ON CLEAN LAURENT SERIES RINGS

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Abstract

Here we prove that, for a 2-primal ring R, the Laurent series ring $R((x))$ is a clean ring if and only if R is a semiregular ring with $J(R)$ nil. This disproves the claim in K. I. Sonin ['Semiprime and semiperfect rings of Laurent series', *Math. Notes* **60** (1996), 222–226] that the Laurent series ring over a clean ring is again clean. As an application of the result, it is shown that, for a 2-primal ring R, $R((x))$ is semiperfect if and only if $R((x))$ is semiregular if and only if R is semiperfect with $J(R)$ nil.

1. Introduction

Throughout rings are associative with identity. For a ring R, $J(R)$ and $U(R)$ denote the Jacobson radical and the set of units of R, respectively. The ring of Laurent series $R((x))$ is defined as a ring that consists of the formal series

$$\sum_{i=l}^{\infty} a_i x^i, \quad l \in \mathbb{Z}, a_i \in R,$$

where x is an independent variable, with the ordinary addition and with multiplication given by the formula

$$\left(\sum_{i=p}^{\infty} a_i x^i\right)\left(\sum_{j=q}^{\infty} a_j x^j\right) = \sum_{k=p+q}^{\infty} \left(\sum_{i+j=k} a_i b_j\right) x^k.$$

We would like to point out that each element of $R((x))$ can be written as $x^l \sum_{i=0}^{\infty} a_i x^i$ or $x^l(a_0 + a_1 x + \cdots)$.

An element of a ring R is called a clean element if it is the sum of an idempotent and a unit, while the ring R is called a clean ring if each of its elements is a clean element (see [4]). A ring R is called 2-primal if the prime radical $N(R)$ is exactly the set of nilpotent elements. Here we prove that, for a 2-primal ring R, $R((x))$ is a clean ring if and only if R is a semiregular ring with $J(R)$ nil. This disproves the claim in [7, Corollary, page 224] that the Laurent series ring over a clean ring is again clean. As an application of the result, it is shown that, for a 2-primal ring R, $R((x))$ is semiperfect if and only if $R((x))$ is semiregular if and only if R is semiperfect with $J(R)$ nil.

2. The results

Recall that a reduced ring is a ring with no nonzero idempotents.

LEMMA 2.1. *Let R be a reduced ring. If $f^2 = f \in R((x))$, then $f \in R$.*

PROOF. Suppose that $f = \sum_{i \geq k} a_i x^i$ is an idempotent of $R((x))$ with $a_k \neq 0$. Then

$$\sum_{i \geq k} a_i x^i = \left(\sum_{i \geq k} a_i x^i\right)\left(\sum_{i \geq k} a_i x^i\right),$$

which implies that $k = 0$ since R is reduced. Suppose that there exists $l > 0$ such that $a_l \neq 0$. We can assume that l is the smallest with this condition. Thus $a_i = 0$ for all $0 < i < l$, and $f = a_0 + a_l x^l + a_{l+1} x^{l+1} + \cdots$. From $f^2 = f$, it follows that $a_0^2 = a_0$ and $a_l = a_0 a_l + a_l a_0$. Multiplying the last equation on the left by a_0 we have

$$a_0 a_l + a_0 a_l a_0 = a_0 a_l$$

and so $a_0 a_l a_0 = 0$. Since R is reduced, $a_0 a_l = a_l a_0 = 0$ and $a_l = 0$ follows, which is a contradiction. Thus $f = a_0 \in R$. □

LEMMA 2.2. *Let R be a reduced ring such that $R((x))$ is a clean ring. Then R is (von Neumann) regular.*

PROOF. Let $a \in R$. Then $-ax^{-1} = e + u$ where $e^2 = e \in R((x))$ and u is a unit of $R((x))$. By Lemma 2.1, $e \in R$. So $ax^{-1} + e$ is a unit of $R((x))$, and hence $a + ex$ is a unit of $R((x))$. There exists $\sum_{i \geq 0} a_i x^i \in R((x))$ such that

$$(a + ex)(a_0 + a_1 x + \cdots) = x^k,$$

where $k \geq 0$. Thus, it follows that

$$aa_0 = 0$$
$$aa_1 + ea_0 = 0$$
$$aa_2 + ea_1 = 0$$
$$\vdots$$
$$aa_{k-1} + ea_{k-2} = 0$$
$$aa_k + ea_{k-1} = 1.$$

Multiplying the second equation by a from the left yields $a^2 a_1 = 0$, and this implies $aa_1 = 0$ as R is reduced. A simple induction shows that $aa_1 = aa_2 = \cdots = aa_{k-1} = 0$. So $a_{k-1}a = 0$. Now multiplying the last equation by a from the right gives $a = aa_k a$. □

