathscr{S}_{-1}$ consisting of connected (resp. étale) formal groups. For any $F \in \mathscr{S}_{-1}$, there is a canonical decomposition $F \cong F_{\mathrm{inf}} \times F_{\text{ét}}$, where $F_{\mathrm{inf}} \in \mathscr{S}_{\mathrm{inf}}$ and $F_{\text{ét}} \in \mathscr{S}_{\text{ét}}$. The functor $\mathrm{Lie}\colon \mathscr{S}_{\mathrm{inf}} \to \mathsf{mod}(k)$ is an equivalence, with a quasi-inverse $V \mapsto V \otimes_k \widehat{\mathbf{G}}_a$, where $\widehat{\mathbf{G}}_a$ denotes the formal completion of $\mathbf{G}_a$.

4.3 Following [14, (5.1.1)], define a *Laumon* 1*-motive* to be a complex $[F \to G]$ in $\mathscr{S}$ such that $F \in \mathscr{S}_{-1}$ (placed at degree −1) and $G \in \mathscr{S}_0$ (placed at degree 0). We denote the category of Laumon 1-motives over k by $\mathscr{M}_1^a$ (or by $\mathscr{M}_1^a(k)$ if we wish to stress the dependency on k). There is an equivalence $(\mathscr{M}_1^a)^{\mathrm{op}} \to \mathscr{M}_1^a$, called the *Cartier duality*.

4.4 A Laumon 1-motive $[F \to G]$ is called a *Deligne* 1*-motive* if $F_{\mathrm{inf}} = 0$ and $G_{\mathrm{uni}} = 0$. Denote by $\mathscr{M}_1$ the full subcategory of $\mathscr{M}_1^a$ consisting of Deligne 1-motives. Along with this, we denote by $\mathscr{M}_1^{\mathrm{uni}}$ (resp. $\mathscr{M}_1^{\mathrm{inf}}$) the essential image of an obvious full faithful functor

$$\mathscr{S}_{\mathrm{uni}} \longrightarrow \mathscr{M}_1^a, \quad U \longmapsto U[0] := [0 \to U],$$
$$(\text{resp.} \quad \mathscr{S}_{\mathrm{inf}} \longrightarrow \mathscr{M}_1^a, \quad F \longmapsto F[1] := [F \to 0]).$$

4.5 Let $M = [F \to G] \in \mathscr{M}_1^a$. We define a filtration on M by

$$\mathrm{fil}^0_{\mathscr{M}} M = M \supset \mathrm{fil}^1_{\mathscr{M}} M = [F_{\text{ét}} \to G] \supset \mathrm{fil}^2_{\mathscr{M}} M = [0 \to G_{\mathrm{uni}}] \supset \mathrm{fil}^3_{\mathscr{M}} M = 0.$$

We put $\mathrm{Gr}^i_{\mathscr{M}} M := \mathrm{fil}^i_{\mathscr{M}} M / \mathrm{fil}^{i+1}_{\mathscr{M}} M$, so that

$$\mathrm{Gr}^0_{\mathscr{M}} M \cong F_{\mathrm{inf}}[1], \quad \mathrm{Gr}^1_{\mathscr{M}} M \cong [F_{\text{ét}} \to G_{\mathrm{sa}}] =: M_{\mathrm{Del}}, \quad \mathrm{Gr}^2_{\mathscr{M}} M = \mathrm{fil}^2_{\mathscr{M}} M = G_{\mathrm{uni}}[0].$$

We have defined functors

$$\mathrm{Gr}^0_{\mathscr{M}}\colon \mathscr{M}_1^a \longrightarrow \mathscr{M}_1^{\mathrm{inf}}, \quad \mathrm{Gr}^1_{\mathscr{M}}\colon \mathscr{M}_1^a \longrightarrow \mathscr{M}_{1,\mathrm{Del}}, \quad \mathrm{Gr}^2_{\mathscr{M}}\colon \mathscr{M}_1^a \longrightarrow \mathscr{M}_1^{\mathrm{uni}}.$$

Note that all these functors are exact, and that $\mathrm{Gr}^0_{\mathscr{M}}$ (resp. $\mathrm{Gr}^2_{\mathscr{M}}$) is a left (resp. right) adjoint to the inclusion $\mathscr{M}_1^{\mathrm{inf}} \hookrightarrow \mathscr{M}_1^a$ (resp. $\mathscr{M}_1^{\mathrm{uni}} \hookrightarrow \mathscr{M}_1^a$). Following [4], we also define (recall that $G_{\mathrm{sa}} = G/G_{\mathrm{uni}}$) $M_\times := M/\mathrm{fil}^2_{\mathscr{M}} M = [F \to G_{\mathrm{sa}}]$. The functor $M \mapsto M_\times$ is a left adjoint of the inclusion $\{G \in \mathscr{M}_1^a \mid G_{\mathrm{uni}} = 0\} \hookrightarrow \mathscr{M}_1^a$.

4.6 We call $M = [F \to G] \in \mathscr{M}_1^a$ *unipotent free* if $G_{\mathrm{uni}} = 0$. For such M, it was shown [4, (2.2.3)] that there is an extension $M^\natural = [F \to G^\natural] \in \mathscr{M}_1^a$ of M by $\mathrm{Ext}_{\mathscr{M}_1^a}(M, \mathbf{G}_a)^*$ such that it is universal among extensions of M by an object of $\mathscr{M}_1^{\mathrm{uni}}$. (Here * denotes k-linear dual. Recall that by convention we identify a k-vector space with an object of $\mathscr{S}_{\mathrm{uni}}$.)

4.7 Now take any $M = [u\colon F \to G] \in \mathscr{M}_1^a$. Note that $M_\times$ and M_{Del} (introduced in §4.5) are unipotent free. By [4, (2.3.2)], an exact sequence $0 \to M_{\mathrm{Del}} \to M_\times \to F_{\mathrm{inf}}[1] \to 0$ induces an exact sequence $0 \to (M_{\mathrm{Del}})^\natural \to (M_\times)^\natural \to \vec{F}_{\mathrm{inf}} \to 0$, where

$$\vec{F}_{\mathrm{inf}} := [F_{\mathrm{inf}} \to \mathrm{Lie}(F_{\mathrm{inf}})] \in \mathscr{M}_1^a.$$

Let us write $(M_{\mathrm{Del}})^\natural = [u^\natural_{\mathrm{Del}}\colon F_{\text{ét}} \to G^\natural_{\mathrm{Del}}]$, $(M_\times)^\natural = [u^\natural_\times\colon F \to G^\natural_\times]$. Then we get an exact sequence

$$0 \to \mathrm{Lie}(G^\natural_{\mathrm{Del}}) \to \mathrm{Lie}(G^\natural_\times) \to \mathrm{Lie}(F_{\mathrm{inf}}) \to 0, \tag{4.1}$$

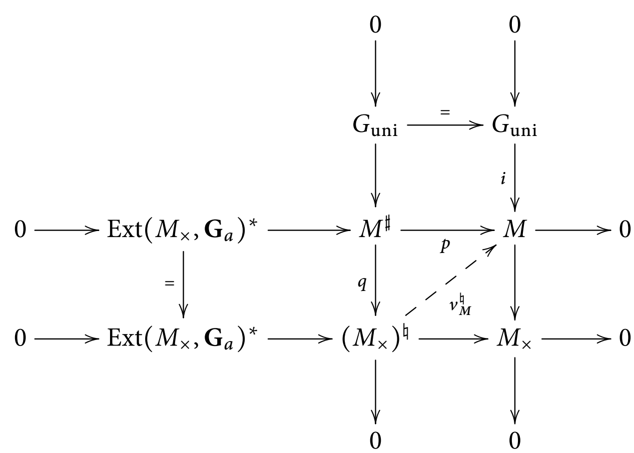
which admits a canonical splitting given by $\mathrm{Lie}(u^\natural_\times)$.

