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Multi-Sided Braid Type Subfactors

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Abstract. We generalise the two-sided construction of examples of pairs of subfactors of the hyperfinite II_1 factor R in [E1]—which arise by considering unitary braid representations with certain properties—to multi-sided pairs. We show that the index for the multi-sided pair can be expressed as a power of that for the two-sided pair. This construction can be applied to the natural examples—where the braid representations are obtained in connection with the representation theory of Lie algebras of types A, B, C, D. We also compute the (first) relative commutants.

1 Introduction

In this paper we construct examples of braid type subfactors of the hyperfinite II_1 factor R by generalising the two-sided construction by the author in [E1].

The method used for constructing these examples consists of generating the inclusions of factors from pairs of ascending sequences of finite dimensional C^* algebras $(A_n)_n \subset (B_n)_n$ satisfying properties related to their inclusion maps and the existence of a unique positive trace on their union (the commuting square property—a notion introduced by S. Popa, see [P]—and the periodicity property of inclusion matrices). Under these assumptions, H. Wenzl, [W1], showed that the index for the resulting pair can be expressed in terms of the weight vectors for the trace on A_n and on B_n. He applied this construction to obtain natural examples, in which the ascending sequences are given by special semisimple quotients of braid groups $\mathbf{B}_n$ arising from Lie theory (see also [W2]).

In the terminology of [E1], Wenzl's examples are called *one-sided braid subfactors* since, if the generators of $\mathbf{B}_n$ are given by $\sigma_1, \dots, \sigma_{n-1}$, then the the inclusion of factors is given by $\langle \hat{g}_0, g_1, \dots, g_n, \dots \rangle'' \subset \langle g_0, g_1, \dots, g_n, \dots \rangle''$, where the g_i's are the images of the σ_i's under the representation considered. In this paper we denote this inclusion by $R(1) \subset R$. The two-sided inclusions defined in [E1], and originally in [Ch], are of the form $\langle \dots, g_{-1}, \hat{g}_0, g_1, \dots \rangle'' \subset \langle \dots, g_{-1}, g_0, g_1, \dots \rangle''$ (denoted here by $R(2) \subset R$), where the braid representation is extended to braid generators labeled by all integers. The latter can be obtained as approximations by inclusions conjugate to $A_n \otimes A_n \subset A_{2n}$, which satisfy periodicity and commuting square properties, and where A_n is the finite dimensional C^* algebra given by the n-braid quotient $\langle g_0, \dots, g_{n-2} \rangle$. The unitary braid representations considered have to satisfy a list of conditions which are satisfied in the natural examples related to Lie theory mentioned above.

In this paper we use the same requirements on the unitary braid representations to construct the multi-sided inclusions. For a fixed integer $s \geq 2$, we define s-sided

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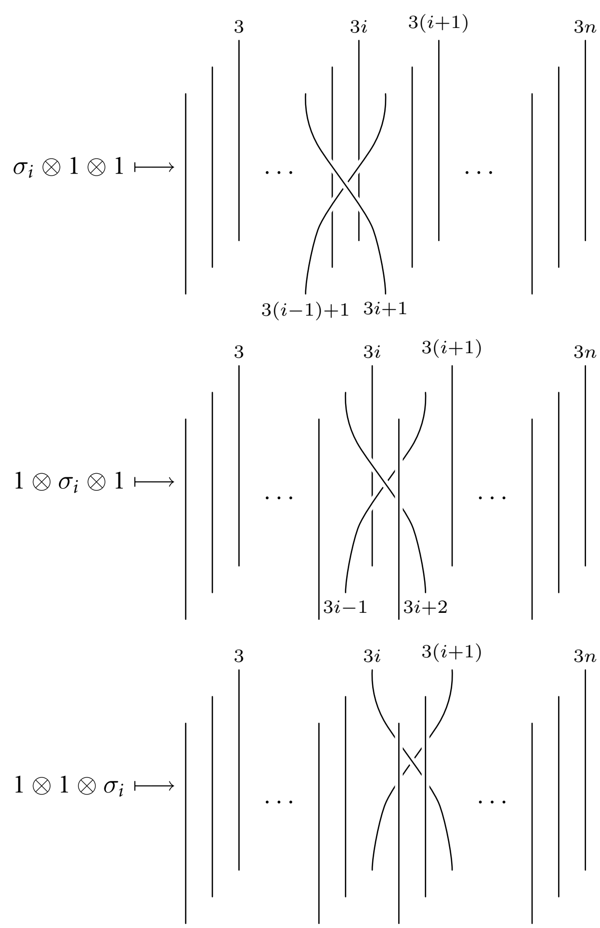
inclusions which shall be denoted by $R(s) \subset R$. If we considered the pair of ascending sequences $(A_n^{\otimes s})_n \subset (A_{sn})_n$, where A_n is the n-braid quotient, the problem would be that the natural inclusions would not even form commuting diagrams. In order to correct this, we consider instead pairs of ascending sequences $(u_n A_n^{\otimes s} u_n^*)_n \subset (A_{sn})_n$, where $u_n = u_n(s)$ is a special unitary in A_{sn}. The embeddings $u_n A_n^{\otimes s} u_n^* \subset A_{sn}$ can be defined from embeddings at the level of the braid groups $\mathbf{B}_n$, and can be viewed intuitively by geometrical diagrams. For example, if $s = 3$, the embedding $\mathbf{B}_n{}^{\otimes 3} \to \mathbf{B}_{3n}$ maps in the following way: For $0 \leq i \leq n-2$,



This map is implemented via conjugation by a braid $\gamma_n \in \mathbf{B}_{3n}$, that is, $x \mapsto \gamma_n x \gamma_n^{-1}$. The algebra $u_n A_n^{\otimes s} u_n^*$ is the representation image of $\gamma_n(\mathbb{C}\mathbf{B}_n{}^{\otimes s})\gamma_n^{-1}$. The inclusion $R(s) \subset R$ arising this way has an index related to that for the inclusion of the two-sided pair $R(2) \subset R$, given by $[R : R(s)] = [R : R(2)]^{s-1}$. Also, as is noted at the end of this article, the two-sided pair $R(2) \subset R$ coincides with the asymptotic inclusion for the one-sided pair $R(1) \subset R$ when the braid representations arise from Lie algebras of types A, B, C, D, see [E2]. As an interesting observation, these in-

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dex values coincide with those for the multiple interval subfactors, as announced by A. Wassermann, [Wa], and computed in [X].

We also consider the following variation, as in [Ch] and [E1]. For a fixed integer $m \geq 0$, we can define new versions of the multi-sided pairs by "excluding" $m+1$ generators. In the context of the one-sided pairs this means the inclusion

$$\langle g_{m+1}, \dots, g_n, \dots \rangle'' \subset \langle g_0, g_1, \dots, g_n, \dots \rangle'',$$

denoted by $R(1)^m \subset R$. In the context of the two-sided pairs it means the inclusion

$$\langle \dots, g_{-m-1}, g_{m+1}, \dots \rangle'' \subset \langle \dots, g_{-1}, g_0, g_1, \dots \rangle'',$$

denoted by $R(2)^m \subset R$. We define an analogous version for the multi-sided pairs, $R(s)^{(\vec{m})} \subset R$, by fixing a nonnegative vector $\vec{m} \in \mathbb{Z}^s$ and by considering the pair generated by the ascending sequence

$$\left(u_n (A_{n-m_1} \otimes \cdots \otimes A_{n-m_s}) u_n^* \right)_n \subset (A_{sn})_n.$$

As with the one- and two-sided cases, the inclusions of factors $R(s)^{(\vec{m})} \subset R$ are no longer irreducible (unless $|\vec{m}| := \sum_i^s m_i \leq 1$), but we may reduce them by minimal projections in the relative commutant, which is isomorphic to $A_{|\vec{m}|}$. The index for $R(s)^{(\vec{m})} \subset R$ is given by

$$[R : R(s)^{(\vec{m})}] = W^{|\vec{m}|} [R : R(2)]^{s-1},$$

where $|\vec{m}| = \sum_{i=1}^s m_i$ and $W = [R : R(1)]$. Finally, we apply Wenzl's formula in [W1] for reduced pairs to obtain the index of $pR(s)^{(\vec{m})}p \subset pRp$, where p is a nonzero projection in the relative commutant $R(s)^{(\vec{m})\prime} \cap R$. The paper is organised as follows:

