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Level structures on abelian varieties and Vojta's conjecture

Dan Abramovich and Anthony Várilly-Alvarado

With an appendix by Keerthi Madapusi Pera

Abstract

Assuming Vojta's conjecture, and building on recent work of the authors, we prove that, for a fixed number field K and a positive integer g, there is an integer m_0 such that for any $m > m_0$ there is no principally polarized abelian variety A/K of dimension g with full level-m structure. To this end, we develop a version of Vojta's conjecture for Deligne–Mumford stacks, which we deduce from Vojta's conjecture for schemes.

Contents

Fix a number field K, a prime p and a positive integer g. Assuming Lang's conjecture, we showed in [AV16] that there exists an integer r such that no principally polarized abelian variety A/K has full level-p^r structure. Recall that, for a positive integer m, a full level-m structure on an abelian variety A/K is an isomorphism of group schemes on the m-torsion subgroup

$$A[m] \xrightarrow{\sim} (\mathbb{Z}/m\mathbb{Z})^g \times (\mu_m)^g. \tag{0.1}$$

Our goal in this note is to show how to dispose of the dependency on a fixed prime p, at the cost of assuming Vojta's conjecture (see [Voj98, Conjecture 2.3] and Conjecture 3.1 below).

Theorem A. *Let K be a number field and let g be a positive integer. Assume Vojta's conjecture. Then there is an integer m_0 such that for any $m > m_0$ no principally polarized abelian variety A/K of dimension g has full level-m structure.*

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Theorem A follows from combining [AV16, Theorem 1.1] and a new result in this note, as follows.

Theorem B. *Let K be a number field and let g be a positive integer. Assume Vojta's conjecture. Then there is an integer m_0 such that for any prime $p > m_0$ no principally polarized abelian variety A/K of dimension g has full level-p structure.*

1. Introduction

1.1 Vojta's conjecture for varieties

Before Merel proved that torsion on elliptic curves over number fields is uniformly bounded [Mer96], it was known that statements related to Masser–Oesterlé's *abc* conjecture [Fre89, Conjecture A-B-C] or Szpiro's conjecture [Szp90, Conjecture 1] imply such bounds; see Frey [Fre89, Corollary 2.2], Hindry–Silverman [HS88, Theorem 7.1] and Flexor–Oesterlé [FO90]. In this paper, we use Vojta's conjecture [Voj98, Conjecture 2.3] as a higher-dimensional analogue of the *abc* conjecture to study level structures on abelian varieties of dimension > 1.

Vojta's conjecture is a quantitative statement, comparing heights $h_{K_X(D)}(x)$, with respect to the log canonical divisor $K_X(D)$, of rational points x in general position on a smooth projective variety X over a number field K, with the truncated counting function $N_K^{(1)}(D,x)$ of such points (see (2.4)) with respect to a normal crossings divisor D. The simplest statement, for K-rational points, says that if $K_X(D)$ is big, then, for small δ,

$$N_K^{(1)}(D,x) \geqslant (1-\delta) h_{K_X(D)}(x) - O(1)$$

for all rational points $x \in X(K)$ outside a Zariski-closed proper subset.

The general notation is, unfortunately, involved, and explained in § 2. The conjecture does have qualitative corollaries, which are easier to explain. The truncated counting function $N_K^{(1)}(D,x)$ measures how often the point x reduces to a point on D modulo primes of K. In particular, when D is empty, then $N_K^{(1)}(D,x) = 0$, in which case the statement says that the height $h_{K_X(D)}(x)$ is bounded. Since the height of a big divisor is a counting function outside a Zariski-closed subvariety, this implies that rational points are not Zariski dense. So, Vojta's conjecture implies Lang's conjecture: the rational points on a positive-dimensional variety of general type are not Zariski dense. More generally, $N_K^{(1)}(D,x) = 0$ whenever x extends to an integral point on $X \smallsetminus D$. This recovers the statement of the Lang–Vojta conjecture: the integral points on a positive-dimensional variety of logarithmic general type are not Zariski dense.