We also need the following remark. The universality of $(M_\times)^\natural$ induces maps v_M and $v^\natural_M$ in the following commutative diagram with exact rows.

$$\begin{CD} 0 @>>> \mathrm{Ext}(M_\times, \mathbf{G}_a)^* @>>> (M_\times)^\natural @>>> M_\times @>>> 0\\ @. @V{v_M}VV @V{v^\natural_M}VV @V{=}VV @.\\ 0 @>>> G_{\mathrm{uni}} @>>> M @>>> M_\times @>>> 0 \end{CD} \tag{4.2}$$

4.8 The *sharp extension* $M^\sharp = [F \to G^\sharp]$ of $M = [F \to G] \in \mathscr{M}_1^a$ is defined to be the pull-back of $(M_\times)^\natural$ by the canonical surjection $M \to M_\times$. (If M is unipotent free, then $M^\sharp = M^\natural$.) There is a commutative diagram with exact rows and columns:

(4.3)



Note that the dotted arrow $v^\natural_M$ makes the lower right triangle commutative by (4.2), but it is *not* necessarily the case for the upper left triangle. The middle vertical exact sequence in (4.3) admits a canonical splitting $s\colon M^\sharp \to G_{\mathrm{uni}}$ characterized by $i \circ s = p - (v^\natural_M \circ q)$. Hence there also is an exact sequence

$$0 \longrightarrow \mathrm{Lie}(G_{\mathrm{uni}}) \longrightarrow \mathrm{Lie}(G^\sharp) \longrightarrow \mathrm{Lie}(G^\natural_\times) \longrightarrow 0$$

equipped with a canonical splitting. Combined with (4.1), we obtain a canonical decomposition

$$\mathrm{Lie}(G^\sharp) \cong \mathrm{Lie}(G^\natural_{\mathrm{Del}}) \oplus \mathrm{Lie}(F_{\mathrm{inf}}) \oplus \mathrm{Lie}(G_{\mathrm{uni}}). \tag{4.4}$$

4.9 Following [4, (3.2.1)], we call an exact functor

$$\mathsf{R}_{\mathrm{dR}}\colon \mathscr{M}_1^a \longrightarrow \mathsf{mod}(k), \quad \mathsf{R}_{\mathrm{dR}}([F \to G]) := \mathrm{Lie}(G^\sharp)$$

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the *sharp de Rham realization*. By (4.4), we have a canonical decomposition

(4.5) $$\mathsf{R}_{\mathrm{dR}}(M) \cong \mathsf{R}_{\mathrm{dR}}(M_{\mathrm{Del}}) \oplus \mathrm{Lie}(F_{\mathrm{inf}}) \oplus \mathrm{Lie}(G_{\mathrm{uni}})$$

for any $M = [F \to G] \in \mathscr{M}_1^a$.

4.10 Let $\mathscr{M}_{1,\mathbb{Q}} := \mathscr{M}_1 \otimes_{\mathbb{Z}} \mathbb{Q}$ be the $\mathbb{Q}$-linear Abelian category of Deligne 1-isomotives (1.1). Recall from §3.5 that $\mathrm{EHM}_1^{\mathbb{Q}}$ is the universal $\mathbb{Q}$-linear category associated with the Betti homology functor (3.5) (with $K = \mathbb{Q}$). L. Barbieri-Viale and J. Ayoub [1] showed the following important result, which will be a key ingredient in the proof of our main result. (Actually, they proved a stronger statement with integral coefficients.)

Theorem 4.1 *We have an equivalence of* $\mathbb{Q}$*-linear Abelian categories* $\mathrm{EHM}_1^{\mathbb{Q}} \xrightarrow{\sim} \mathscr{M}_{1,\mathbb{Q}}$.

This functor is induced by a functor $\mathrm{Crv} \to \mathscr{M}_1$ via universality (see Remark 5.3). We will construct its modulus version in the next section.

5 1-motives of a Curve With Modulus and the Main Theorem

In this section, we associate a Laumon 1-motive $\mathsf{LM}(X,Y,Z) \in \mathscr{M}_1^a$ with a smooth proper k-curve X and two effective divisors Y, Z on X with disjoint support. We shall see functorial properties that yield two functors

$$\overline{\mathsf{L}}\mathsf{M}\colon \overline{\mathrm{M}}\mathrm{Crv} \longrightarrow \mathscr{M}_1^a, \quad \underline{\mathsf{L}}\mathsf{M}\colon \underline{\mathrm{M}}\mathrm{Crv} \longrightarrow \mathscr{M}_1^a.$$

5.1 Let X be a smooth proper k-curve and Y an effective divisors on X. We denote by $J(X,Y) \in \mathscr{S}_0$ the *generalized Jacobian* of X with modulus Y in the sense of Rosenlicht and Serre [18, 21]. Recall that $J(X,Y)$ is the connected component of the Picard scheme $\underline{\mathrm{Pic}}(X_Y)$ of a proper k-curve X_Y that is obtained by collapsing Y into a single (usually singular) point [21, Chapter IV, §3–4]. It can also be defined as the Albanese variety attached to a pair (X,Y) [19, Example 2.34], [20, §3.3].

Let X' be another smooth proper k-curve and Y' an effective divisor on it. Let $f\colon X \to X'$ be a k-morphism. When $Y \leqslant f^*Y'$, we have a pull-back $f^*\colon J(X',Y') \to J(X,Y)$ deduced by the functoriality of the Picard scheme. When

$$Y - Y_{\mathrm{red}} \geqslant f^*(Y' - Y'_{\mathrm{red}}), \quad Y_{\mathrm{red}} \geqslant (f^*Y')_{\mathrm{red}},$$

we have a push-forward $f_*\colon J(X,Y) \to J(X',Y')$ by [20, Proposition 3.22].

Lemma 5.1 *There exists a canonical isomorphism (Proposition 3.3)*

(5.1) $$\mathrm{Lie}\, J(X,Y)_{\mathrm{uni}} \cong U(X,Y).$$

Proof If $Y = \varnothing$, then $J(X,Y)$ is an Abelian variety so that $J(X,Y)_{\mathrm{uni}} = 0$, and hence the lemma holds. We suppose $Y \neq \varnothing$ in what follows. Consider an exact sequence of sheaves on X: $0 \to \mathfrak{I}_Y \to \mathfrak{I}_{Y_{\mathrm{red}}} \to \mathfrak{I}_{Y_{\mathrm{red}}}/\mathfrak{I}_Y \to 0$. We have $H^0(X, \mathfrak{I}_{Y_{\mathrm{red}}}) = 0$ since Y is a non-empty effective divisor. It follows that

$$H^0(X, \mathfrak{I}_{Y_{\mathrm{red}}}/\mathfrak{I}_Y) \cong \ker(H^1(X,\mathfrak{I}_Y) \to H^1(X, \mathfrak{I}_{Y_{\mathrm{red}}})).$$

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By [21, Chapter V, §10, Proposition 5], there are canonical isomorphisms

$$H^1(X, \mathfrak{I}_Y) \cong \operatorname{Lie} J(X, Y), \quad H^1(X, \mathfrak{I}_{Y_{\mathrm{red}}}) \cong \operatorname{Lie} J(X, Y_{\mathrm{red}}).$$

Now the lemma follows from an exact sequence

$$0 \longrightarrow \operatorname{Lie} J(X, Y)_{\mathrm{uni}} \longrightarrow \operatorname{Lie} J(X, Y) \longrightarrow \operatorname{Lie} J(X, Y)_{\mathrm{sa}} \longrightarrow 0$$

and a canonical isomorphism $J(X, Y)_{\mathrm{sa}} = J(X, Y_{\mathrm{red}})$. ∎

5.2 Let X be a smooth proper k-curve and Z an effective divisor on X. We construct an object $F(X, Z) := F(X, Z)_{\mathrm{inf}} \times F(X, Z)_{\text{ét}} \in \mathscr{S}_{-1}$ as follows. First, we define

$$F(X, Z)_{\text{ét}} := \ker[\pi_0(Z) \longrightarrow \pi_0(X)],$$

where the map is the one induced by the closed immersion $Z \to X$. Here, for any k-variety V, we define $\pi_0(V) \in \mathscr{S}_{-1}$ by declaring $\pi_0(V)(U)$ is the free Abelian group on the set of connected components of $U \times_k V$ for $U \in \mathrm{Aff}$. This depends only on the reduced part of V. Next we define (Proposition 3.3, see also [13, §5.3])

$$F(X, Z)_{\mathrm{inf}} := V(X, Z) \otimes_k \widehat{\mathbf{G}}_a. \tag{5.2}$$

Let X' be another smooth proper k-curve and Z' an effective divisor on it. Let $f: X \to X'$ be a k-morphism. There is a pull-back $f^*: F(X', Z') \to F(X, Z)$ (resp. a push-forward $f_*: F(X, Z) \to F(X', Z')$) when $Z - Z_{\mathrm{red}} \geqslant f^*(Z' - Z'_{\mathrm{red}})$ and $Z_{\mathrm{red}} \geqslant (f^*Z')_{\mathrm{red}}$ (resp. $Z \leqslant f^*Z'$). On the infinitesimal (resp. étale) part, they are defined by Corollary 3.5 (resp. pull-back and push-forward of cycles).