2. Preliminaries
   - 2.1 Basic Definitions and Properties of Braid Groups
   - 2.2 Assumptions On the Braid Group Representations
3. The Construction of Multi-Sided Braid Type Subfactors
   - 3.1 The Definition of the Unitaries at the Braid Group Level
   - 3.2 Lifting the Unitaries to the Braid Quotients. Periodicity Commuting Square Properties
   - 3.3 Definition of the Pairs $R(s) \subset R$. The Relative Commutant.
   - 3.4 Definition of the Pairs $R(s)^{(\vec{m})} \subset R$. The Relative Commutant.
4. A Formula For the Index
5. References

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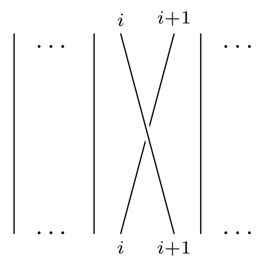
2 Preliminaries

2.1 Basic Definitions and Properties of Braid Groups

Recall that the braid group $\mathbf{B}_n$ on n strands is defined by generators $\sigma_1, \ldots, \sigma_{n-1}$ and the braid relations

(B1) $\sigma_{i+1}\sigma_i\sigma_{i+1} = \sigma_i\sigma_{i+1}\sigma_i$, for $i = 1, \ldots, n-2$,
(B2) $\sigma_i\sigma_j = \sigma_j\sigma_i$, for $|i-j| \geq 2$.

A geometric picture of the standard generator σ_i is given by the following diagram:



and multiplication is given by concatenation of such diagrams (see [Bi] for more details). $\mathbf{B}_n$ is embedded into $\mathbf{B}_{n+1}$ by adding one vertical strand at the end of each generator of $\mathbf{B}_n$. Denote $\bigcup \mathbf{B}_n$ by $\mathbf{B}_\infty$.

We state below some well known basic relations for elements in $\mathbf{B}_n$ that will be needed in the next sections. If $1 \leq i, j \leq n-1$, we denote by

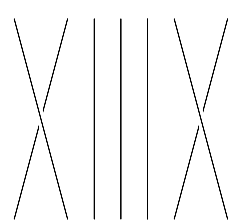
$$(\sigma_i \cdots \sigma_j)$$

the element of $\mathbf{B}_n$ given by the increasing product $\sigma_i\sigma_{i+1}\cdots\sigma_j$ if $i < j$, or by the decreasing product $\sigma_i\sigma_{i-1}\cdots\sigma_j$ if $i > j$.

Lemma 1 *Let $t, r \in \mathbb{N}$. Then the following relations hold:*

(B3) $(\sigma_t \cdots \sigma_{t+r})\sigma_{t+r+1}^{\pm 1}(\sigma_{t+r}^{-1} \cdots \sigma_t^{-1}) = (\sigma_{t+r+1}^{-1} \cdots \sigma_{t+1}^{-1})\sigma_t^{\pm 1}(\sigma_{t+1} \cdots \sigma_{t+r+1})$.
(B4) $(\sigma_t \cdots \sigma_{t+r})\sigma_{t+r-1}^{\pm 1}(\sigma_{t+r}^{-1} \cdots \sigma_{t+1}^{-1}\sigma_t^{-1}) = \sigma_{t+r}^{\pm 1}$.
(B5) $(\sigma_{t+r}^{-1} \cdots \sigma_t^{-1})\sigma_{t+1}^{\pm 1}(\sigma_t \cdots \sigma_{t+r}) = \sigma_t^{\pm 1}$.
(B6) $(\sigma_{t+r+1} \cdots \sigma_{t+1})\sigma_t^{\pm 1}(\sigma_{t+1}^{-1} \cdots \sigma_{t+r+1}^{-1}) = (\sigma_t^{-1} \cdots \sigma_{t+r}^{-1})\sigma_{t+r+1}^{\pm 1}(\sigma_{t+r} \cdots \sigma_t)$.
(B7) $(\sigma_{t+r+1} \cdots \sigma_t)\sigma_{t+1}^{\pm 1}(\sigma_t^{-1} \cdots \sigma_{t+r+1}^{-1}) = \sigma_t^{\pm 1}$
(B8) $(\sigma_t^{-1} \cdots \sigma_{t+r+1}^{-1})\sigma_{t+r}^{\pm 1}(\sigma_{t+r+1} \cdots \sigma_t) = \sigma_{t+r+1}^{\pm 1}$.

The algebraic tensor product $\mathbb{C}\mathbf{B}_n \otimes \mathbb{C}\mathbf{B}_m$ can be seen as a subalgebra of $\mathbb{C}\mathbf{B}_{n+m}$ via the embedding defined by juxtaposition. For example, if $n = 3$, $m = 4$

$$\sigma_1 \otimes \sigma_3 \longmapsto \overline{\sigma_1 \otimes \sigma_3} =$$  .

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In other words, the embedding $\mathbb{C}\mathbf{B}_n \otimes \mathbb{C}\mathbf{B}_m \hookrightarrow \mathbb{C}\mathbf{B}_{n+m}$ is defined by

$$\beta \otimes \gamma \longmapsto \overline{\beta \otimes \gamma} = \beta \,\mathrm{shift}_n\, \gamma,$$

where $\mathrm{shift}_n\, \sigma_i = \sigma_{i+n}$. Thus, the embedding $\mathbb{C}\mathbf{B}_n^{\otimes s} \hookrightarrow \mathbb{C}\mathbf{B}_{sn}$ is given by

$$\beta_1 \otimes \cdots \otimes \beta_s \longmapsto \overline{\beta_1 \otimes \cdots \otimes \beta_s} = \beta_1 \,\mathrm{shift}_n(\beta_2) \cdots \mathrm{shift}_{n(s-1)}(\beta_n). \tag{2.1}$$

2.2 Assumptions On the Braid Group Representations

We shall work with representations ρ of $\mathbb{C}\mathbf{B}_\infty$ that satisfy the following properties as in [E1]:

(i) ρ is locally finite dimensional: For every $n \in \mathbb{N}$, $\rho(\mathbb{C}\mathbf{B}_n)$ is a finite dimensional C*-algebra, so that we can write $\rho(\mathbb{C}\mathbf{B}_n) \simeq \bigoplus_{\lambda \in \Lambda_n} M_{a_\lambda}(\mathbb{C})$, for some index set Λ_n. Set $A_n = \rho(\mathbb{C}\mathbf{B}_n)$.

(ii) ρ is unitary: That is, $g_i = \rho(\sigma_i)$ is a unitary for all i.

(iii) The ascending sequence of finite dimensional C*-algebras $(A_n) = \big(\rho(\mathbb{C}\mathbf{B}_n)\big)$ is periodic, in the sense of Wenzl, [W1, Lemma 1.4].

(iv) Any element $x \in A_{n+1}$ can be written as a sum of elements $a g_n^{\pm 1} b + c$ with $a, b, c \in A_n$.

(v) The unique positive faithful normalised trace tr on $\bigcup A_n$ has the Markov property:

$$\mathrm{tr}(g_n^{\pm 1} x) = \eta^{(\pm)}\, \mathrm{tr}(x) \quad \text{for all } x \in A_n, \text{ for all } n,$$

where $\eta^{(+)}, \eta^{(-)}$ are fixed complex numbers. Given condition (iv), the Markov condition implies the multiplicativity property for the trace:

$$\mathrm{tr}(xy) = \mathrm{tr}(x)\,\mathrm{tr}(y),$$

if x and y are in subalgebras generated by disjoint subsets of generators $g_i^{\pm 1}$.

(vi) Existence of a projection p with the *contraction property*: $p \in A_k$ has the *contraction property* if for all $n \in \mathbb{N}$,

$$pA_{n+k}p \simeq pA_{k+1,n+k} \simeq A_{k+1,n+k},$$

where $A_{s,t}$ is the algebra generated by $\{1, g_s^{\pm 1}, \ldots, g_{t-1}^{\pm 1}\}$. Note that since we already have the multiplicative property of the trace by (iv) and (v), the second isomorphism above is always true.