Campana studied varieties where divisors between K_X and $K_X + D$ are big and, for algebraic curves, stated qualitative conjectures interpolating between Faltings' and Siegel's theorems. We will study these intermediate conjectures in higher dimensions in a follow-up note. These statements are qualitative consequences of Vojta's conjecture.

Our arguments below use Vojta's conjecture for points of bounded degree, which requires an additional discriminant term $d_K(K(x))$: for small δ the inequality

$$N_K^{(1)}(D,x) + d_K(K(x)) \geqslant h_{K_X(D)}(x) - \delta h_H(x) - O_{[K(x):K]}(1)$$

is conjectured to hold, away from a Zariski-closed subset, where H is a big divisor. Note that when $x \in X(K)$, we have $d_K(K(x)) = 0$.

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1.2 Vojta's conjecture for stacks

Theorem B is decidedly about rational points on stacks, not varieties. Specifically, an abelian variety A/K corresponds to a rational point on the moduli *stack* $\widetilde{\mathcal{A}}_g$ of principally polarized abelian varieties. It should thus come as no surprise that, to prove Theorem B, we require a version of Vojta's conjecture for Deligne–Mumford stacks (Proposition 3.2), which we deduce from Vojta's original conjecture.

Surprisingly, unlike the case of varieties, Vojta's conjecture for stacks requires a discriminant term even for a K-rational point: the image of such a point x in X is naturally a stack $\mathcal{T}_x$ that is in general ramified over the ring of integers $\mathcal{O}_K$. The corresponding inequality

$$N_K^{(1)}(D,x) + d_K(\mathcal{T}_x) \geqslant h_{K_X(D)}(x) - \delta h_H(x) - O(1) \tag{1.1}$$

holds away from a Zariski-closed proper subset, conditional on Vojta's conjecture for varieties.

Proposition 3.2 is proved by passing to a branched covering $Y \to X$ by a variety. Such a covering was constructed by Kresch and Vistoli in [KV04, Theorem 1]; we adapt their construction to stacks with normal crossings divisors in Proposition 2.3.

While the discriminant term comes naturally from Vojta's statement for points of bounded degree, one might contemplate doing away with it. It is, however, indispensable, at least if one is to state a conjecture that is not patently false. Consider the root stack $X = \mathbb{P}^2(\sqrt{C})$, where C is a curve of degree $\geqslant 7$, and let $D = \emptyset$. Then K_X is ample, $N_K^{(1)}(D,x) \equiv 0$ and yet there is a dense collection of rational points on the open subset $\mathbb{P}^2 \smallsetminus C \subset X$.

1.3 Abelian varieties, counting functions and discriminants

Fix an integer m_0 and consider the set $\widetilde{\mathcal{A}}_g(K)_{p \geqslant m_0}$ of points corresponding to abelian varieties admitting full level-p structures for primes $p \geqslant m_0$. Our task is to show that for large m_0 this set is empty. To this end, it is natural to focus on an irreducible component $X \subset \overline{\widetilde{\mathcal{A}}_g(K)_{p \geqslant m_0}}$ of the Zariski closure. This leads to the following setup: consider a closed substack $X \subset \widetilde{\mathcal{A}}_g$, a resolution of singularities $X' \to X$ and a normal crossings compactification $\overline{X}'$ with boundary divisor D. Following Zuo [Zuo00, Theorem 0.1(ii)], we showed in [AV16, Theorem 1.9] that $K_{\overline{X}'}(D)$ is big.

With a version of Vojta's conjecture for stacks in hand, the key to proving Theorem B is to show that, for points $x \in X'(K)$ corresponding to abelian varieties with full level-p structure, the terms $N_K^{(1)}(D,x)$ and $d_K(\mathcal{T}_x)$ on the left-hand side of (1.1) are small compared to the height $h_{K_{\overline{X}'}(D)}(x)$, as soon as p is large enough.