5.3 We recall Russell's results [19, §2.1]. Let V be a Noetherian reduced scheme. Define $\underline{\mathrm{Div}}_V \in \mathscr{S}$ to be the sheaf that associates with $\operatorname{Spec}(R) \in \mathrm{Aff}$ the group of all Cartier divisors on $V \otimes_k R$ generated locally on $\operatorname{Spec}(R)$ by effective Cartier divisors which are flat over R. There is a canonical "class" map

$$\mathrm{cl}: \underline{\mathrm{Div}}_V \longrightarrow \underline{\mathrm{Pic}}_V \tag{5.3}$$

to the Picard scheme $\underline{\mathrm{Pic}}_V$ of V. Let $\underline{\mathrm{Div}}_V^0$ denote the inverse image under cl of the connected component $\underline{\mathrm{Pic}}_X^0$ of $\underline{\mathrm{Pic}}_X$. We have $\underline{\mathrm{Div}}_V^0(k) = H^0(V, \mathscr{K}_V^\times / \mathscr{O}_V^\times)$ (the group of Cartier divisors on V) and $\operatorname{Lie}(\underline{\mathrm{Div}}_V^0) = H^0(V, \mathscr{K}_V / \mathscr{O}_V)$, where $\mathscr{K}_V$ is the sheaf of the total ring of fractions of $\mathscr{O}_V$. In [19, Proposition 2.13] it was shown that for any $F \in \mathscr{S}_{-1}$ and a pair of maps $a_{\mathrm{inf}}: \operatorname{Lie}(F) \to \operatorname{Lie}(\underline{\mathrm{Div}}_V^0)$, and $a_{\text{ét}}: F(k) \to \underline{\mathrm{Div}}_V^0(k)$, there exists a unique map

$$a = (a_{\mathrm{inf}}, a_{\text{ét}}): F \longrightarrow \underline{\mathrm{Div}}_V \tag{5.4}$$

that induces a map a_{inf} (resp. $a_{\text{ét}}$) via Lie (resp. by taking sections over $\operatorname{Spec} k$).

Let X be a smooth proper k-curve and let Y, Z be two effective divisors on X with disjoint support. We apply the above argument to $V = X_Y$, where X_Y is the curve we discussed in §5.1. Since Y and Z are disjoint, we may identify Z as a closed subscheme of X_Y. We define

$$\begin{aligned}\tau'_{\mathrm{inf}}: \operatorname{Lie}(F(X, Z)_{\mathrm{inf}}) = H^0(X, \mathfrak{I}_Z^{-1}\mathfrak{I}_{Z_{\mathrm{red}}}/\mathscr{O}_X) = H^0(X_Z, \mathfrak{I}_Z^{-1}\mathfrak{I}_{Z_{\mathrm{red}}}/\mathscr{O}_{X_Y}) \\ \longrightarrow H^0(X, \mathscr{K}_{X_Y}/\mathscr{O}_{X_Y}) = \operatorname{Lie}(\underline{\mathrm{Div}}_{X_Y}^0)\end{aligned}$$

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to be the map induced by the inclusion $\mathcal{I}_Z^{-1}\mathcal{I}_{Z_{\mathrm{red}}} \subset \mathcal{K}_{X_Y}$. Also, we define

$$\pi_0(Z)(k) = Z_0(Z) \longrightarrow \underline{\mathrm{Div}}_{X_Y}(k) = \mathrm{Div}(X_Y)$$

by sending $D \in Z_0(Z)$ to $\mathcal{O}_{X_Y}(D)$. It restricts to

$$\tau'_{\text{ét}}: F(X,Z)_{\text{ét}}(k) = \ker[\pi_0(Z) \to \pi_0(X)] \longrightarrow \underline{\mathrm{Div}}^0_{X_Y}(k).$$

Using them, we define

$$\tau(X,Y,Z) := \mathrm{cl} \circ (\tau'_{\mathrm{inf}},\ \tau'_{\text{ét}}): F(X,Z) \longrightarrow \underline{\mathrm{Pic}}^0_{X_Y} = J(X,Y),$$

where we used the notations from (5.3) and (5.4). We then define a Laumon 1-motive attached to (X,Y,Z) by

$$\mathsf{LM}(X,Y,Z) := [F(X,Z) \xrightarrow{\tau(X,Y,Z)} J(X,Y)] \in \mathcal{M}_1^a. \tag{5.5}$$

From this definition it is evident that

$$\mathsf{LM}(X,Y,Z)_{\mathrm{Del}} = \mathsf{LM}(X,Y_{\mathrm{red}},Z_{\mathrm{red}}). \tag{5.6}$$

5.4 Let X' be another smooth proper k-curve and let Y', Z' be two effective divisors on X' with disjoint support. Let $f: X \to X'$ be a k-morphism. If f defines a morphism in $\overline{\mathrm{M}}\mathrm{Crv}$, then the square

$$\begin{array}{ccc} F(X',Z') & \xrightarrow{\tau(X',Y',Z')} & J(X',Y') \\ \downarrow f^* & & \downarrow f^* \\ F(X,Z) & \xrightarrow{\tau(X,Y,Z)} & J(X,Y) \end{array}$$

commutes. Similarly if f defines a morphism in $\underline{\mathrm{M}}\mathrm{Crv}$, then the square

$$\begin{array}{ccc} F(X,Z) & \xrightarrow{\tau(X,Y,Z)} & J(X,Y) \\ \downarrow f_* & & \downarrow f_* \\ F(X',Z') & \xrightarrow{\tau(X',Y',Z')} & J(X',Y') \end{array}$$

commutes. This enables us to make the following definition.

Definition 5.2 We define a functor $\overline{\mathsf{L}}\mathsf{M}: \overline{\mathrm{M}}\mathrm{Crv}^{\mathrm{op}} \to \mathcal{M}_1^a$ (resp. $\underline{\mathsf{L}}\mathsf{M}: \underline{\mathrm{M}}\mathrm{Crv} \to \mathcal{M}_1^a$) by setting $\overline{\mathsf{L}}\mathsf{M}(X,Y,Z) = \underline{\mathsf{L}}\mathsf{M}(X,Y,Z) = \mathsf{LM}(X,Y,Z)$, and $\overline{\mathsf{L}}\mathsf{M}(f) = f^*$ (resp. $\underline{\mathsf{L}}\mathsf{M}(f) = f_*$) for a morphism f in $\overline{\mathrm{M}}\mathrm{Crv}$ (resp. in $\underline{\mathrm{M}}\mathrm{Crv}$).

Remark 5.3. The composition of $\underline{\mathsf{L}}\mathsf{M}$ with $\mathrm{Crv} \to \underline{\mathrm{M}}\mathrm{Crv}$ from (3.8) factors through $\mathcal{M}_1$ (see §4.4). This induces the functor in Theorem 4.1 via universality.

Proposition 5.4 *There is an isomorphism of functors* $\mathsf{R}_{\mathrm{dR}} \circ \overline{\mathsf{L}}\mathsf{M} \to \mathbf{H}^1_{\mathrm{dR}}$.

Proof Let $(X,Y,Z) \in \overline{\mathrm{M}}\mathrm{Crv}$. By (4.5), (5.1), (5.2), and (5.6), we have

$$\mathsf{R}_{\mathrm{dR}} \circ \overline{\mathsf{L}}\mathsf{M}(X,Y,Z) = \mathsf{R}_{\mathrm{dR}}(X,Y_{\mathrm{red}},Z_{\mathrm{red}}) \oplus U(X,Y) \oplus V(X,Z).$$

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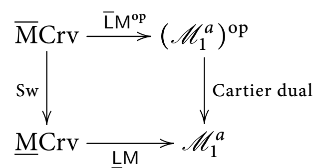
Moreover, by [5, Corollary 2.6.4] there is a canonical isomorphism

$$\mathsf{R}_{\mathrm{dR}}(X, Y_{\mathrm{red}}, Z_{\mathrm{red}}) \cong \mathbf{H}^1_{\mathrm{dR}}(X, Y_{\mathrm{red}}, Z_{\mathrm{red}}).$$

Now the proposition follows from (3.4). ■

Remark 5.5. There is also an isomorphism of functors $\mathsf{R}_{\mathrm{dR}} \circ \underline{\mathsf{L}}\mathsf{M} \to {}^t\mathbf{H}^1_{\mathrm{dR}}$ considered as functors $\underline{\mathrm{M}}\mathrm{Crv} \to \mathsf{mod}(k)$, see (3.3).