Given a locally finite representation ρ of the braid group, there is an associative, commutative, graded product on $\bigoplus_n K_0(A_n)$ defined as follows (see [GW] for details). For projections $x \in A_n$ and $y \in A_m$ define $x \otimes y = x\,\mathrm{shift}_n(y) \in A_{n+m}$, where $\mathrm{shift}_n\colon \mathbb{C}\mathbf{B}_m \to \mathbb{C}\mathbf{B}_{n+m}$ is determined by $\sigma_i \mapsto \sigma_{i+n}$. Then $[x] \otimes [y] = [x \otimes y]$ defines the multiplication in $\bigoplus_n K_0(A_n)$. Denote the structure constants of this multiplication by $c^\nu_{\lambda\mu}$. That is, if p_λ and p_μ are minimal projections in the classes labelled by

$\lambda \in \Lambda_n$ and $\mu \in \Lambda_m$, then

$$[p_\lambda] \otimes [p_\mu] = \sum_{\nu \in \Lambda_{n+m}} c^{\nu}_{\lambda\mu} [p_\nu].$$

The following conditions are equivalent to the existence of a projection with the contraction property (see [W3] or [E1]).

Lemma 2 *With the same notation as throughout this section, the following are equivalent:*

(a) There exists a projection $p \in A_k$ with the contraction property, i.e., for all $n \in \mathbb{N}$, $pA_{n+k}p \simeq pA_{k+1,n+k} \simeq A_{k+1,n+k}$, where $A_{s,t}$ is the algebra generated by $\{1, g_s^{\pm 1}, \dots, g_{t-1}^{\pm 1}\}$.

(b) There exists a projection $p \in A_k$ such that for every minimal projection $p_\lambda \in A_n$, and for all $n \in \mathbb{N}$, the projection $p \otimes p_\lambda$ remains minimal in A_{n+k}. Moreover, if $\lambda \neq \lambda'$ then $p \otimes p_\lambda$ and $p \otimes p_{\lambda'}$ are not equivalent.

(c) For all $n \in \mathbb{N}_0$ there exists an injective map $j \colon \Lambda_n \to \Lambda_{n+k}$ that preserves the structure coefficients for the multiplication in $\bigoplus_n K_0(A_n)$, that is, such that $c^{\nu}_{\lambda\mu} = c^{j(\nu)}_{j(\lambda)\mu} = c^{j(\nu)}_{\lambda j(\mu)}$ for all $\lambda \in \Lambda_n$, $\mu \in \Lambda_m$, $\nu \in \Lambda_{n+m}$, and also such that $c^{\epsilon}_{\lambda j(\mu)} = 0$ if $\epsilon \notin j(\Lambda_{n+m})$. (Here, $\Lambda_0 := \{[\varnothing]\}$, "the empty diagram", and $p_{[\varnothing]} := \mathrm{id}$, so that $c^{\mu}_{[\varnothing]\lambda} = \delta_{\lambda,\mu}$, for $\lambda, \mu \in \Lambda_n$.)

Remarks Assume that (A_n) has periodicity k and that there exists a projection $p \in A_{k'}$ with the contraction property.

(1) The periodicity condition on the ascending sequence (A_n) forces the injective map $j \colon \Lambda_n \to \Lambda_{n+k'}$ (from the contraction property) to be a bijection for large $n \in \mathbb{N}$.

(2) The sequence (A_n) is also k'-periodic, so one can assume without loss of generality that $k = k'$.

(3) Let I be given by the set $\{k \in \mathbb{N}$ such that (A_n) is k-periodic$\}$. Let k_0 be given by $\min I$. Then $I = \{nk_0$ such that $n \in \mathbb{N}\}$.

(4) $p^{\otimes s} \in A_{sk'}$ has the contraction property for all $s \in \mathbb{N}$.

3 The Construction of Multi-Sided Braid Type Subfactors

The goal is to build for each integer $s \in \mathbb{N}_{\geq 2}$ a pair of ascending sequences of finite dimensional C* algebras $(A^s_n)_n \subset (A_{sn})_n$ such that

(i) A_n is an n-braid quotient as in [E1],

(ii) $A^s_n = u^s_n (A_n^{\otimes s}) u^{s\,*}_n$ for a unitary $u^s_n \in A_{sn}$,

(iii) $(A^s_n)_n \subset (A_{sn})_n$ is periodic and has the commuting square property.

The task is to find a sequence of unitaries $(u^s_n)_n$ for which (iii) above holds. We shall proceed by defining these elements in the braid groups $\mathbf{B}_{sn}$ in Section 3.1. In Section 3.2 we shall lift them to the quotients A_{sn} and show that (iii) holds.

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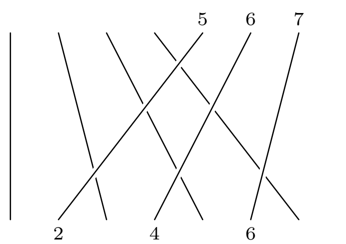
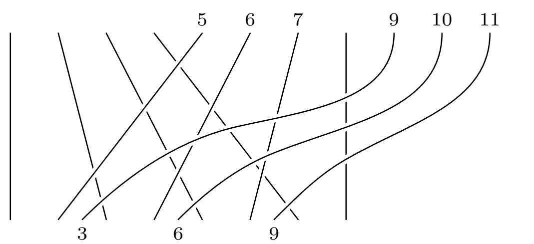
3.1 The Definition of the Unitaries at the Braid Group Level

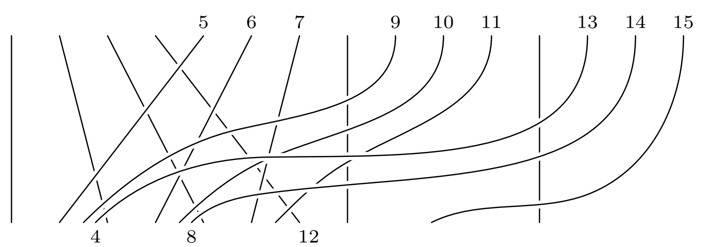
Fix $s \in \mathbb{N}_{\geq 2}$. Define for $n \in \mathbb{N}_{\geq 2}$ the following elements in $\mathbf{B}_{sn}$:

$$\gamma_n = \gamma_n(s) = \Phi_1^{-1}\Phi_2^{-1}\cdots\Phi_{n-1}^{-1},$$

where $\Phi_t = (\sigma_{(s-1)(t+1)}\cdots\sigma_{s(t+1)-2})(\sigma_{(s-2)(t+1)}\cdots\sigma_{s(t+1)-3})\cdots(\sigma_{t+1}\cdots\sigma_{st})$. Notice that $\gamma_{n+1} = \gamma_n\Phi_n^{-1}$ for every n. To illustrate the γ_n's pictorially, below we draw the diagrams for γ_2, γ_3, and γ_4 corresponding to $s = 4$:

$\gamma_2 = \Phi_1^{-1} =$ 

$\gamma_3 = \gamma_2\Phi_2^{-1} =$ 

$\gamma_4 = \gamma_3\Phi_3^{-1} =$ 

Proposition 3 *Define $\Psi_n\colon \mathbf{B}_n^{\otimes s} \to \mathbf{B}_{ns}$ by $\Psi_n(\beta) = \gamma_n\bar{\beta}\gamma_n^{-1}$ (where $\beta \mapsto \bar{\beta}$ is the juxtaposition embedding described in (2.1)). Then the diagram below is a commuting diagram for every $n \in \mathbb{N}_{\geq 2}$,*

$$\begin{array}{ccc} \mathbf{B}_n^{\otimes s} & \xrightarrow{\Psi_n} & \mathbf{B}_{ns} \\ \iota_n\downarrow & & \downarrow\jmath_n \\ \mathbf{B}_{n+1}^{\otimes s} & \xrightarrow[\Psi_{n+1}]{} & \mathbf{B}_{(n+1)s} \end{array}$$

where ι_n and $\jmath_n$ are given by $\iota_n(\beta_1\otimes\cdots\otimes\beta_s) = \iota(\beta_1)\otimes\cdots\otimes\iota(\beta_s)$ and $\jmath_n(\beta) = \iota^{(s)}(\beta)$, and where $\iota\colon \mathbf{B}_n \to \mathbf{B}_{n+1}$ is the canonical embedding.