To this end, we show that each one of these terms is bounded by a small fraction of the height $h_D(x)$; see Lemmas 4.4 and 4.5. First, to bound the truncated counting function $N_K^{(1)}(D,x)$, we use the fact that the compactified moduli space $\overline{\widetilde{\mathcal{A}}}_g^{[p]}$ of abelian varieties with full level-p structure is highly ramified over the compactification $\overline{\widetilde{\mathcal{A}}}_g$ along the boundary. This is well known away from characteristic p; see [AV16, Proposition 4.1]. The remaining case of characteristic p is proven in Proposition A.4 as part of the Appendix by Madapusi Pera, where the structure of the boundary is described using Mumford's construction.

Second, using standard discriminant bounds, we show that the discriminant term $d_k(\mathcal{T}_x)$ grows at most like $\log p$. Meanwhile, the height $h_D(x)$ grows at least linearly in p. For this we use a point-counting argument of Flexor–Oesterlé [FO90, Théorème 3] and Silverberg [Sil92, Theorem 3.3] (see also Kamienny [Kam82, § 6(2a)]) to show that x reduces to D modulo a small prime, whose contribution to $h_D(x)$ is at least proportional to p, since $\overline{\widetilde{\mathcal{A}}}_g^{[p]} \to \overline{\widetilde{\mathcal{A}}}_g$ is highly ramified.

Together, these two bounds can be leveraged to show that the totality of points $x \in X'(K)$ corresponding to abelian varieties over K with full level-p structure for $p \gg 0$ is contained in a Zariski-closed proper subset of X'. A Noetherian induction argument allows us to deduce Theorem B from this result.

2. Preliminaries

In this section, we set up notation that will remain in force throughout. Let K be a number field and let $\overline{K}$ be a fixed algebraic closure of K. We write $\mathcal{O}_K$ for the ring of integers of K and $\mathrm{Disc}(\mathcal{O}_K)$ for its discriminant. We denote by M_K^0 the set of non-zero primes of $\mathcal{O}_K$; for $p \in M_K^0$, we write $\mathcal{O}_{K,p}$ for the localization of $\mathcal{O}_K$ at p and $\kappa(p)$ for the residue field. We use S to denote a finite set of places of K that includes the infinite places, and $\mathcal{O}_{K,S}$ for the ring of S-integers of K.

For a finite extension L/K, we write $\Omega_{\mathcal{O}_L/\mathcal{O}_K}$ for the module of Kähler differentials.

2.1 Discriminants of fields

For a finite extension E/K, following Vojta, define the relative logarithmic discriminant as

$$d_K(E) = \frac{1}{[E:K]} \log|\mathrm{Disc}(\mathcal{O}_E)| - \log|\mathrm{Disc}(\mathcal{O}_K)|.$$

Noting that $(\mathrm{Disc}(\mathcal{O}_K)) = N_{K/\mathbb{Q}} \det \Omega_{\mathcal{O}_K/\mathbb{Z}}$ as ideals, we have

$$d_K(E) = \frac{1}{[E:K]} \deg \Omega_{\mathcal{O}_E/\mathcal{O}_K};$$

see [Voj98, p. 1106]. The right-hand side can be decomposed into a sum of local contributions

$$\deg \Omega_{\mathcal{O}_E/\mathcal{O}_K} = \sum_{\mathfrak{p} \in M_E^0} \deg_{\mathfrak{p}} \Omega_{\mathcal{O}_E/\mathcal{O}_K} = \sum_{\mathfrak{p} \in M_E^0} \mathrm{length}(\Omega_{\mathcal{O}_{E_p}/\mathcal{O}_{K_p}}) \log|\kappa(\mathfrak{p})|.$$

For $p \in M_K^0$, we write

$$d_K(E)_p := \frac{1}{[E:K]} \sum_{\mathfrak{p}|p} \deg_{\mathfrak{p}} \Omega_{\mathcal{O}_E/\mathcal{O}_K}$$

for the contribution of the primes above p, so that $d_K(E) = \sum_{p \in M_K^0} d_K(E)_p$.