LEMMA 2.3. *Let $a \in R$ and $e^2 = e \in R$ such that both $ea(1 - e)$ and $(1 - e)ae$ are contained in $J(R)$. If eae is clean in eRe and $(1 - e)a(1 - e)$ is clean in $(1 - e)R(1 - e)$, then a is clean in R.*

PROOF. Write $eae = u + f$ and $(1 - e)a(1 - e) = v + h$, where $u \in U(eRe)$, $f^2 = f \in eRe$, $v \in U((1 - e)R(1 - e))$ and $g^2 = g \in (1 - e)R(1 - e)$. Then $f + g$ is an idempotent of R and $u + v \in U(R)$. As $ea(1 - e), (1 - e)ae \in J(R)$, $(u + v) + (ea(1 - e) + (1 - e)ae) \in U(R)$. So, by the Pierce decomposition,

$$a = eae + ea(1 - e) + (1 - e)ae + (1 - e)a(1 - e)$$
$$= (f + g) + [(u + v) + (ea(1 - e) + (1 - e)ae)]$$

is a clean element of R. □

Recall that the prime radical $N(R)$ of a ring R is defined to be the intersection of all the prime ideals in R. A 2-primal ring is a ring R such that the prime radical $N(R)$ is exactly the set of nilpotent elements. It is known that a ring R is 2-primal if and only if R/I is a domain for every minimal prime ideal I of R. Examples of 2-primal rings include commutative rings and reduced rings.

For a subset S of a ring R, let $S((x))$ be the set of Laurent series with coefficients in S.

LEMMA 2.4. *Let R be a 2-primal ring. Then $J(R((x))) = N(R)((x))$.*

PROOF. If R is 2-primal, then $R/N(R)$ is reduced, and hence $J(R((x))) = N(R)((x))$ by [8, Theorem 2]. □

Exchange rings were introduced by Warfield [9] via the exchange property of modules. Here we use an equivalent condition of an exchange ring obtained independently by Goodearl–Warfield [1] and Nicholson [4]: a ring R is an exchange ring if and only if for each $a \in R$ there exists $e^2 = e \in R$ such that $e \in aR$ and $1 - e \in (1 - a)R$. By [4], every clean ring is an exchange ring, and the converse holds if all idempotents of the ring are central.

We are now ready to prove the main theorem of the paper. Recall that a ring is said to be strongly regular if it is reduced and regular.

THEOREM 2.5. *Let R be a* 2*-primal ring. The following assertions are equivalent:*

(1)   *$R((x))$ is a clean ring;*
(2)   *$R((x))/J(R((x)))$ is a clean ring;*
(3)   *$R((x))$ is an exchange ring;*
(4)   *$R((x))/J(R((x)))$ is an exchange ring;*
(5)   *R is semiregular with $J(R)$ nil;*
(6)   *$R/J(R)$ is strongly regular with $J(R)$ nil.*

PROOF. $(1) \Rightarrow (2) \Rightarrow (4)$ and $(1) \Rightarrow (3) \Rightarrow (4)$. These are clear.

$(4) \Rightarrow (5)$. By Lemma 2.4, $J(R((x))) = N(R)((x))$. So

$$(R/N(R))((x)) \cong R((x))/N(R)((x)) = R((x))/J(R((x)))$$

is an exchange ring. Since R is 2-primal, $R/N(R)$ is reduced; so all idempotents of $(R/N(R))((x))$ are central by Lemma 2.1. Thus $(R/N(R))((x))$ is a clean ring by [4, Proposition 1.8]. Hence $R/N(R)$ is regular by Lemma 2.2. It follows that $J(R) \subseteq N(R)$. So $J(R) = N(R)$, which is nil.

$(5) \Rightarrow (6)$. As R is 2-primal, the assumption implies $J(R) = N(R)$, so $R/J(R)$ is reduced and regular.

$(6) \Rightarrow (1)$. In view of Lemma 2.4, (6) implies $J(R((x))) = J(R)((x))$.

Let $f \in R((x))$, and write $f = x^{-k}(a_0 + a_1 x + \cdots)$, where $k \geq 0$. We show next that f is clean in $R((x))$. As $\overline{R} := R/J(R)$ is strongly regular, $\bar{a}_0 = \bar{e}_0 \bar{u}_0$, where $\bar{e}_0$ is a central idempotent of $\overline{R}$ and $\bar{u}_0$ is a unit of $\overline{R}$. As idempotents of $\overline{R}$ can be lifted to idempotents of R, we can assume that $e_0^2 = e_0$. Thus, $a_0 = e_0 u_0 e_0 + j_0$, where $e_0 u_0 e_0$ is a unit of $e_0 R e_0$ and $j_0 \in J(R)$. Then