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Proof

$$\jmath_n\big(\Psi_n(\beta)\big) = \Psi_{n+1}\big(\iota_n(\beta)\big) \iff \jmath_n(\gamma_n\bar{\beta}\gamma_n^{-1}) = \gamma_{n+1}\overline{\iota_n(\beta)}\gamma_{n+1}^{-1}$$
$$\iff \gamma_n\bar{\beta}\gamma_n^{-1} = \gamma_{n+1}\overline{\iota_n(\beta)}\gamma_{n+1}^{-1}$$
$$\iff \bar{\beta} = \gamma_n^{-1}\gamma_{n+1}\overline{\iota_n(\beta)}\gamma_{n+1}^{-1}\gamma_n$$
$$(\clubsuit) \qquad \iff \bar{\beta} = \Phi_n^{-1}\overline{\iota_n(\beta)}\Phi_n.$$

We want to show the above equality for every generator β of $\mathbf{B}_n^{\otimes s}$, that is, for elements of the form $\beta = 1_n \otimes \cdots \otimes \underbrace{\sigma_i}_{j\text{-th pos.}} \otimes \cdots \otimes 1_n$, with $i = 1, \ldots, n-1$ and $j = 1, \ldots, s$. (Remark: By looking at the geometric pictures in last page, we could omit the algebraic proof that follows below by just observing that conjugating by Φ_n^{-1} pulls the $n+1$-st strand of each tensor factor in $1_{n+1} \otimes \cdots \otimes \sigma_i \otimes \cdots \otimes 1_{n+1} = \iota_n(\beta)$ to the last s strands of $B_{(n+1)s}$, and so ($\clubsuit$) would follow.)

$$\bar{\beta} = \Phi_n^{-1}\overline{\iota_n(\beta)}\Phi_n \iff \mathrm{shift}_{n(j-1)}(\sigma_i) = \Phi_n^{-1}\,\mathrm{shift}_{(n+1)(j-1)}(\sigma_i)\Phi_n$$
$$\iff \sigma_{n(j-1)+i} = \Phi_n^{-1}\sigma_{(n+1)(j-1)+i}\Phi_n$$
$$(*) \qquad \iff \Phi_n\sigma_{n(j-1)+i}\Phi_n^{-1} = \sigma_{(n+1)(j-1)+i}.$$

We shall prove $(*)$ by induction on $s \geq 2$.

Case $s = 2$ In this case, $\Phi_n = (\sigma_{n+1}\cdots\sigma_{2n})$, and $j = 1$ or $j = 2$. If $j = 1$ then for any $i = 1, \ldots, n-1$ $\Phi_n\sigma_i\Phi_n^{-1} = \sigma_i$ by the defining braid relation (B2). If $j = 2$, then $\Phi_n\sigma_{n+i}\Phi_n^{-1} = (\sigma_{n+1}\cdots\sigma_{n+i+1})\sigma_{n+i}(\sigma_{n+i+1}^{-1}\cdots\sigma_{n+1}^{-1}) = \sigma_{i+n+1}$ by the braid relations (B2), (B4) (Lemma 1).

Inductive step, $s \Rightarrow s+1$ In this case,

$$\Phi_n^{(s)} = (\sigma_{s(n+1)}\cdots\sigma_{(s+1)(n+1)-2})(\sigma_{(s-1)(n+1)}\cdots\sigma_{(s+1)(n+1)-3})\cdots(\sigma_{n+1}\cdots\sigma_{(s+1)n}),$$
$$\Phi_n^{(s+1)} = (\sigma_{(s-1)(n+1)}\cdots\sigma_{s(n+1)-2})(\sigma_{(s-2)(n+1)}\cdots\sigma_{s(n+1)-3})\cdots(\sigma_{n+1}\cdots\sigma_{sn}).$$

It is not difficult to see that

$$\Phi_n^{(s+1)} = \mathrm{shift}_{n+1}(\Phi_n^{(s)})(\sigma_{n+1}\cdots\sigma_{(s+1)n}). \tag{3.1}$$

By the inductive hypothesis, for any $n \in \mathbb{N}$, $j = 1, \ldots, s$ and $i = 1, \ldots, n-1$,

$$\sigma_{i+(n+1)(j-1)} = \Phi_n^{(s)}\sigma_{i+n(j-1)}\Phi_n^{(s)\,-1}, \tag{3.2}$$

and we must show that this implies

$$\sigma_{i+(n+1)(j-1)} = \Phi_n^{(s+1)}\sigma_{i+n(j-1)}\Phi_n^{(s+1)\,-1},$$

for $j = 1, \ldots, s+1$, and $i = 1, \ldots, n-1$. By (3.1),

$$\begin{aligned}
&\Phi_n^{(s+1)^{-1}} \sigma_{i+(n+1)(j-1)} \Phi_n^{(s+1)} \\
&\quad = (\sigma_{(s+1)n}^{-1} \cdots \sigma_{n+1}^{-1}) \operatorname{shift}_{n+1}(\Phi_n^{(s)})^{-1} \sigma_{i+(n+1)(j-1)} \operatorname{shift}_{n+1}(\Phi_n^{(s)})(\sigma_{n+1} \cdots \sigma_{(s+1)n}) \\
&\quad = (\sigma_{(s+1)n}^{-1} \cdots \sigma_{n+1}^{-1}) \operatorname{shift}_{n+1}(\Phi_n^{(s)^{-1}} \sigma_{i+(n+1)(j-2)} \Phi_n^{(s)})(\sigma_{n+1} \cdots \sigma_{(s+1)n}).
\end{aligned}$$

By inductive hypothesis, (3.2), if $j = 1, \ldots, s+1$, then $j - 1 = 0, \ldots, s$ and so

$$\Phi_n^{(s)^{-1}} \sigma_{i+(n+1)(j-2)} \Phi_n^{(s)} = \begin{cases} \sigma_{i+n(j-2)} & \text{if } j \geq 2 \\ \sigma_i & \text{if } j = 1. \end{cases}$$

(Note that if $j = 1$, then σ_i commutes with $\Phi_n^{(s+1)}$ by (B2)). Thus,

$$\begin{aligned}
&\Phi_n^{(s+1)^{-1}} \sigma_{i+(n+1)(j-1)} \Phi_n^{(s+1)} \\
&\quad = \begin{cases} \sigma_i & \text{if } j = 1, \\ (\sigma_{(s+1)n}^{-1} \cdots \sigma_{n+1}^{-1}) \sigma_{i+(n+1)(j-2)} (\sigma_{n+1} \cdots \sigma_{(s+1)n}) & \text{if } j \geq 2. \end{cases}
\end{aligned} \tag{3.3}$$

If $j \geq 2$, then $i + n(j-1) + 1 \geq i + n + 1 \geq n + 1$, so that by braid relations (B2), (B5),

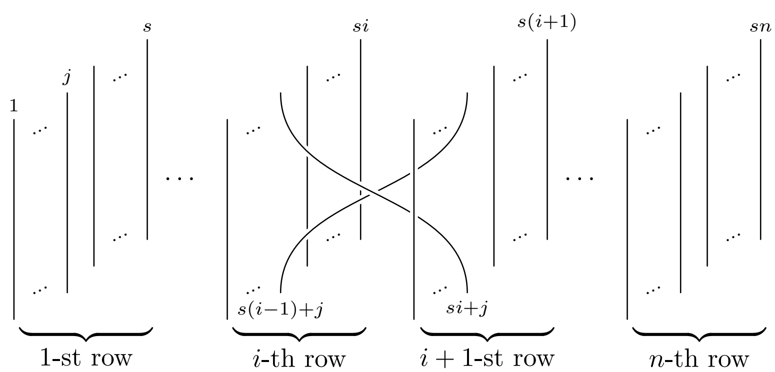
$$\begin{aligned}
&(\sigma_{(s+1)n}^{-1} \cdots \sigma_{n+1}^{-1}) \sigma_{i+(n+1)(j-2)} (\sigma_{n+1} \cdots \sigma_{(s+1)n}) \\
&\qquad = (\sigma_{(s+1)n}^{-1} \cdots \sigma_{i+n(j-1)}^{-1}) \sigma_{i+(n+1)(j-2)} (\sigma_{i+n(j-1)} \cdots \sigma_{(s+1)n}) \\
&\qquad = \sigma_{i+n(j-1)}
\end{aligned} \tag{3.4}$$

Therefore, by (3.3) and (3.4), for all $j = 1, \ldots, s+1$,

$$\Phi_n^{(s+1)^{-1}} \sigma_{i+(n+1)(j-1)} \Phi_n^{(s+1)} = \sigma_{i+n(j-1)}. \qquad \blacksquare$$

Remark 4 One could show by induction that, pictorially, Ψ_n maps generators of $\mathbf{B}_n^{\otimes s}$ into $\mathbf{B}_{sn}$ in the following way: For $1 \leq i \leq n-1$ and $1 \leq j \leq s$,

$$(1 \otimes \cdots \otimes \underbrace{\sigma_i}_{j\text{-th pos.}} \otimes \cdots \otimes 1) \mapsto$$