If L/E is a further finite extension, the formula for discriminants in the tower $L/E/K$ gives

$$d_K(L) = \frac{1}{[E:K]} d_E(L) + d_K(E) = \frac{1}{[L:K]} \deg \Omega_{\mathcal{O}_L/\mathcal{O}_E} + d_K(E), \tag{2.1}$$

$$d_K(L)_p = \frac{1}{[E:K]} d_E(L)_p + d_K(E)_p = \frac{1}{[L:K]} \deg \Omega_{\mathcal{O}_{L,p}/\mathcal{O}_{E,p}} + d_K(E)_p. \tag{2.2}$$

In particular, if L/E is unramified above $p \in M_K^0$, then $d_K(L)_p = d_K(E)_p$.

2.2 Discriminants of stacks

We shall need analogous definitions where $\mathrm{Spec}\, \mathcal{O}_E$ is replaced by a normal separated Deligne–Mumford stack $\mathcal{T}$ with coarse moduli scheme $\mathrm{Spec}\, \mathcal{O}_E$:

$$d_K(\mathcal{T}) = \frac{1}{\deg(\mathcal{T}/\mathcal{O}_K)} \deg(\Omega_{\mathcal{T}}) - \log|\mathrm{Disc}(\mathcal{O}_K)| = \frac{1}{\deg(\mathcal{T}/\mathcal{O}_K)} \deg(\Omega_{\mathcal{T}/\mathrm{Spec}\, \mathcal{O}_K}),$$

$$d_K(\mathcal{T})_p = \frac{1}{\deg(\mathcal{T}/\mathcal{O}_K)} \deg(\Omega_{\mathcal{T}_p}) - \log|\mathrm{Disc}(\mathcal{O}_K)| = \frac{1}{\deg(\mathcal{T}/\mathcal{O}_K)} \deg(\Omega_{\mathcal{T}_p/\mathrm{Spec}\, \mathcal{O}_{K,p}}).$$

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The quantity $\deg(\mathcal{T}/\mathcal{O}_K)$ is in general rational, as the fiber over $\operatorname{Spec} K$ might be a gerbe over $\operatorname{Spec} E$.[1] However, we still have $d_K(\mathcal{T}) = \sum_{p \in M_K^0} d_K(\mathcal{T})_p$.

Choose a morphism $\operatorname{Spec} \mathcal{O}_F \to \mathcal{T}$ unramified above p, so that $(\Omega_{\mathcal{T}/\operatorname{Spec}\mathcal{O}_K})_{\operatorname{Spec}\mathcal{O}_{F,p}} = \Omega_{\mathcal{O}_{F,p}/\mathcal{O}_{K,p}}$. Since $[F:K] = \deg(\mathcal{T}/\mathcal{O}_K) \cdot \deg(\operatorname{Spec}\mathcal{O}_F/\mathcal{T})$, we can compute $d_K(\mathcal{T})_p$ entirely with schemes.

LEMMA 2.1. *For* $\operatorname{Spec}\mathcal{O}_F \to \mathcal{T}$ *unramified above* p, *we have* $d_K(\mathcal{T})_p = d_K(F)_p$.

We deduce analogues of (2.1) and (2.2).

LEMMA 2.2. *Let* L/E *be a finite extension field and* $\pi\colon \operatorname{Spec}\mathcal{O}_L \to \mathcal{T}$ *a morphism. Then*

$$d_K(L)_p = \frac{1}{[L:K]} \deg(\Omega_{\operatorname{Spec}(\mathcal{O}_{L,p})/\mathcal{T}_p}) + d_K(\mathcal{T})_p$$

and

$$d_K(L) = \frac{1}{[L:K]} \deg(\Omega_{\operatorname{Spec}(\mathcal{O}_L)/\mathcal{T}}) + d_K(\mathcal{T}).$$