Remark 5.6. (This remark will not be used in the sequel.) For any $(X, Y, Z) \in \overline{\mathrm{M}}\mathrm{Crv}$, we find that $\mathsf{LM}(X, Y, Z)$ and $\mathsf{LM}(X, Z, Y)$ are Cartier dual to each other. In other words, using a functor $\mathrm{Sw}\colon \underline{\mathrm{M}}\mathrm{Crv} \to \overline{\mathrm{M}}\mathrm{Crv}$ defined by $\mathrm{Sw}(X, Y, Z) = (X, Z, Y)$, we get a commutative diagram.



5.5 Let $\mathscr{M}^a_{1,\mathbb{Q}} := \mathscr{M}^a_1 \otimes_{\mathbb{Z}} \mathbb{Q}$ be the $\mathbb{Q}$-linear Abelian category of Laumon 1-isomotives (1.1).

Proposition 5.7 *Any Laumon* 1*-motive* $M = [F \xrightarrow{u} G]$ *is a quotient in* $\mathscr{M}^a_{1,\mathbb{Q}}$ *of* $\overline{\mathsf{L}}\mathsf{M}(X, Y, Z)$ *for some object* (X, Y, Z) *of* $\overline{\mathrm{M}}\mathrm{Crv}$.

Remark 5.8. If M is such that $F = 0$, then (X, Y, Z) can be chosen as $Z = \varnothing$ Similarly, if M is such that $G_l = 0$, then (X, Y, Z) can be chosen as $Y = \varnothing$. This will be apparent from the proof given below.

Proof We divide the proof into three steps.

Step 1. (*Cf.* [21, Chapter VII, §2, no. 13, Theorem. 4].) We first prove the proposition assuming that k is algebraically closed, and that both $u_{\mathrm{inf}}\colon \mathrm{Lie}(F_{\mathrm{inf}}) \to \mathrm{Lie}(G)$ and $u_{\mathrm{\acute{e}t}}\colon F_{\mathrm{\acute{e}t}} \to G$ are injective. Choose a $\mathbb{Z}$-basis $e_1, \dots, e_r$ of $F_{\mathrm{\acute{e}t}}$, and put $p_i := u_{\mathrm{\acute{e}t}}(e_i) \in G$ $(i = 1, \dots, r)$. Let $p_0 \in G$ be the identity element. We take a one-dimensional closed integral subscheme C'_0 on G that contains $p_0, p_1, \dots, p_r$ as regular points. Also, choose a k-basis $t_1, \dots, t_{s'}$ of $\mathrm{Lie}(F_{\mathrm{inf}})$, and put $v_i := u_{\mathrm{inf}}(t_i) \in \mathrm{Lie}(G)$, $i = 1, \dots, s'$. We extend $v_1, \dots, v_{s'}$ to a k-basis $v_1, \dots, v_{s'}, \dots, v_s$ of $\mathrm{Lie}(G)$. For each $i = 1, \dots, s$, we take a one-dimensional closed integral subscheme C'_i on G that passes p_0 regularly and that has tangent v_i at p_0. For $i = 0, 1, \dots, s$, we let $C_i \to C'_i$ be the normalization. We denote the preimage of p_j in C_i by the same letter p_j. (Here $j = 0, \dots, r$ for $i = 0$, and $j = 0$ for $i = 1, \dots, s$.) Let X_i be the smooth completion of C_i. Let Y_i be a modulus for the morphism $C_i \to C'_i \hookrightarrow G$. This means that Y_i is an effective divisor supported on $X_i \smallsetminus C_i$ and that $C_i \to G$ factors as $C_i \to J(X_i, Y_i) \xrightarrow{g_i} G$. We also define effective divisors $Z_0 := (p_0) + (p_1) + \cdots + (p_r) \in \mathrm{Div}(X_0)$, $Z_i := 2(p_0) \in \mathrm{Div}(X_i)$, $i = 1, \dots, s'$, and $Z_i := 0$, $i = s' + 1, \dots, s$. Let X be the disjoint union of $X_0, \dots, X_s$, and let $Y = Y_0 + \cdots + Y_s$, $Z = Z_0 + \cdots + Z_s$.

By definition, we have $F(X, Z)_{\mathrm{\acute{e}t}} = F(X_0, Z_0) = \mathrm{Div}^0_{Z_0}(X_0)$, hence we can define an isomorphism $F(X, Z)_{\mathrm{\acute{e}t}} \to F_{\mathrm{\acute{e}t}}$ by $\sum_{i=1}^r n_i(p_i - p_0) \mapsto \sum_{i=1}^r n_i e_i$, $n_1, \dots, n_r \in \mathbb{Z}$.

Also, by definition, we have $F(X,Z)_{\mathrm{inf}} = \bigoplus_{i=1}^{s'} F(X_i, Z_i) = \bigoplus_{i=1}^{s'} k\cdot v_i$, hence we can define an isomorphism $F(X,Z)_{\mathrm{inf}} \to F_{\mathrm{inf}}$ by $\sum_{i=1}^{s'} a_i v_i \mapsto \sum_{i=1}^{s'} a_i t_i$, $a_1,\dots,a_{s'} \in k$. We have defined an isomorphism $f\colon F(X,Z) \to F$. Finally, we define $g\colon J(X,Y) \to G$ as the sum of $g_i\colon J(X_i,Y_i) \to G$ over $i = 0,\dots,s$. Since the image of

$$\mathrm{Lie}(g_i)\colon \mathrm{Lie}(J(X_i,Y_i)) \longrightarrow \mathrm{Lie}(G)$$

contains v_i, we find $\mathrm{Lie}(g)\colon \mathrm{Lie}(J(X,Y)) \to \mathrm{Lie}(G)$ is surjective, hence $g\colon J(X,Y) \to G$ itself is also surjective. It is straightforward to see that f and g define an epimorphism $\overline{\mathsf{L}}\mathsf{M}(X,Y,Z) \to M$ in $\mathscr{M}_1^a$. (Here we do not need to tensor with $\mathbb{Q}$.)

Step 2. We drop the assumption that k is algebraically closed, but keep the assumption that both u_{inf} and $u_{\mathrm{\acute{e}t}}$ are injective. By Step 1, we can find a finite extension k'/k such that the base change of M to k' satisfies the conclusion of the proposition. The Weil restriction functor

$$R_{k'/k}\colon \mathscr{M}_{1,\mathbb{Q}}^a(k') \longrightarrow \mathscr{M}_{1,\mathbb{Q}}^a(k), \qquad R_{k'/k}([F\to G]) = [R_{k'/k}(F) \to R_{k'/k}(G)]$$

is exact. (Here we denote by $\mathscr{M}_{1,\mathbb{Q}}^a(k)$ and $\mathscr{M}_{1,\mathbb{Q}}^a(k')$ for the category of Laumon 1-isomotives over k and over k'.) Moreover, for any $(X,Y,Z) \in \overline{\mathrm{M}}\mathrm{Crv}_{k'}$ we have $R_{k'/k}\overline{\mathsf{L}}\mathsf{M}_{k'}(X,Y,Z) = \overline{\mathsf{L}}\mathsf{M}_k(X_k,Y_k,Z_k)$, where for a k'-scheme S we write S_k for the k-scheme S with structure morphism $S \to \operatorname{Spec} k' \to \operatorname{Spec} k$. (This follows from a general fact that the Picard functor commutes with base change.) This proves the proposition in this case.

Step 3. We prove the proposition in the general case. Let $F_2 := \ker(u)$, $M_1 := [F/F_2 \to G]$, $M_2 := [F_2 \to 0]$. Then there is a non-canonical isomorphism $M \cong M_1 \oplus M_2$ in $\mathscr{M}_{1,\mathbb{Q}}^a$. Now we apply the result from Step 2, and we are done. ∎

5.6 Henceforth, we suppose that k is a number field. Note that $\mathscr{M}_{1,\mathbb{Q}}^a$ is a $\mathbb{Q}$-linear Abelian category. By Propositions 2.1 and 5.4, we obtain a $\mathbb{Q}$-linear exact faithful functor

$$\mathbf{LM}\colon \mathrm{ECMM}_1 \to \mathscr{M}_{1,\mathbb{Q}}^a, \tag{5.7}$$

and two invertible natural transformations $\mathbf{LM} \circ \overline{\mathbf{H}}^1_{\mathrm{dR}} \to \overline{\mathsf{L}}\mathsf{M}$, $\mathsf{R}_{\mathrm{dR}} \circ \mathbf{LM} \to F_{\mathrm{dR}}^a$. The main result of this article is the following theorem.