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Note that we are arranging the sn strands in n rows with s strands each, so that $\Psi_n(1\otimes\cdots\otimes\underbrace{\sigma_i}_{j\text{-th pos.}}\otimes\cdots\otimes 1)$ can be seen as a crossover between the j-th dots belonging to the i-th and $i+1$-st rows. One can see from this that $\Psi_n(1\otimes\cdots\otimes\underbrace{\sigma_i}_{j\text{-th pos.}}\otimes\cdots\otimes 1)\in\langle\sigma_{(i-1)s+j},\dots,\sigma_{is+j-1}\rangle$, which is a fact we shall need later on. More precisely,

$$\Psi_n(1\otimes\cdots\otimes\underbrace{\sigma_i}_{j\text{-th pos.}}\otimes\cdots\otimes 1)=v_{i,j}\sigma_{is+j-1}{v_{i,j}}^{-1}, \tag{3.5}$$

where $v_{i,j}\in\langle\sigma_{(i-1)s+j},\dots,\sigma_{is+j-1}\rangle$ is given by

$$v_{i,j}=\begin{cases}\mathrm{shift}_{s(i-1)}\big((\sigma_j\cdots\sigma_{s-1})(\sigma_s^{-1}\cdots\sigma_{s+j-2}^{-1})\big) & \text{if } 1<j<s\\ \mathrm{shift}_{s(i-1)}\big((\sigma_1\cdots\sigma_{s-1})\big) & \text{if } j=1\\ \mathrm{shift}_{s(i-1)}\big((\sigma_s^{-1}\cdots\sigma_{2s-2}^{-1})\big) & \text{if } j=s.\end{cases}$$

3.2 Lifting the Unitaries to the Braid Quotients. Periodicity and Commuting Square Properties

From now on we shall consider s to be a fixed integer not smaller than 2. We shall lift the maps Ψ_n defined before to the braid quotients. If ρ is a fixed unitary braid representation as in the preliminaries, define for every $n\in\mathbb{N}$ the finite dimensional C* algebras $A_n=\rho(\mathbb{C}\mathbf{B}_n)$. If we set the unitary $u_n=\rho(\gamma_n)\in A_{sn}$, where $\gamma_n\in\mathbf{B}_{sn}$ implements Ψ_n as in Proposition 3, then

$$u_nA_n^{\otimes s}{u_n}^*=\rho\big(\Psi_n(\mathbb{C}\mathbf{B}_n^{\otimes s})\big)\subset A_{sn}.$$

(Observe that by multiplicativity of the trace, property (v) in the preliminaries, $\rho(\mathbb{C}\mathbf{B}_n^{\otimes s})\simeq A_n^{\otimes s}$). By Proposition 3 the diagram below is a commuting diagram:

$$\begin{array}{ccccc}\mathbb{C}\mathbf{B}_n^{\otimes s} & \xrightarrow{\Psi_n} & \mathbb{C}\mathbf{B}_{ns} & \xrightarrow{\rho} & A_{sn}\\ {\scriptstyle\iota_n}\downarrow & & {\scriptstyle\jmath_n}\downarrow & & \downarrow{\scriptstyle\rho\jmath_n}\\ \mathbb{C}\mathbf{B}_{n+1}^{\otimes s} & \xrightarrow[\Psi_{n+1}]{} & \mathbb{C}\mathbf{B}_{(n+1)s} & \xrightarrow[\rho]{} & A_{s(n+1)}\end{array}$$

In consequence, the following diagram is also a commuting diagram:

$$(*)\qquad\begin{array}{ccc}A_n^{\otimes s} & \xrightarrow{\hat u_n} & A_{sn}\\ {\scriptstyle\bar\iota_n}\downarrow & & \downarrow{\scriptstyle\bar\jmath_n}\\ A_{n+1}^{\otimes s} & \xrightarrow[\hat u_{n+1}]{} & A_{s(n+1)},\end{array}$$

where $\hat u_n(x):=u_nxu_n^*$ is the lifting of Ψ_n, and where $\bar\iota_n=\rho\iota_n$ and $\bar\jmath_n=\rho\jmath_n$ are the liftings of the natural inclusions ι_n and $\jmath_n$ defined in Proposition 3.

Proposition 5 *For every $n\in\mathbb{N}$, the commuting diagram $(*)$ above has the commuting square property.*

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Proof We must show that if $E_{A_n^{\otimes s}}$ and $E_{A_{sn}}$ are the trace preserving conditional expectations onto $A_n^{\otimes s}$ and A_{sn} respectively, then the diagram below is a commuting diagram (see *e.g.* [GHJ, Proposition 4.2.1.] for equivalent definitions of a commuting square):

$$\begin{array}{ccc} A_n^{\otimes s} & \xrightarrow{\hat{u}_n} & A_{sn} \\ {\scriptstyle E_{A_n^{\otimes s}}}\uparrow & & \uparrow{\scriptstyle E_{A_{sn}}} \\ A_{n+1}^{\otimes s} & \xrightarrow[\hat{u}_{n+1}]{} & A_{s(n+1)} \end{array}$$

For this we shall use the same technique as in [E1] for the 2-sided construction, where we intercalate two intermediate algebras. Here we shall intercalate $2(s-1)$ intermediate algebras. This consists of decomposing the commuting diagram $(*)$ as the concatenation of s connected commuting diagrams in the following way, where for $j = 1, \dots, s$ we set $T_j = (1 \otimes \cdots 1 \otimes \underset{j\text{-th pos.}}{g_n} \otimes 1 \cdots \otimes 1)$:

$$\begin{array}{ccc} A_n^{\otimes s} & \xrightarrow{\hat{u}_n} & A_{sn} \\ \downarrow & & \downarrow \\ \langle A_n^{\otimes s}, T_1 \rangle & \xrightarrow{\hat{u}_{n+1}} & A_{sn+1} \\ \downarrow & & \downarrow \\ \langle A_n^{\otimes s}, T_1, T_2 \rangle & \xrightarrow{\hat{u}_{n+1}} & A_{sn+2} \\ \downarrow & & \downarrow \\ \vdots & & \vdots \\ \langle A_n^{\otimes s}, T_1, T_2, \dots, T_{s-1} \rangle & \xrightarrow{\hat{u}_{n+1}} & A_{sn+s-1} \\ \downarrow & & \downarrow \\ A_{n+1}^{\otimes s} & \xrightarrow[\hat{u}_{n+1}]{} & A_{s(n+1)}, \end{array}$$

where $g_n = \rho(\sigma_n)$ as in the preliminaries. Let us observe that

$$A_{n+1}^{\otimes s} = \langle A_n^{\otimes s}, T_1, T_2, \dots, T_s \rangle,$$

by property **(iv)** in the preliminaries, and that

$$u_{n+1} \langle A_n^{\otimes s}, T_1, T_2, \dots, T_j \rangle u_{n+1}^* \subset A_{sn+j}$$

for $j = 1, \dots, s$, which is a consequence of Remark 4 (at the end of Section 3.1). It follows that each of the subdiagrams above is a commuting diagram since so is the diagram $(*)$. We shall proceed to show that each of these subdiagrams has the commuting square property: For $j = 1, \dots, s-1$, let $E_{\langle A_n^{\otimes s}, T_1, T_2, \dots, T_j \rangle} \colon \langle A_n^{\otimes s}, T_1, T_2, \dots, T_{j+1} \rangle \to \langle A_n^{\otimes s}, T_1, T_2, \dots, T_j \rangle$ and $E_{A_{sn+j}} \colon A_{sn+j+1} \to A_{sn+j}$ be the unique trace preserving conditional expectations onto $\langle A_n^{\otimes s}, T_1, T_2, \dots, T_j \rangle$ and onto A_{sn+j}, respectively. We should show that

$$(\hat{u}_{n+1} \circ E_{\langle A_n^{\otimes s}, T_1, T_2, \dots, T_j \rangle})(y) = (E_{A_{sn+j}} \circ \hat{u}_{n+1})(y)$$