$$e_0 f e_0 = x^{-k}((e_0 u_0 e_0 + e_0 j_0 e_0) + e_0 a_1 e_0 x + e_0 a_2 e_0 x^2 + \cdots)$$

is a unit (and hence a clean element) of $(e_0 R e_0)((x))$ $(= e_0 R((x)) e_0)$. Since $\bar{e}_0$ is central in $\overline{R}$, $e_0 f - f e_0 \in J(R)((x)) = J(R((x)))$. So

$$e_0 f(1 - e_0), (1 - e_0) f e_0 \in J(R((x))).$$

By Lemma 2.3, to show that f is clean in $R((x))$ it suffices to show that $(1-e_0)f(1-e_0)$ is clean in $(1 - e_0)R(1 - e_0)((x))$ $(= (1 - e_0)R((x))(1 - e_0))$. Note that

$$\begin{aligned} f_1 &:= (1 - e_0)f(1 - e_0) \\ &= x^{-k}((1 - e_0)j_0(1 - e_0) + (1 - e_0)a_1(1 - e_0)x + \cdots). \end{aligned}$$

As $R_1 := (1 - e_0)R(1 - e_0)$ is again 2-primal, strongly regular modulo $J(R_1)$ with $J(R_1)$ nil, as argued above we have $(1 - e_0)a_1(1 - e_0) = e_1 u_1 e_1 + j_1$, where e_1 is an

idempotent of R_1 which is central modulo $J(R_1)$, $e_1 u_1 e_1$ is a unit of $e_1 R_1 e_1$ and $j_1 \in J(R_1)$. We see that, as $e_1 j_0 e_1 \in J(R_1) \subseteq J(R_1)((x)) = J(R_1((x)))$,

$$e_1 f_1 e_1 = x^{-k}(e_1 j_0 e_1 + (e_1 u_1 e_1 + e_1 j_1 e_1)x + e_1 a_2 e_1 x^2 + e_1 a_3 e_1 x^3 + \cdots)$$

is a unit of $R_1((x))$. Since $e_1 a - a e_1 \in J(R_1)$ for all $a \in R_1$, we have $e_1 f_1 (1 - e_0 - e_1)$, $(1 - e_0 - e_1)f_1 e_1 \in J(R_1((x)))$. So to show that f_1 is clean in $R_1((x))$, by Lemma 2.3 it suffices to show that $f_2 := (1 - e_0 - e_1)f_1(1 - e_0 - e_1)$ is clean in $(1 - e_0 - e_1)R(1 - e_0 - e_1)((x))$.

Let us write $(1 - e_0 - \cdots - e_{i-1})a_i(1 - e_0 - \cdots - e_{i-1}) = e_i u_i e_i + j_i$, where e_i is an idempotent of $R_i := (1 - e_0 - \cdots - e_{i-1})R(1 - e_0 - \cdots - e_{i-1})$ which is central modulo $J(R_i)$, $e_i u_i e_i$ is a unit of $e_i R_i e_i$ and $j_i \in J(R_i)$ for $i = 1, 2, \ldots, k - 1$. Let $f_i := (1 - e_0 - e_1 - \cdots - e_{i-1})f(1 - e_0 - e_1 - \cdots - e_{i-1})$ for $i = 1, 2, \ldots, k - 1$. The argument above shows that

$$f \text{ is clean in } R((x))$$
$$\Leftrightarrow f_1 \text{ is clean in } R_1((x))$$
$$\Leftrightarrow f_2 \text{ is clean in } R_2((x))$$
$$\vdots$$
$$\Leftrightarrow f_k \text{ is clean in } R_k((x)).$$

Note that $f_k = g + h$, where

$$g = x^{-k}(1 - e_0 - \cdots - e_{k-1})(j_0 + j_1 x + \cdots + j_{k-1}x^{k-1})(1 - e_0 - \cdots - e_{k-1}),$$
$$h = (1 - e_0 - \cdots - e_{k-1})(a_k + a_{k+1}x + \cdots)(1 - e_0 - \cdots - e_{k-1}).$$

As $j_0, \ldots, j_{k-1} \in J(R)$, $g \in J(R_k)((x)) = J(R_k((x)))$. But h is an element of the power series ring $R_k[[x]]$. As $R_k[[x]]$ is clean by [2, Proposition 5], h is a clean element of $R_k[[x]]$. So h is a clean element of $R_k((x))$. Thus $h = w + e$, where w is a unit of $R_k((x))$ and e is an idempotent of $R_k((x))$. As $g \in J(R_k((x)))$, $g + w$ is a unit of $R_k((x))$, so $f_k = (g + w) + e$ is a clean element of $R_k((x))$. The proof is complete. □

COROLLARY 2.6. *Let R be a reduced ring. Then $R((x))$ is clean if and only if R is strongly regular.*

Thus, for a domain R, $R((x))$ is clean if and only if R is a division ring. If $R = \mathbb{Z}_{(p)}$ is the localization of $\mathbb{Z}$ at a prime p, then R is a local domain that is not a division ring, and hence $R((x))$ is not clean. So this is a counter-example to [7, Corollary, page 224].