Theorem 5.9 *Suppose that k is a number field. The functor* $\mathbf{LM}\colon \mathrm{ECMM}_1 \to \mathscr{M}_{1,\mathbb{Q}}^a$ *in* (5.7) *is an equivalence.*

6 Filtration on Nori Motives With Modulus

We continue to assume that k is a number field. In this section, we construct on every object of ECMM_1 a two steps filtration that mirrors the one on Laumon 1-motives defined in §4.5.

6.1 Consider the morphism of quivers

$$\overline{\mathrm{M}}\mathrm{Crv} \longrightarrow \overline{\mathrm{M}}\mathrm{Crv}, \quad (X,Y,Z) \longmapsto (X,Y,Z_{\mathrm{red}}). \tag{6.1}$$

Note that if a morphism $f: X \to X'$ of k-curves defines a morphism $(X, Y, Z) \to (X', Y', Z')$ in $\overline{\mathrm{M}}\mathrm{Crv}$, then it also defines a morphism $(X, Y, Z_{\mathrm{red}}) \to (X', Y', Z'_{\mathrm{red}})$ in $\overline{\mathrm{M}}\mathrm{Crv}$, by our definition of $\overline{\mathrm{M}}\mathrm{Crv}$ (see §3.2).

If (X, Y, Z) is a k-curve with modulus, let us observe that by construction

$$\mathrm{fil}^1_{\mathscr{M}} \mathsf{LM}(X, Y, Z) = \mathsf{LM}(X, Y, Z_{\mathrm{red}}).$$

Hence the square

$$\begin{array}{ccc} \overline{\mathrm{M}}\mathrm{Crv}^{\mathrm{op}} & \xrightarrow{\mathsf{LM}} & \mathscr{M}^a_{1,\mathbb{Q}} \\ \Big\downarrow{\scriptstyle (-,-,-_{\mathrm{red}})} & & \Big\downarrow{\scriptstyle \mathrm{fil}^1_{\mathscr{M}}} \\ \overline{\mathrm{M}}\mathrm{Crv}^{\mathrm{op}} & \xrightarrow{\mathsf{LM}} & \mathscr{M}^a_{1,\mathbb{Q}} \end{array} \tag{6.2}$$

commutes and Proposition 2.4 shows the existence of a $\mathbb{Q}$-linear exact functor

$$\mathrm{fil}^1 : \mathrm{ECMM}_1 \to \mathrm{ECMM}_1$$

and two invertible natural transformations

$$\rho: \mathrm{fil}^1_{\mathscr{M}} \circ \mathbf{LM} \longrightarrow \mathbf{LM} \circ \mathrm{fil}^1 \quad \rho: \mathrm{fil}^1 \circ \overline{\mathbf{H}}^1_{\mathrm{dR}} \longrightarrow \overline{\mathbf{H}}^1_{\mathrm{dR}} \circ (-,-,-_{\mathrm{red}}),$$

such that the corresponding diagram as in (2.1) is commutative.

Let us now show that there exists a natural transformation $\mathrm{fil}^1 \to \mathrm{Id}$ that is a monomorphism for every object in ECMM_1. Let (X, Y, Z) be a k-curve with modulus. Since $Z_{\mathrm{red}} \leqslant Z$, the identity of X defines an edge $(X, Y, Z) \to (X, Y, Z_{\mathrm{red}})$ that provides a natural transformation $\iota: (-,-,-_{\mathrm{red}}) \to \mathrm{Id}$ of functors from $\overline{\mathrm{M}}\mathrm{Crv}^{\mathrm{op}}$ with values in $\overline{\mathrm{M}}\mathrm{Crv}^{\mathrm{op}}$. Note that this transformation induces the monomorphism $\mathrm{fil}^1 \mathsf{LM}(X, Y, Z) \to \mathsf{LM}(X, Y, Z)$ in $\mathscr{M}^a_{1,\mathbb{Q}}$ and that the square

$$\begin{array}{ccc} \mathrm{fil}^1_{\mathscr{M}} \circ \mathsf{LM} & = & \mathsf{LM} \circ (-,-,-_{\mathrm{red}}) \\ \Big\downarrow{\scriptstyle \iota_{\mathscr{M}} \star \mathsf{LM}} & & \Big\downarrow{\scriptstyle \mathsf{LM} \star \iota} \\ \mathsf{LM} & = & \mathsf{LM} \end{array}$$

is commutative. We may therefore apply Proposition 2.5 to obtain a natural transformation $\bar{\iota}: \mathrm{fil}^1 \to \mathrm{Id}$ that makes the squares

$$\begin{array}{ccc} \mathrm{fil}^1 \circ \overline{\mathbf{H}}^1_{\mathrm{dR}} & \xrightarrow{\rho} & \overline{\mathbf{H}}^1_{\mathrm{dR}} \circ (-,-,-_{\mathrm{red}}) \\ \Big\downarrow{\scriptstyle \bar{\iota} \star \overline{\mathbf{H}}^1_{\mathrm{dR}}} & & \Big\downarrow{\scriptstyle \overline{\mathbf{H}}^1_{\mathrm{dR}} \star \iota} \\ \overline{\mathbf{H}}^1_{\mathrm{dR}} & = & \overline{\mathbf{H}}^1_{\mathrm{dR}} \end{array} \qquad \begin{array}{ccc} \mathrm{fil}^1_{\mathscr{M}} \circ \mathbf{LM} & \xrightarrow{\rho} & \mathbf{LM} \circ \mathrm{fil}^1 \\ \Big\downarrow{\scriptstyle \iota \star \mathbf{LM}} & & \Big\downarrow{\scriptstyle \mathbf{LM} \star \bar{\iota}} \\ \mathbf{LM} & = & \mathbf{LM} \end{array} \tag{6.3}$$

commutative. Note that by Remark 2.6, for every object A in ECMM_1 the morphism $\bar{\iota}: \mathrm{fil}^1 A \to A$ is a monomorphism.

6.2 So far we have constructed the first step of the filtration. Let us now construct the second one. Let $\mathscr{D}$ be the full subquiver of $\overline{\mathrm{M}}\mathrm{Crv}$ whose vertices are the k-curves with modulus (X,Y,Z) such that Z is *reduced*.

We denote by $\mathscr{M}_{1,\mathbb{Q}}^{\mathrm{inf}=0}$ the kernel of the exact functor $\mathrm{Gr}^0_{\mathscr{M}}$. This is the category of Laumon 1-isomotives without infinitesimal part and by definition it is the full subcategory of $\mathscr{M}_{1,\mathbb{Q}}^{a}$ of objects M such that $\mathrm{Gr}^0_{\mathscr{M}}(M)=0$, *i.e.*, such that $\iota_{\mathscr{M}}\colon \mathrm{fil}^1_{\mathscr{M}}(M)\to M$ is an isomorphism. Similarly we denote by $\mathrm{ECMM}_1^{\mathrm{inf}=0}$ the kernel of the exact functor $\mathrm{Gr}^0\colon \mathrm{ECMM}_1\to\mathrm{ECMM}_1$ constructed in §6.1. The compatibility given in (6.3) ensures that the functor (5.7) induces an exact functor $\mathbf{LM}\colon \mathrm{ECMM}_1^{\mathrm{inf}=0}\to\mathscr{M}_{1,\mathbb{Q}}^{\mathrm{inf}=0}$.

Proposition 6.1 *The universal $\mathbb{Q}$-linear Abelian category associated with the representation $\mathbf{H}^1_{\mathrm{dR}}\colon \mathscr{D}^{\mathrm{op}}\to\mathsf{mod}(\mathbb{Q})$ is equivalent to* $\mathrm{ECMM}_1^{\mathrm{inf}=0}$.