for every $y \in \langle A_n^{\otimes s}, T_1, T_2, \dots, T_{j+1} \rangle$. If $y \in \langle A_n^{\otimes s}, T_1, T_2, \dots, T_{j+1} \rangle$, by property (iv) in the preliminaries, y is a sum of elements of the form $aT_{j+1}^{\pm 1} b + c$, with $a, b, c \in \langle A_n^{\otimes s}, T_1, T_2, \dots, T_j \rangle$. Thus, by linearity and the bimodule property of the conditional expectations (that is, $E \colon M \to N \Rightarrow E(axb) = aE(x)b$ if $a, b \in N$) it is enough to consider $y = T_{j+1}^{\pm 1}$. By Remark 4, (3.5),

$$u_{n+1} T_{j+1}^{\pm 1} u_{n+1}^* = v g_{sn+j}^{\pm 1} v^{-1}$$

for some $v \in A_{sn+j}$. Therefore, if $x \in A_{sn+j}$, by property (v) in the preliminaries

$$\begin{aligned} \operatorname{tr}(u_{n+1} T_{j+1}^{\pm 1} u_{n+1}^* x) &= \operatorname{tr}(v g_{sn+j}^{\pm 1} v^{-1} x) \\ &= \operatorname{tr}(g_{sn+j}^{\pm 1} v^{-1} x v) = \operatorname{tr}(g_{sn+j}^{\pm 1}) \operatorname{tr}(v^{-1} x v) \\ &= \eta^{(\pm)} \operatorname{tr}(x), \end{aligned}$$

so that $E_{A_{sn+j}}(u_{n+1} T_{j+1}^{\pm 1} u_{n+1}^*) = \eta^{(\pm)} \cdot 1$. On the other hand, if $x = (x_1 \otimes \cdots \otimes x_s) \in A_n^{\otimes s}$, then by property (v) in the preliminaries

$$\begin{aligned} \operatorname{tr}(T_{j+1}^{\pm 1} x) &= \operatorname{tr}\big((1 \otimes \cdots \otimes \underset{j+1\text{-st pos.}}{g_n^{\pm 1}} \otimes \cdots 1)(x_1 \otimes \cdots \otimes x_s)\big) \\ &= \operatorname{tr}(x_1) \cdots \operatorname{tr}(x_j) \operatorname{tr}(g_n^{\pm 1} x_{j+1}) \operatorname{tr}(x_{j+2}) \cdots \operatorname{tr}(x_s) \\ &= \operatorname{tr}(x_1) \cdots \operatorname{tr}(x_j) \operatorname{tr}\big(E_{A_n}(g_n^{\pm 1}) x_{j+1}\big) \operatorname{tr}(x_{j+2}) \cdots \operatorname{tr}(x_s) \\ &= \eta^{(\pm)} \operatorname{tr}(x), \end{aligned}$$

so that $(\hat{u}_n \circ E_{\langle A_n^{\otimes s}, T_1, T_2, \dots, T_j \rangle})(T_{j+1}^{\pm 1}) = \eta^{(\pm)} \cdot 1 = (E_{A_{sn+j}} \circ \hat{u}_{n+1})(T_{j+1}^{\pm 1})$.

Exactly the same argument as above can be used to show that the very top subdiagram has the commuting square property; that is, that

$$(\hat{u}_n \circ E_{A_n^{\otimes s}})(y) = (E_{A_{sn}} \circ \hat{u}_{n+1})(y) \quad \forall y \in \langle A_n^{\otimes s}, T_1 \rangle.$$

Finally, since all the commuting concatenated diagrams have the commuting square property, so does the diagram $(*)$. ∎

Proposition 6 *The ascending sequence $(A_n^{\otimes s})_n \subset_{\hat{u}_n} (A_{sn})_n$ is k-periodic, where k is the period of $(A_n)_n \subset (A_{n+1})_n$.*

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Proof The proof is analogous to that for the 2-sided case in [E1]. By assumption on the braid quotients ((iii) in the preliminaries), the sequence $(A_n)_n$ is k-periodic. Since the inclusion matrix for $A_n^{\otimes s} \subset A_{n+1}^{\otimes s}$ is given by $G_n^{\otimes s}$, where G_n is the inclusion matrix for $A_n \subset A_{n+1}$, then $G_n^{\otimes s} = G_{n+k}^{\otimes s}$, so that $(A_n^{\otimes s})_n$ is k-periodic. Similarly, the inclusion matrix for $A_{sn} \subset A_{s(n+1)}$ is given by $G_{sn}G_{sn+1}\cdots G_{sn+s-1} = G_{sn+sk}G_{sn+sk+1}\cdots G_{sn+sk+s-1}$, so that $(A_{sn})_n$ is k-periodic. It remains to show that the inclusion $A_n^{\otimes s} \subset_{\hat{u}_n} A_{sn}$ is k-periodic, which is equivalent to showing that $A_n^{\otimes s} \subset A_{sn}$ is k-periodic with the natural inclusion from braid juxtaposition. The inclusion data for $A_n^{\otimes s} \subset A_{sn}$ are given in terms of the structure constants for the multiplication $K_0(A_n)^s \to K_0(A_{sn})$, that is, the coefficients for the inclusion matrix are given by

$$g_{(\lambda_1,\dots,\lambda_s),\mu} = \sum_{i=1}^{s-2} \sum_{\mu_i \in \Lambda_{(i+1)n}} c^{\mu}_{\lambda_1,\mu_{s-2}} c^{\mu_{s-2}}_{\lambda_2,\mu_{s-3}} \cdots c^{\mu_2}_{\lambda_{s-2},\mu_1} c^{\mu_1}_{\lambda_{s-1},\lambda_s}.$$

But for large n, the structure constants coincide with those for the multiplication $K_0(A_{n+k})^s \to K_0(A_{s(n+k)})$, by s applications of Lemma 2(c) and Remark (1) in the preliminaries. Thus, the inclusion matrices for $A_n^{\otimes s} \subset A_{sn}$ and $A_{n+k}^{\otimes s} \subset A_{s(n+k)}$ agree, and so $A_n^{\otimes s} \subset A_{sn}$ is k-periodic. ■

3.3 Definition of the Pairs $R(s) \subset R$. The Relative Commutant

It follows from periodicity that there is a unique positive faithful normalised trace on $\bigcup_n (u_n A_n^{\otimes s} u_n^*)$ and $\bigcup_n A_{sn}$. Define $R(s)$ and R to be the weak closure of these unions in the GNS representation with respect to the trace. We obtain a pair of hyperfinite II_1 factors $R(s) \subset R$.

Proposition 7 *The relative commutant $R(s)' \cap R$ is isomorphic to $\mathbb{C}$.*

Proof Because of the periodicity and commuting square properties we can apply Wenzl's estimate [W1, Theorem 1.6]: For $n \in \mathbb{N}$ sufficiently large, and for any projection $\tilde{p} \in u_n A_n^{\otimes s} u_n^*$

$$\dim R(s)' \cap R \leq \dim \tilde{p}\big((u_n A_n^{\otimes s} u_n^*)' \cap A_{sn}\big).$$

By the assumptions on the braid representations we can find a projection with the contraction property $p \in A_n$, for n large. Then, the projection $p^{\otimes s} \in A_{sn}$ also has the contraction property (see Remarks in the preliminaries). In particular,

$$\mathbb{C} \simeq p^{\otimes s} A_n^{\otimes s} p^{\otimes s} \simeq p^{\otimes s} A_{sn} p^{\otimes s},$$

so that $p^{\otimes s}\big((A_n^{\otimes s})' \cap A_{sn}\big) \simeq \mathbb{C}$. Therefore,

$$\dim R(s)' \cap R \leq \dim \tilde{p}\big((u_n A_n^{\otimes s} u_n^*)' \cap A_{sn}\big) \simeq \mathbb{C},$$

where $\tilde{p} = \hat{u}_n(p^{\otimes s})$. ■

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3.4 Definition of the Pairs $R(s)^{(\vec{m})} \subset R$

As mentioned in the introduction, we can introduce a variation in the construction of the pairs. We assume again that $s \geq 2$ is a fixed integer, and we also fix a non-negative vector $\vec{m} = (m_1, \dots, m_s) \in \mathbb{Z}^s$. For n large, consider the finite dimensional C* algebra

$$\begin{aligned} A_n^{\vec{m}} &:= A_{m_1+1,n} \otimes A_{m_2+1,n} \otimes \cdots \otimes A_{m_s+1,n} \\ &\simeq A_{n-m_1} \otimes A_{n-m_2} \otimes \cdots \otimes A_{n-m_s}, \end{aligned}$$

where $A_{s,t}$ is the algebra generated by $\{1, g_s^{\pm 1}, \dots, g_{t-1}^{\pm 1}\}$.