Proof. To prove the local statement, choose $\psi\colon \operatorname{Spec}\mathcal{O}_F \to \mathcal{T}$ unramified above p, and let $\mathcal{U} = \operatorname{Spec}(\mathcal{O}_F) \times_{\mathcal{T}} \operatorname{Spec}(\mathcal{O}_L)$ with projection $\phi\colon \mathcal{U} \to \operatorname{Spec}(\mathcal{O}_L)$. These objects fit together in the commutative diagram

$$\begin{array}{ccccc} \mathcal{U} & \xrightarrow{\phi} & \operatorname{Spec}\mathcal{O}_L & & \\ \downarrow & & \downarrow{\scriptstyle\pi} & & \\ \operatorname{Spec}\mathcal{O}_F & \xrightarrow{\psi} & \mathcal{T} & \xrightarrow{\tau} & \operatorname{Spec}\mathcal{O}_K \end{array}$$

We have

$$\Omega_{\mathcal{U}_p/\operatorname{Spec}(\mathcal{O}_{F,p})} = \Omega_{\mathcal{U}_p/\mathcal{T}_p} = \phi^*\Omega_{\operatorname{Spec}(\mathcal{O}_{L,p})/\mathcal{T}_p},$$
$$\Omega_{\mathcal{U}_p/\operatorname{Spec}(\mathcal{O}_{K,p})} = \phi^*\Omega_{\mathcal{O}_{L,p}/\mathcal{O}_{K,p}}$$

and

$$\Omega_{\mathcal{O}_{F,p}/\mathcal{O}_{K,p}} = \psi^*\Omega_{\mathcal{T}_p/\operatorname{Spec}(\mathcal{O}_{K,p})}.$$

The projection formula gives

$$\deg(\Omega_{\mathcal{U}_p/\mathcal{T}_p}) = \deg(\psi)\deg(\Omega_{\operatorname{Spec}(\mathcal{O}_{L,p})/\mathcal{T}_p}),$$
$$\deg(\Omega_{\mathcal{O}_{F,p}/\mathcal{O}_{K,p}}) = \deg(\psi)\deg(\Omega_{\mathcal{T}_p/\operatorname{Spec}(\mathcal{O}_{K,p})})$$

and finally

$$\begin{aligned} d_K(L)_p &= \frac{1}{[L:K]}\deg(\Omega_{\mathcal{O}_{L,p}/\mathcal{O}_{K,p}}) = \frac{1}{[L:K]\deg\psi}\deg(\Omega_{\mathcal{U}_p/\operatorname{Spec}(\mathcal{O}_{K,p})}) \\ &= \frac{1}{[L:K]\deg\psi}(\deg(\Omega_{\mathcal{U}_p/\operatorname{Spec}(\mathcal{O}_{F,p})}) + (\deg\pi)\deg(\Omega_{\mathcal{O}_{F,p}/\mathcal{O}_{K,p}})) \\ &= \frac{1}{[L:K]\deg\psi}\deg(\Omega_{\mathcal{U}_p/\mathcal{T}_p}) + \frac{1}{\deg(\mathcal{T}/\operatorname{Spec}\mathcal{O}_K)}\deg(\Omega_{\mathcal{T}_p/\operatorname{Spec}(\mathcal{O}_{K,p})}) \\ &= \frac{1}{[L:K]}\deg(\Omega_{\operatorname{Spec}(\mathcal{O}_{L,p})/\mathcal{T}_p}) + d_K(\mathcal{T})_p, \end{aligned}$$

as required. The global formula follows by summing over $p \in M_K^0$. □

[1] We can redefine $\mathcal{T}_x$ to be the normalization in the field E, so that $\deg(\mathcal{T}/\mathcal{O}_K) = [E:K]$, an integer.