Proof Let us denote by $\mathscr{C}$ the associated category and by $\mathscr{D}^{\mathrm{op}}\xrightarrow{\overline{\mathbf{H}}^1_{\mathscr{C}}}\mathscr{C}\xrightarrow{F_{\mathscr{C}}}\mathsf{mod}(\mathbb{Q})$ the canonical factorization of the restriction of $\mathbf{H}^1_{\mathrm{dR}}$ to $\mathscr{D}^{\mathrm{op}}$. Since the restriction of $\overline{\mathbf{H}}^1_{\mathrm{dR}}$ to $\mathscr{D}^{\mathrm{op}}$ takes its values in the Abelian subcategory $\mathrm{ECMM}_1^{\mathrm{inf}=0}$, the universal property of Nori's category ensures the existence of a $\mathbb{Q}$-linear exact faithful functor $I_{\mathscr{C}}\colon\mathscr{C}\to\mathrm{ECMM}_1^{\mathrm{inf}=0}$ and two invertible natural transformations $\gamma\colon I_{\mathscr{C}}\circ\overline{\mathbf{H}}^1_{\mathscr{C}}\to\overline{\mathbf{H}}^1_{\mathrm{dR}}$ and $\delta\colon F^a_{\mathrm{dR}}\circ I_{\mathscr{C}}\to F_{\mathscr{C}}$ such that the square

$$\begin{array}{ccc} F^a_{\mathrm{dR}}\circ I_{\mathscr{C}}\circ\overline{\mathbf{H}}^1_{\mathscr{C}} & \xrightarrow{F^a_{\mathrm{dR}}\star\gamma} & F^a_{\mathrm{dR}}\circ\overline{\mathbf{H}}^1_{\mathrm{dR}} \\ \Big\downarrow{\scriptstyle \delta\star\overline{\mathbf{H}}^1_{\mathscr{C}}} & & \Big\| \\ F_{\mathscr{C}}\circ\overline{\mathbf{H}}^1_{\mathscr{C}} & = & \mathbf{H}^1_{\mathrm{dR}} \end{array}$$

is commutative. To construct a quasi-inverse to the functor $I_{\mathscr{C}}$ let us go back to the construction of fil^1 in §6.1. Observe that (6.1) takes its values in $\mathscr{D}$ and that the square (6.2) can be refined in a square

$$\begin{array}{ccc} \overline{\mathrm{M}}\mathrm{Crv}^{\mathrm{op}} & \xrightarrow{\mathsf{LM}} & \mathscr{M}^a_{1,\mathbb{Q}} \\ {\scriptstyle (-,-,-_{\mathrm{red}})}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{fil}^1_{\mathscr{M}}} \\ \mathscr{D}^{\mathrm{op}} & \xrightarrow{\mathsf{LM}|_{\mathscr{D}}} & \mathscr{M}^{\mathrm{inf}=0}_{1,\mathbb{Q}}. \end{array}$$

By Propositions 2.4 and 2.5, this shows the existence of a $\mathbb{Q}$-linear exact functor $\mathrm{fil}^1_{\mathscr{C}}\colon \mathrm{ECMM}_1\to\mathscr{C}$ and an invertible natural transformation $I_{\mathscr{C}}\circ\mathrm{fil}^1_{\mathscr{C}}\to\mathrm{fil}^1$.

Let us denote by $I_{\mathrm{inf}=0}$ the inclusion functor of $\mathrm{ECMM}_1^{\mathrm{inf}=0}$ into ECMM_1. Since $\mathrm{fil}^1\circ I_{\mathrm{inf}=0}$ is isomorphic to the identity, the composition $I_{\mathscr{C}}\circ\mathrm{fil}^1_{\mathscr{C}}\circ I_{\mathrm{inf}=0}$ is isomorphic to the identity. This shows that the faithful functor $I_{\mathscr{C}}$ is an equivalence and that $\mathrm{fil}^1_{\mathscr{C}}\circ I_{\mathrm{inf}=0}$ is a quasi-inverse. ■

Now consider the morphism of quivers

$$\mathscr{D}\longrightarrow\mathscr{D},\quad (X,Y,Z)\longmapsto(X,Y_{\mathrm{red}},Z).$$

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(This is indeed a morphism, because if $f\colon X \to X'$ is a morphism of k-curves and if effective divisors $Y \subset X$ and $Y' \subset X'$ satisfy $Y \leqslant f^*Y'$, then we have $Y_{\mathrm{red}} \leqslant (f^*Y')_{\mathrm{red}} \leqslant f^*(Y'_{\mathrm{red}})$.) Since the square

$$\begin{array}{ccc} \mathscr{D}^{\mathrm{op}} & \xrightarrow{\mathsf{LM}} & \mathscr{M}_{1,\mathbb{Q}}^{\mathrm{inf}=0} \\ {\scriptstyle (-,-_{\mathrm{red}},-)}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Gr}^1_{\mathscr{M}}} \\ \mathscr{D}^{\mathrm{op}} & \xrightarrow{\mathsf{LM}} & \mathscr{M}_{1,\mathbb{Q}}^{\mathrm{inf}=0} \end{array}$$

is commutative, Propositions 2.4 and univinf=0 show the existence of a $\mathbb{Q}$-linear exact functor[3] $\mathrm{Gr}^1\colon \mathrm{ECMM}_1^{\mathrm{inf}=0} \to \mathrm{ECMM}_1^{\mathrm{inf}=0}$ and two invertible natural transformations

$$\rho\colon \mathrm{Gr}^1_{\mathscr{M}} \circ \mathbf{LM} \longrightarrow \mathbf{LM} \circ \mathrm{Gr}^1, \quad \rho\colon \mathrm{Gr}^1 \circ \overline{\mathbf{H}}^1_{\mathrm{dR}} \longrightarrow \overline{\mathbf{H}}^1_{\mathrm{dR}} \circ (-,-_{\mathrm{red}},-),$$

such that the corresponding diagram, as in (2.1), is commutative.

Note that for every Laumon 1-isomotive M, there is a canonical epimorphism $\mathrm{fil}^1_{\mathscr{M}}(M) \to \mathrm{Gr}^1_{\mathscr{M}}(M)$. In particular, if M is without infinitesimal part, there is a canonical epimorphism $\pi_{\mathscr{M}}\colon M \to \mathrm{Gr}^1_{\mathscr{M}}(M)$. Since $Y_{\mathrm{red}} \leqslant Y$, the identity of X induces an edge from (X, Y_{red}, Z) to (X, Y, Z) in $\mathscr{D}$.

Remark 6.2. Note that if Y is not reduced, then the identity of X does not define an edge from (X, Y, Z) to $(X, Y_{\mathrm{red}}, Z_{\mathrm{red}})$ in $\overline{\mathrm{M}}\mathrm{Crv}$. This is the main reason for introducing the subquiver $\mathscr{D}$.

This provides a natural transformation $\pi_{\mathscr{D}}\colon \mathrm{Id}_{\mathscr{D}^{\mathrm{op}}} \to (-,-_{\mathrm{red}},-)$ of functors from $\mathscr{D}^{\mathrm{op}}$ with values in $\mathscr{D}^{\mathrm{op}}$. Note that the square

$$\begin{array}{ccc} \mathsf{LM}|_{\mathscr{D}} & =\!=\!=\!= & \mathsf{LM}|_{\mathscr{D}} \\ {\scriptstyle \pi_{\mathscr{M}} \star \mathsf{LM}|_{\mathscr{D}}}\big\downarrow & & \big\downarrow{\scriptstyle \mathsf{LM}|_{\mathscr{D}} \star \pi_{\mathscr{D}}} \\ \mathrm{Gr}^1_{\mathscr{M}} \circ \mathsf{LM}|_{\mathscr{D}} & =\!=\!=\!= & \mathsf{LM}|_{\mathscr{D}} \circ (-,-_{\mathrm{red}},-) \end{array}$$

commutes. We may therefore apply Proposition 2.5 to obtain a natural transformation $\overline{\pi}\colon \mathrm{Id} \to \mathrm{Gr}^1$ that makes the squares

$$\begin{array}{ccc} \overline{\mathbf{H}}^1_{\mathrm{dR}} & =\!=\!=\!= & \overline{\mathbf{H}}^1_{\mathrm{dR}} \\ {\scriptstyle \overline{\pi} \star \overline{\mathbf{H}}^1_{\mathrm{dR}}}\big\downarrow & & \big\downarrow{\scriptstyle \overline{\mathbf{H}}^1_{\mathrm{dR}} \star \pi_{\mathscr{D}}} \\ \mathrm{Gr}^1 \circ \overline{\mathbf{H}}^1_{\mathrm{dR}} & \xrightarrow{\rho} & \overline{\mathbf{H}}^1_{\mathrm{dR}} \circ (-,-_{\mathrm{red}},-) \end{array} \qquad \begin{array}{ccc} \mathbf{LM} & =\!=\!=\!= & \mathbf{LM} \\ {\scriptstyle \pi_{\mathscr{M}} \star \mathbf{LM}}\big\downarrow & & \big\downarrow{\scriptstyle \mathbf{LM} \star \overline{\pi}} \\ \mathrm{Gr}^1_{\mathscr{M}} \circ \mathbf{LM} & \xrightarrow{\rho} & \mathbf{LM} \circ \mathrm{Gr}^1 \end{array}$$

commutative. Note that in the above squares, all natural transformations are between functors on $\mathscr{D}^{\mathrm{op}}$ or $\mathrm{ECMM}_1^{\mathrm{inf}=0}$. By Remark 2.6, for every object A in $\mathrm{ECMM}_1^{\mathrm{inf}=0}$, the morphism $\overline{\pi}\colon A \to \mathrm{Gr}^1(A)$ is an epimorphism.