Proposition 8 *The pair of ascending sequences $(A_n^{\vec{m}})_n \subset_{\hat{u}_n} (A_{sn})_n$ has the periodicity and commuting square properties.*

Proof This is an easy consequence of Propositions 5 and 6. The commuting square property follows from the fact that the two commuting subdiagrams below have the commuting square property:

$$\begin{array}{ccccc} A_n^{\vec{m}} & \hookrightarrow & A_n^{\otimes s} & \xrightarrow{\hat{u}_n} & A_{sn} \\ \downarrow & & \downarrow & & \downarrow \\ A_{n+1}^{\vec{m}} & \hookrightarrow & A_{n+1}^{\otimes s} & \xrightarrow[\hat{u}_{n+1}]{} & A_{sn+1} \end{array}.$$

Periodicity follows from Proposition 6: The inclusion matrix for $u_n A_n^{\vec{m}} u_n^* \subset A_{sn}$ is the composition of the k-periodic inclusion matrices for $A_n^{\vec{m}} \subset A_n^{\otimes s}$ and for $A_n^{\otimes s} \subset A_{sn}$ respectively. And the inclusion matrix for $A_n^{\vec{m}} \subset A_{n+1}^{\vec{m}}$ is given by $G_{n-m_1} \otimes G_{n-m_2} \otimes \cdots \otimes G_{n-m_s}$, where G_n is the inclusion matrix for $A_n \subset A_{n+1}$. ■

Thus, we can define $R(s)^{(\vec{m})}$ and R to be the hyperfinite II_1 factors obtained by taking the weak operator closure of $\bigcup_n (u_n A_n^{\vec{m}} u_n^*)$ and $\bigcup_n A_{sn}$ in the GNS representation with respect to the unique positive faithful trace.

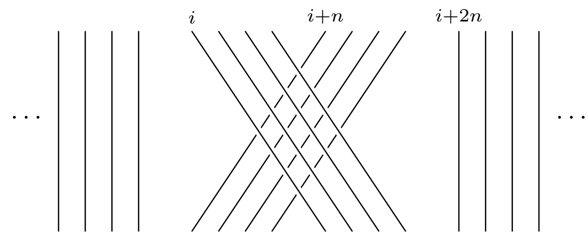
Lemma 9 *If $\delta \in \mathbb{S}_s$ is a permutation and if $\vec{m}_\delta := (m_{\delta(1)}, \dots, m_{\delta(s)}) \in \mathbb{Z}^s$, then the pair $R(s)^{(\vec{m})} \subset R$ is conjugate to $R(s)^{(\vec{m}_\delta)} \subset R$.*

Proof We only sketch a proof. The two pairs are conjugate because for every n sufficiently large there exists a unitary $w_n \in A_{sn}$ such that

(i) $\hat{w}_n(A_n^{(\vec{m})}) := w_n(A_n^{(\vec{m})}) w_n^* = A_n^{(\vec{m}_\delta)}$,

(ii) $\hat{w}_n$ is compatible with the natural inclusions, that is, $\hat{w}_{n+1} \circ \jmath_n = \jmath_n \circ \hat{w}_n$ and $\hat{w}_{n+1} \circ \bar{\iota}_n = \bar{\iota}_n \circ \hat{w}_n$.

The unitaries $w_n \in A_{sn}$ are defined in the following way. Take any braid $\beta \in \mathbf{B}_s$ such that the permutation δ can be obtained from β via the natural epimorphism $\mathbf{B}_s \to \mathbb{S}_s$ (pictorially, one can regard this map as gluing the crossings in the diagram of a braid so as to not distinguish which string lies on top or under the others). Define for each generator $\sigma_i \in \mathbf{B}_s$ the element $\sigma_i^{(n)} \in \mathbf{B}_{sn}$ given by (pictorially)



If we write now β as a product of generators σ_i, $\beta = \sigma_{i_1} \cdots \sigma_{i_t}$ (not necessarily in an increasing order) then set $\beta^{(n)} := \sigma_{i_1}^{(n)} \cdots \sigma_{i_t}^{(n)}$. It is easy to check that if $(\alpha_1 \otimes \cdots \otimes \alpha_s) \in \mathbf{B}_n^{\otimes s}$ then $\beta^{(n)}(\alpha_1 \otimes \cdots \otimes \alpha_s)\beta^{(n)^{-1}} = (\alpha_{\sigma(1)} \otimes \cdots \otimes \alpha_{\sigma(s)})$. It can be shown that the unitary $w_n := \rho(\beta^{(n)}) \in A_{sn}$ has the properties (i) and (ii). ■

Proposition 10 *The relative commutant $R(s)^{(\vec{m})'} \cap R$ is isomorphic to the algebra $A_{|\vec{m}|}$, where $|\vec{m}| := \sum_{i=1}^{s} m_i$.*

Proof We split this proof in parts (i) and (ii). In (i) we show that $\dim R(s)^{(\vec{m})'} \cap R \leq \dim A_{|\vec{m}|}$, and in (ii) we show that $R(s)^{(\vec{m})'} \cap R$ contains a subalgebra isomorphic to $A_{|\vec{m}|}$:

(i) Note that the pair $R(s)^{(\vec{m})} \subset R$ can also be obtained via approximating by the following finite dimensional inclusions (which satisfy the periodicity and the commuting square properties as well):

$$\tilde{A}_n^{\vec{m}} := A_{m_1+1,n+m_1-1} \otimes A_{m_2+1,n+m_2-1} \otimes \cdots \otimes A_{m_s+1,n+m_s-1} \subset_{\hat{u}_n} A_{sn+|\vec{m}|}. \tag{3.6}$$

We can use Wenzl's estimate [W1, Theorem 1.6] as in Proposition 7: For $n \in \mathbb{N}$ sufficiently large and for any non-zero projection $p \in u_n \tilde{A}_n^{\vec{m}} u_n^*$,

$$\dim R(s)^{(\vec{m})'} \cap R \leq \dim p\big((u_n \tilde{A}_n^{\vec{m}} u_n^*)' \cap A_{sn}\big).$$

The inclusion (3.6) is conjugate (via a special unitary in $A_{sn+|\vec{m}|}$ that relabels the generator indices) to

$$A_n^{\otimes s} \subset A_{sn+|\vec{m}|}.$$

For large n there exists a projection $p \in A_n$ with the contraction property, so that $p^{\otimes s}$ has the contraction property in A_{sn} (see the remarks in the preliminaries). Thus

$$\mathbb{C} \simeq p^{\otimes s}(A_n^{\otimes s})p^{\otimes s} \subset p^{\otimes s} A_{sn+|\vec{m}|} p^{\otimes s} \simeq A_{|\vec{m}|},$$

so that

$$\dim R(s)^{(\vec{m})'} \cap R \leq \dim A_{|\vec{m}|}. \tag{3.7}$$

(ii) By construction, the algebra $A_{m_1} \otimes \cdots \otimes A_{m_s}$ is isomorphic to the subalgebra of $R(s)^{(\vec{m})'} \cap R$ given by $\varinjlim u_n(A_{m_1} \otimes \cdots \otimes A_{m_s}\rangle u_n^*$.

Without loss of generality, by Lemma 9, one can assume that the vector $\vec{m}$ has its coordinates ordered decreasingly, $m_1 \geq m_2 \geq \cdots \geq m_s$, by considering the appropriate permutation $\delta \in \mathcal{S}_s$. Take $l := \max\{i : m_i \neq 0\}$, if it exists. If $m_i = 0$ for $i = 1, \ldots, s$ or if $l = 1$, then $A_{m_1} \otimes \cdots \otimes A_{m_s} \simeq A_{|\vec{m}|}$, and from (3.7) and the above paragraph we obtain that $R(s)^{(\vec{m})'} \cap R \simeq A_{|\vec{m}|}$. Thus, we may assume that l exists and $l \geq 2$.

We shall show first that the elements g_i, with $i = 1, \ldots, l-1$, commute with $u_n A_n^{\vec{m}} u_n^*$ for every n sufficiently large (and thus they will be contained in $R(s)^{(\vec{m})'} \cap R$). Afterwards we shall show that the algebra generated by these g_i's and by $\varinjlim u_n(A_{m_1} \otimes \cdots \otimes A_{m_s}\rangle u_n^*$ has a subalgebra isomorphic to $A_{|\vec{m}|}$, so that by (3.7) the proof of the proposition shall be complete.