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2.3 Heights on stacks

For a divisor H on a smooth projective scheme Y, we denote by $h_H(x)$ the Weil height of x with respect to H, which is well defined up to a bounded function on $Y(\overline{K})$. To define a notion of height on a Deligne–Mumford stack, we pull back to a cover by a scheme and work there instead. Let X/K be a smooth proper Deligne–Mumford stack with projective coarse moduli scheme and let $H \subset X$ be a divisor. Let $f\colon Y \to X$ be the finite flat surjective morphism from a smooth projective scheme Y guaranteed by [KV04, Theorem 1] or Proposition 2.3 below. For a point $x \in X(\overline{K})$, let $y \in Y(\overline{K})$ be a point over x and define

$$h_H(x) := h_{f^*(H)}(y).$$

This definition has the advantage of having functoriality properties of heights built into it. It is also compatible with passing to the coarse moduli space, at the price of working with $\mathbb{Q}$-Cartier divisors on slightly singular schemes: any divisor H on X is the pullback of a $\mathbb{Q}$-Cartier divisor $\underline{H}$ on the coarse moduli space $\underline{X}$ and, if $\underline{x} \in \underline{X}$ is the image of x, then

$$h_H(x) = h_{\underline{H}}(\underline{x}).$$

Our definition has the disadvantage that there can be infinitely many rational points (with the same image in $\underline{X}$) with the same height. In a forthcoming paper, Ellenberg *et al.* construct an alternative notion of height on a stack, with the property that there are only finitely many non-isomorphic points with bounded height.

2.4 Normal crossings models

Let $(\mathscr{X}, \mathscr{D})$ be a pair with $\mathscr{X} \to \operatorname{Spec} \mathcal{O}_{K,S}$ a smooth proper morphism from a scheme or a Deligne–Mumford stack, and $\mathscr{D}$ a fiber-wise normal crossings divisor on $\mathscr{X}$. Let (X, D) be the generic fiber of the pair $(\mathscr{X}, \mathscr{D})$; we say that $(\mathscr{X}, \mathscr{D})$ is a normal crossings model of the pair (X, D). Write $\mathscr{D} = \sum_i \mathscr{D}_i$ and let D_i be the generic fiber of $\mathscr{D}_i$.

2.5 Intersection multiplicities on schemes and stacks

For R an integral extension of $\mathcal{O}_{K,S}$, and $q \subset R$ a non-zero prime ideal, let R_q be the localization of R at q, with maximal ideal $\mathfrak{m}_q$ and residue field $\kappa(q)$.

We first define multiplicities for integral points. Let $x \in \mathscr{X}(R_q)$, and define $n_q(\mathscr{D}_i, x)$ as the intersection multiplicity of x and $\mathscr{D}_i$. In other words, letting $I_{\mathscr{D}_i}$ denote the ideal of $\mathscr{D}_i$, we have an equality of ideals in R_q:

$$I_{\mathscr{D}_i}\big|_x = \mathfrak{m}_q^{n_q(\mathscr{D}_i, x)}.$$

Note that if R' is an integral extension of R, with maximal ideal $\mathfrak{q} \mid q$, and if $y \in \mathscr{X}(R'_{\mathfrak{q}})$ is the composite of $\operatorname{Spec} R'_{\mathfrak{q}} \to \operatorname{Spec} R_q \to \mathscr{X}$, then we have $n_{\mathfrak{q}}(\mathscr{D}_i, y) = e(\mathfrak{q} \mid q) n_q(\mathscr{D}_i, x)$, where $e(\mathfrak{q} \mid q)$ is the ramification index of $\mathfrak{q}$ over q.

This observation prompts the following extension of the definition of $n_q(\mathscr{D}_i, x)$ to a rational point x of $\mathscr{X}$. Denoting by $K(R)$ and $K(R')$ the respective fraction fields of R and R', if $x \in \mathscr{X}(K(R))$ and if $y \in \mathscr{X}(R')$ is an integral point over x, then the quantity

$$n_q(\mathscr{D}_i, x) := \frac{1}{e(\mathfrak{q} \mid q)} n_{\mathfrak{q}}(\mathscr{D}_i, y) \tag{2.3}$$

is well defined.

Finally, define $n_q(\sum a_i \mathscr{D}_i, x) := \sum_i a_i n_q(\mathscr{D}_i, x)$.