A ring R is called strongly π-regular if for each $a \in R$ there exists $n \geq 1$ such that $a^n \in a^{n+1}R$. A commutative ring R is strongly π-regular if and only if $R/J(R)$ is (strongly) regular with $J(R)$ nil (see [3, Exercise 4.15]).

COROLLARY 2.7. *Let R be a commutative ring. Then $R((x))$ is clean if and only if R is strongly π-regular.*

The following example shows that in Theorem 2.5 the equivalence '(1) $\Longleftrightarrow$ (6)' may not hold if the ring R is not a 2-primal ring.

EXAMPLE 2.8. Let S be any strongly regular ring and $R = \mathbb{M}_n(S)$ with $n \geq 2$. Then R is not strongly regular. But $R((x))$ is clean. In fact, $R((x)) = \mathbb{M}_n(S)((x)) \cong \mathbb{M}_n(S((x)))$ (where $\sum_{l \geq k}(s_{ij}^{(l)})x^l \mapsto (\sum_{l \geq k} s_{ij}^{(l)} x^l)$ is the required isomorphism). One sees that $S((x))$ is clean by Theorem 2.5, so $\mathbb{M}_n(S((x)))$ is clean by [2].

LEMMA 2.9. *If $R = R_1 \times R_2$ is a direct product of rings, then $R((x)) \cong R_1((x)) \times R_2((x))$.*

PROOF. The map $\sum_{i \geq k}(a_i, b_i)x^i \mapsto (\sum_{i \geq k} a_i x^i, \sum_{i \geq k} b_i x^i)$ is the required isomorphism from $R((x))$ onto $R_1((x)) \times R_2((x))$. □

In [10, Theorem], it was proved that $R((x))$ being semilocal implies that R is semiperfect with $J(R)$ nil.

THEOREM 2.10. *Let R be a 2-primal ring. The following assertions are equivalent:*

(1)   *$R((x))$ is semiperfect;*
(2)   *$R((x))$ is semilocal;*
(3)   *$R((x))$ is semiregular;*
(4)   *$R((x))/J(R((x)))$ is regular;*
(5)   *R is semiperfect with $J(R)$ nil;*
(6)   *$R/J(R)$ is a finite direct product of division rings with $J(R)$ nil.*

PROOF. (1) $\Rightarrow$ (2) $\Rightarrow$ (4) and (1) $\Rightarrow$ (3) $\Rightarrow$ (4). These are clear.

(4) $\Rightarrow$ (5). By Lemma 2.4, $J(R((x))) = N(R)((x))$. So

$$(R/N(R))((x)) \cong R((x))/N(R)((x)) = R((x))/J(R((x)))$$

is regular. By [6], the ring $R/N(R)$ is semisimple Artinian. It follows that $J(R) = N(R)$. So (5) holds.

(5) $\Rightarrow$ (6). As R is 2-primal, $R/N(R)$ is reduced, so $J(R) = N(R)$. Since R is semiperfect, it follows that $R/J(R)$ is a finite direct product of division rings.

(6) $\Rightarrow$ (1). By Theorem 2.5, $R((x))$ is clean. So idempotents of the ring $R((x))/J(R((x)))$ can be lifted to idempotents of $R((x))$. Since $J(R)$ is nil, $J(R) = N(R)$. Hence $J(R((x))) = J(R)((x))$ by Lemma 2.4. Write $R/J(R) = D_1 \times \cdots \times D_k$, where the D_i are division rings. So, in view of Lemma 2.9,

$$R((x))/J(R((x))) = R((x))/J(R)((x)) \cong (R/J(R))((x))$$
$$\cong D_1((x)) \times \cdots \times D_k((x)),$$

which is semisimple Artinian. So $R((x))$ is semiperfect. □

A ring is called uniquely clean if each of its elements can be uniquely expressed as the sum of an idempotent and a unit. It is known that R is uniquely clean if and only if $R[[x]]$ is uniquely clean (see [5, Corollary 10]).

PROPOSITION 2.11. *The Laurent series ring $R((x))$ is not uniquely clean for any nontrivial ring R.*

PROOF. If $T := R((x))$ is uniquely clean, then $T/J(T)$ is boolean by [5, Theorem 20]. So $\bar{1}$ is the only unit of $T/J(T)$. Thus, $\bar{x} = \bar{1}$ in $T/J(T)$, that is, $1 - x \in J(T)$. But $1 - x$ is clearly a unit of T. This is a contradiction.   □

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