By Remark 4 at the end of Section 3.1 and the braid relation (B2), it is immediate that if $i = 1, \ldots, l-1$ then

$$[g_i, u_n(1 \otimes \cdots \otimes \underset{j\text{-th pos.}}{g_r} \otimes \cdots \otimes 1)u_n^*] = 0 \tag{3.8}$$

for $r \geq 2$. So we only need to show (3.8) in the case that $r = 1$ and $m_j = 0$ (for we take $(1 \otimes \cdots \otimes \underset{j\text{-th pos.}}{g_1} \otimes \cdots \otimes 1) \in A_n^{\vec{m}}$). If there exists j with $m_j = 0$, then $j > l$ by definition of l. In this case, using the formula (3.5):

$$u_n(1 \otimes \cdots \otimes \underset{j\text{-th pos.}}{g_1} \otimes \cdots \otimes 1)u_n^*$$
$$= \begin{cases} (g_j \cdots g_{s-1})(g_s^{-1} \cdots g_{s+j-2}^{-1})g_{s+j-1}(g_{s+j-2} \cdots g_s)(g_{s-1}^{-1} \cdots g_j^{-1}) & \text{if } j < s \\ (g_s^{-1} \cdots g_{2s-2}^{-1})g_{s+j-1}(g_{2s-2} \cdots g_s) & \text{if } j = s. \end{cases}$$

Thus, by the braid relation (B2), (3.8) is true if $j = s$. Also by (B2), (3.8) holds if $j < s$, since $|i - j| \geq 2$ (because $i \leq l-1$ and $j > l$).

It remains to show that for large n, the algebra generated by $u_n(A_{m_1} \otimes \cdots \otimes A_{m_s}\rangle u_n^*$ and by the elements g_i, for $i = 1, \ldots, l-1$, contains a subalgebra isomorphic to $A_{|\vec{m}|}$. For this, let us define the map

$$\Gamma \colon A_{|\vec{m}|} \to \langle u_n(A_{m_1} \otimes \cdots \otimes A_{m_s}\rangle u_n^*, g_i, i = 1, \ldots, l-1\rangle,$$

in the following way: For $j = 1, \ldots, s$,

$$\Gamma|_{(1\otimes\cdots 1\otimes A_{m_j}\otimes 1\cdots\otimes 1)} = \hat{u}_n|_{(1\otimes\cdots 1\otimes A_{m_j}\otimes 1\cdots\otimes 1)}.$$

For $r \leq l-1$ and even,

$$\Gamma(g_{m_1+\cdots+m_r}) = g_r.$$

For $r \le l-1$ and odd,

$$\Gamma(g_{m_1+\cdots+m_r}) = u_n(1\otimes\cdots\otimes 1\otimes \underbrace{g_{m_r-1}^{-1}\cdots g_1^{-1}}_{r\text{-th pos.}}\otimes 1\cdots\otimes 1)u_n^* g_r u_n$$

$$(1\otimes\cdots\otimes 1\otimes \underbrace{g_1\cdots g_{m_{r+1}-1}}_{r+1\text{-st pos.}}\otimes 1\cdots\otimes 1)u_n^*.$$

To check that this map is well defined, one should check that the braid relations are preserved, which is straightforward but long and tedious, so we shall omit it this time. The fact that the morphism is injective follows from the trace properties. ■

4 A Formula For the Index

Theorem 11 $[R:R(s)^{(\vec{m})}] = W^{|\vec{m}|}[R:R(2)]^{s-1}$, *where* $W := [R:R(1)]$.

Proof By [W1, Theorem 1.5(iii)], the index for the pair $R(s)^{(\vec{m})} \subset R$ is

$$[R:R(s)^{(\vec{m})}] = \frac{\|\vec{s}^{(n)}\|^2}{\|\vec{v}^{(n)}\|^2},$$

for large n, where $\vec{s}^{(n)}$ and $\vec{v}^{(n)}$ are the weight vectors for the trace restricted to $u_n A_n^{\vec{m}} u_n^*$ and to A_{sn}, respectively. Because of the multiplicativity of the trace,

$$s^{(n)}_{i_1,i_2,\dots,i_s} = t^{(n-m_1)}_{i_1}\cdots t^{(n-m_s)}_{i_s} \quad\text{and } \vec{v}^{(n)} = \vec{t}^{(sn)},$$

where $\vec{t}^{(n)}$ is the weight vector for the trace on A_n. Also by [W1, Theorem 1.5(iii)], $W = \frac{\|\vec{t}^{(n)}\|^2}{\|\vec{t}^{(n+1)}\|^2}$ for large n. Hence,

$$\begin{aligned}
[R:R(s)^{(\vec{m})}] &= \frac{\prod_{i=1}^{s}\|\vec{t}^{(n-m_i)}\|^2}{\|\vec{t}^{(sn)}\|^2}\\
&= \prod_{j=0}^{n(s-1)+m_1-1}\frac{\|\vec{t}^{(n-m_1+j)}\|^2}{\|\vec{t}^{(n-m_1+j+1)}\|^2}\prod_{i=2}^{s}\|\vec{t}^{(n-m_i)}\|^2\\
&= W^{n(s-1)+m_1}\prod_{i=2}^{s}\|\vec{t}^{(n-m_i)}\|^2\\
&= W^{m_1}(\|\vec{t}^{(n)}\|^2W^n)^{s-1}\prod_{i=2}^{s}\frac{\|\vec{t}^{(n-m_i)}\|^2}{\|\vec{t}^{(n)}\|^s}\\
&= W^{m_1}(\|\vec{t}^{(n)}\|^2W^n)^{s-1}\prod_{i=2}^{s}\prod_{j=0}^{m_i-1}\frac{\|\vec{t}^{(n-m_i+j)}\|^2}{\|\vec{t}^{(n-m_i+j+1)}\|^2}\\
&= W^{m_1}(\|\vec{t}^{(n)}\|^2W^n)^{s-1}\prod_{i=2}^{s}W^{m_i}\\
&= (\|\vec{t}^{(n)}\|^2W^n)^{s-1}W^{|\vec{m}|}.
\end{aligned}$$

On the other hand, by [E1, (3.3.2)], $[R : R(2)] = \|\vec{t}^{(n)}\|^2 W^n$, for large n, and so the desired formula follows. ■

Corollary 12 *If $p \in R(s)^{(\vec{m})'} \cap R$ is a nonzero projection, the index for the pair of reduced factors $pR(s)^{(\vec{m})}p \subset pRp$ is*

$$[pRp : pR(s)^{(\vec{m})}p] = [R : R(2)]^{s-1}[pRp : pR(1)^{|\vec{m}|}p].$$

Proof By [W1, Theorem 1.5(iii)] and Theorem 11, $[pRp : pR(s)^{(\vec{m})}p] = [R : R(s)^{(\vec{m})}]\cdot \mathrm{tr}(p)^2 = [R : R(2)]^{s-1}W^{|\vec{m}|}\,\mathrm{tr}(p)^2$. Denote by $R(1)^{|\vec{m}|} \subset R$ the one-sided inclusion given by $\langle g_{|\vec{m}|+1}, \ldots, g_n, \ldots\rangle'' \subset \langle g_1, \ldots, g_n, \ldots\rangle''$ (as in the introduction). Because $R(s)^{(\vec{m})'} \cap R \simeq A_{|\vec{m}|} \simeq R(1)^{|\vec{m}|'} \cap R$ (see [E1]), and by [W1, Theorem 1.5(iii)], $[pRp : pR(1)^{|\vec{m}|}p] = \mathrm{tr}(p)^2 W^{|\vec{m}|}$. ■

Remarks When the braid representations considered are the ones associated with classical Lie algebras (as described in [W1], [W2]), the two-sided inclusions $R(2) \subset R$ are conjugate to the asymptotic inclusions (see [O]) of the corresponding one-sided inclusions, as it was shown in [E1], [E2], and also by Goto in [G]. Also, the family of multi-sided subfactors forms a descending sequence of intermediate subfactors $\cdots R(n+1) \subset R(n) \subset \cdots \subset R(2) \subset R$, where for $n \geq 2$ the inclusions $R(n+1) \subset R(n)$ are conjugate to $R(2) \subset R$ (work in preparation). The two-sided inclusions have finite depth, see for instance [EK]. We expect that all the multi-sided inclusions have finite depth as well, and we shall look at whether there is any relation to asymptotic inclusions when $s > 2$.

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