Let A be an object in ECMM_1. Then $\mathrm{fil}^1(A)$ belongs to $\mathrm{ECMM}_1^{\mathrm{inf}=0}$ and we set

$$\mathrm{fil}^2(A) := \mathrm{Ker}[\mathrm{fil}^1(A) \to \mathrm{Gr}^1(\mathrm{fil}^1(A))].$$

[3]Note that the notation might be misleading: Gr^1 is not yet the set of graded pieces associated to a filtration.

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Note that by definition $\mathrm{Gr}^1(A) := \mathrm{Gr}^1(\mathrm{fil}^1(A))$.

7 Proof of the Main Theorem

In this section, we assume that k is a number field. We complete the proof of Theorem 5.9.

7.1 Recall from Proposition 3.15 that we have a fully faithful functor $I_{\mathrm{ECM}}\colon \mathrm{ECM}_1^{\mathrm{dR}} \to \mathrm{ECMM}_1$. The composition of I_{ECM} with $\mathbf{LM}\colon \mathrm{ECMM}_1 \to \mathscr{M}_{1,\mathbb{Q}}^a$ factors through the category $\mathscr{M}_{1,\mathbb{Q}}$ of Deligne 1-isomotives. This induces a functor $\mathrm{ECM}_1^{\mathrm{dR}} \to \mathscr{M}_{1,\mathbb{Q}}$ by universality.

Proposition 7.1 *The functor* $\mathrm{ECM}_1^{\mathrm{dR}} \to \mathscr{M}_{1,\mathbb{Q}}$ *is an equivalence.*

Proof This follows from (3.6), Theorem 4.1, Proposition 3.12, and the Cartier duality for $\mathscr{M}_{1,\mathbb{Q}}$. ∎

Let $\mathrm{ECMM}_1^{\mathrm{uni}}$ be the intersection of the kernel of the exact functors Gr^0 and Gr^1 constructed in §6.1 and §6.2. An object A in ECMM_1 belongs to the full subcategory $\mathrm{ECMM}_1^{\mathrm{uni}}$ if and only if the canonical monomorphism $\mathrm{fil}^2(A) \to A$ is an isomorphism. Since the functor LM is compatible with the filtration, it induces a $\mathbb{Q}$-linear exact faithful functor (see §4.4) $\mathsf{LM}\colon \mathrm{ECMM}_1^{\mathrm{uni}} \to \mathscr{M}_{1,\mathbb{Q}}^{\mathrm{uni}}$. Note that $\mathscr{M}_{1,\mathbb{Q}}^{\mathrm{uni}}$ is simply the category of unipotent commutative algebraic groups over k and that the functor $\mathscr{M}_{1,\mathbb{Q}}^{\mathrm{uni}} \to \mathsf{mod}(k)$ given by the restriction of the de Rham realization R_{dR} is nothing but the functor that associates with a unipotent commutative algebraic k-group its Lie algebra and is therefore an equivalence.

For the proof of the next proposition, we need an elementary lemma.

Lemma 7.2 *For any* $\mu \in \mathbb{Z}_{>0}$, *k is generated by* $\{a^\mu \mid a \in k\}$ *as a* $\mathbb{Q}$*-algebra.*

Proof Write $k = \mathbb{Q}(\gamma)$ with $\gamma \in k$. Then γ can be written as a $\mathbb{Q}$-linear combination of $\gamma^\mu, (\gamma+1)^\mu, \dots, (\gamma+\mu-1)^\mu$. The lemma follows from this. ∎

Proposition 7.3 *The functor* $\mathsf{LM}\colon \mathrm{ECMM}_1^{\mathrm{uni}} \to \mathscr{M}_{1,\mathbb{Q}}^{\mathrm{uni}}$ *is an equivalence.*

Proof We define a subquiver $\overline{\mathrm{M}}\mathrm{Po}$ of $\overline{\mathrm{M}}\mathrm{Crv}$ as follows. The vertices are given by $\mathrm{P}_n := (\mathbb{P}^1, n[\infty], \varnothing) \in \overline{\mathrm{M}}\mathrm{Crv}$ for any integer $n \geq 2$. The edges from P_n to P_m are of two types:

(7.1) an automorphism of $\mathbb{P}^1$ that fixes ∞, when $n = m \geq 2$.

(7.2) the identity map on $\mathbb{P}^1$, when $m \geq n \geq 2$.

Let $w\colon \mathsf{mod}(k) \to \mathsf{mod}(\mathbb{Q})$ be the forgetful functor. Consider the representation $T = w \circ R_{\mathrm{dR}} \circ \mathsf{LM}|_{\overline{\mathrm{M}}\mathrm{Po}}\colon \overline{\mathrm{M}}\mathrm{Po}^{\mathrm{op}} \to \mathsf{mod}(\mathbb{Q})$ and its canonical factorization

$$\overline{\mathrm{M}}\mathrm{Po}^{\mathrm{op}} \xrightarrow{\overline{T}} \mathscr{U} \xrightarrow{F_T} \mathsf{mod}(\mathbb{Q}),$$

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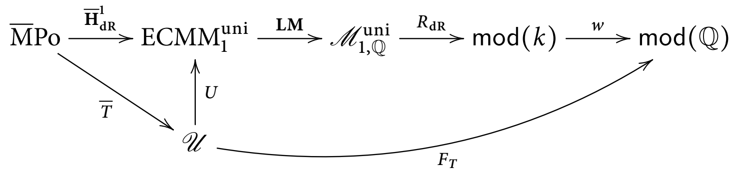
where $\mathscr{U} = \mathsf{comod}(\mathcal{C}_T)$ is Nori's universal category (see Remark 2.2). Note that the restriction of the representation $\overline{\mathbf{H}}^1_{\mathrm{dR}}$ to the subquiver $\overline{\mathrm{M}}\mathrm{Po}$ takes its values in $\mathrm{ECMM}_1^{\mathrm{uni}}$. Hence, by the universal property of Nori's construction (see Theorem 2.1), there exist a $\mathbb{Q}$-linear exact faithful functor $U: \mathscr{U} \to \mathrm{ECMM}_1^{\mathrm{uni}}$, and two invertible natural transformations $\alpha: U \circ \overline{T} \to \overline{\mathbf{H}}^1_{\mathrm{dR}}$ and $\beta: w \circ R_{\mathrm{dR}} \circ \mathbf{LM} \circ U \to F_T$ such that the diagram



is commutative. Since the functor $\mathbf{LM}$ is faithful, to show the proposition it is enough to show that $R_{\mathrm{dR}} \circ \mathbf{LM} \circ U: \mathscr{U} \to \mathsf{mod}(k)$ is an equivalence of categories (note that the functor U will then also be an equivalence). It suffices to see that $\mathcal{C}_T$ is the $\mathbb{Q}$-linear dual of the algebra k, and this amounts to checking that for every full subquiver $\mathscr{E}$ of $\overline{\mathrm{M}}\mathrm{Po}$ with finitely many objects, $\mathrm{End}_{\mathbb{Q}}(T|_{\mathscr{E}}) = k$. We may assume $\mathscr{E}$ of the form $\{\mathrm{P}_2, \dots, \mathrm{P}_n\}$ for some integer $n \geqslant 2$. Write $\mathbb{P}^1 = \mathrm{Proj}(k[T,S])$ and put $t = T/S$, $s = S/T$ so that $\infty \in \mathbb{P}^1$ is defined by $s = 0$. By (5.1) and (5.5), the representation T maps P_n to the $\mathbb{Q}$-vector space $sk[s]/(s^n)$. We compute the action of morphisms on this space in three instances:

(a) Let $n \geqslant 2$ and consider the edge $e: \mathrm{P}_n \to \mathrm{P}_n$ of type (7.1) given by $t \mapsto at$, where a is a fixed element in $k^\times$. Then $T(e)$ is the k-linear map represented by a diagonal matrix $(a^{-1}, a^{-2}, \dots, a^{1-n})$ with respect to the k-basis $\{s, s^2, \dots, s^{n-1}\}$,

(b) Let $n \geqslant 2$ and consider the edge $e: \mathrm{P}_n \to \mathrm{P}_n$ of type (7.1) given by $t \mapsto t - 1$. Then $T(e)$ maps $s = 1/t$ to $1/(t-1) = s + s^2 + \cdots + s^{n-1}$. (We will not need to know $T(e)(s^i)$ for $i > 1$.)

(c) Let $m \geqslant n \geqslant 2$ and consider the edge of type (7.2). Then $T(e)$ is the map

$$sk[s]/(s^m) \longrightarrow sk[s]/(s^n)$$

induced by the identity on $sk[s]$.

Let α be an element in $\mathrm{End}_{\mathbb{Q}}(T|_{\mathscr{E}})$. Then α is given by a family

$$(\alpha^{(i)})_{i=2}^n \in \prod_{i=2}^n \mathrm{End}_{\mathbb{Q}}(T(\mathrm{P}_i))$$

such that for every edge $e: \mathrm{P}_i \to \mathrm{P}_j$ in $\overline{\mathrm{M}}\mathrm{Po}$

$$\alpha_i \circ T(e) = T(e) \circ \alpha_j. \tag{7.3}$$

We write $\alpha^{(i)} = (\alpha^{(i)}_{\mu\nu})_{\mu,\nu=1,\dots,i}$ with $\alpha^{(i)}_{\mu\nu} \in \mathrm{End}_{\mathbb{Q}}(k) (\cong M_d(\mathbb{Q})$ with $d = [k:\mathbb{Q}])$. Let us define a $\mathbb{Q}$-algebra embedding $m: k \to \mathrm{End}_{\mathbb{Q}}(k)$ by $m(a)(x) = ax$ $(a, x \in k)$.

The condition (7.3) for all edges of the form (a) implies that

$$m(a)^{-\mu} \alpha^{(i)}_{\mu\nu} = \alpha^{(i)}_{\mu\nu} m(a)^{-\nu} \quad \text{for all } a \in k^\times. \tag{7.4}$$

Since $m(a)$ for $a \in \mathbb{Q}$ lies in the center of $\mathrm{End}_{\mathbb{Q}}(k)$, (7.4) applied to, say, $a = 2$ yields $\alpha^{(i)}_{\mu\nu} = 0$ if $\mu \neq \nu$. In view of Lemma 7.2, it also yields that $\alpha^{(i)}_{\mu\mu}$ belongs to the centralizer of the image of m, which is k itself as k is a maximal commutative subring of $\mathrm{End}_{\mathbb{Q}}(k)$. We write $\alpha_{\mu\mu}^{(i)} = m(a_\mu^{(i)})$ with $a_\mu^{(i)} \in k$. Applying the condition (7.3) for all edges of the form (b), we obtain $a_1^{(i)} = a_\mu^{(i)}$ for all μ. Finally, (7.3) for all edges of type (c) yields $a_1^{(i)} = a_1^{(1)}$ for all i. We have shown that $\alpha = a_1^{(1)} \in k$. This completes the proof. ∎

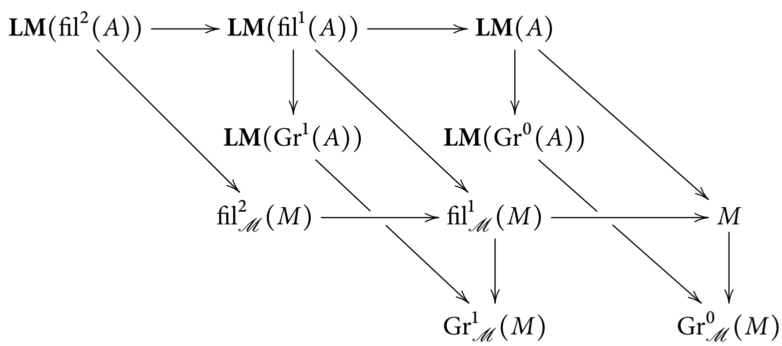
Let $\mathrm{ECMM}_1^{\mathrm{inf}}$ be the kernel of the exact functor fil^1 constructed in §6.1. By a dual argument, we obtain the following proposition.

Proposition 7.4 *The restriction of the functor* $\mathbf{LM}\colon \mathrm{ECMM}_1 \to \mathscr{M}_{1,\mathbb{Q}}^a$ *to the subcategory* $\mathrm{ECMM}_1^{\mathrm{inf}}$ *induces an equivalence of categories between* $\mathrm{ECMM}_1^{\mathrm{inf}}$ *and* $\mathscr{M}_{1,\mathbb{Q}}^{\mathrm{inf}}$ *(see §4.4).*

7.2 We finally prove our main theorem.

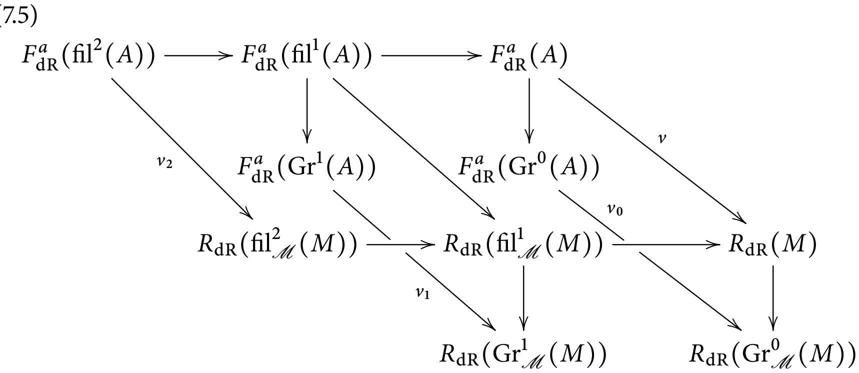
Proof of Theorem 5.9 To prove Theorem 5.9 it is enough to show that we are in a situation where the criteria of Proposition 2.3 apply. The first condition is obviously satisfied and the second one follows from Proposition 5.7. It remains to prove that the third condition is also satisfied.

Let A be an object in ECMM_1, M an object in $\mathscr{M}_{1,\mathbb{Q}}^a$, and $u\colon \mathbf{LM}(A) \to M$ a morphism in $\mathscr{M}_{1,\mathbb{Q}}^a$. By applying the functor R_{dR}, we get a morphism $R_{\mathrm{dR}}(u)\colon F_{\mathrm{dR}}^a(A) \to R_{\mathrm{dR}}(M)$ of $\mathbb{Q}$-vector spaces. Note that u induces a commutative diagram in $\mathscr{M}_{1,\mathbb{Q}}^a$:



Applying the functor R_{dR} yields a commutative diagram:

(7.5)



where we set $v = R_{\mathrm{dR}}(u)$, $v_0 := R_{\mathrm{dR}}(\mathrm{Gr}^0_{\mathscr{M}}(u))$, $v_1 := R_{\mathrm{dR}}(\mathrm{Gr}^1_{\mathscr{M}}(u))$, and $v_2 := R_{\mathrm{dR}}(\mathrm{fil}^2_{\mathscr{M}}(u))$ to simplify notations.

By construction of the category ECMM_1 (see Remark 2.2), there exists a finite subquiver $\mathscr{E}$ of $\overline{\mathrm{M}}\mathrm{Crv}$ such that in the diagram

$$\begin{array}{ccccc} F^a_{\mathrm{dR}}(\mathrm{fil}^2(A)) & \longrightarrow & F^a_{\mathrm{dR}}(\mathrm{fil}^1(A)) & \longrightarrow & F^a_{\mathrm{dR}}(A) \\ & & \downarrow & & \downarrow \\ & & F^a_{\mathrm{dR}}(\mathrm{Gr}^1(A)) & & F^a_{\mathrm{dR}}(\mathrm{Gr}^0(A)) \end{array}$$

all objects are canonically endowed with an $\mathrm{End}(H^1_{\mathrm{dR}}|_{\mathscr{E}})$-module structure and all morphisms are $\mathrm{End}(H^1_{\mathrm{dR}}|_{\mathscr{E}})$-linear. Using Propositions 7.3, 7.4, and 7.1, by allowing $\mathscr{E}$ to be bigger, we may assume that the kernels of the maps v_0, v_1, v_2 are sub-$\mathrm{End}(T|_{\mathscr{E}})$-modules. An easy diagram chase in (7.5) shows that the kernel of v is a sub-$\mathrm{End}(T|_{\mathscr{E}})$-module of F^a_{dR} as well. This concludes the proof. ■

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