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2.6 Counting functions

Following Vojta [Voj98, p. 1106], for $x \in \mathscr{X}(\overline{K})$, with residue field $K(x)$, define the truncated counting function

$$N_K^{(1)}(D, x) = \frac{1}{[K(x):K]} \sum_{\substack{q \in \operatorname{Spec} \mathcal{O}_{K(x),S} \\ n_q(\mathscr{D}, x) > 0}} \log |\kappa(q)|. \tag{2.4}$$

The quantity on the right-hand side of (2.4) depends on the model $(\mathscr{X}, \mathscr{D})$ and the finite set S only up to a bounded function on $X(\overline{K})$. However, we are interested in this quantity only up to such functions. Hence, the notation $N_K^{(1)}(D, x)$ does not reflect the model $(\mathscr{X}, \mathscr{D})$ or the finite set S.

By [Voj98, p. 1113] or [HS00, Theorem B.8.1(e)], we have the bound

$$N_K^{(1)}(D, x) \leqslant \frac{1}{[K(x):K]} \sum_{\mathfrak{q}} n_{\mathfrak{q}}(D, x) \log |\kappa(\mathfrak{q})| \leqslant h_D(x) + O(1), \tag{2.5}$$

which can be further improved whenever we bound the multiplicities $n_{\mathfrak{q}}(D, x)$ from below.

2.7 Coverings of stacks

We require the following version of [KV04, Theorem 1], due to Kresch and Vistoli, adapted to the case of a stack with a normal crossings divisor.

Proposition 2.3. *Suppose that X/K is a smooth proper Deligne–Mumford stack with projective moduli scheme, with a normal crossings divisor $D \subset X$. Then there exists a finite surjective morphism $\pi : Y \to X$ such that Y is a smooth projective irreducible scheme, $D_Y := \pi^* D \subset Y$ is a normal crossings divisor and the ramification divisor R of $Y \to X$ meets every stratum of D_Y properly.*

The proof of this proposition requires the following slicing lemma.

Lemma 2.4 (See [KV04, Lemma 1]). *Let $f : U \to V$ be a morphism of quasi-projective varieties over an infinite field, with constant fiber dimension $r > 0$; let $D \subset V$ be a divisor. Assume that U is smooth and $D_U = f^{-1} D$ is a simple normal crossings divisor. Let $U \subset \mathbb{P}^N$ be a projective embedding. Denote by D_U^I the closed strata of (U, D_U), and assume further that $D_U^I \to D^I := f(D_U^I)$ is generically smooth for each I. Then, for sufficiently high d, the intersection $D_U^I \cap H^{(d)}$ of each stratum D_U^I with a general hypersurface $H^{(d)} \subset \mathbb{P}^N$ of degree d is a smooth Cartier divisor in D_U^I, generically smooth and of constant fiber dimension $r - 1$ over D^I.*

Proof of the lemma. For each I, [KV04, Lemma 1] applied to $U \to V$ replaced by $D_U^I \to D^I$ provides an integer d_I and, for each $d \geqslant d_I$, an open subset of $\Gamma(\mathbb{P}^N, \mathcal{O}(d))$, where $D_U^I \cap H^{(d)}$ is a smooth Cartier divisor in D_U^I of constant fiber dimension $r - 1$ over D^I. By Bertini's theorem, after possibly enlarging d_I we may replace it by a smaller open subset, where $D_U^I \cap H^{(d)} \to D^I$ is also generically smooth. Take $d \geqslant \max_I d_I$. □

Proof of Proposition 2.3. First, we note that X is a quotient stack: to see this, one combines [KV04, Theorem 2] together with a result of Gabber implying that the Azumaya–Brauer group of a quasi-projective scheme over a field coincides with the cohomological Brauer group [dJon]. Next, proceeding as in the proof of [KV04, Theorem 1], one can construct a smooth projective morphism of stacks $\pi : P \to X$ with a representable open substack $Q \subset P$, whose fiber dimension is greater than $\dim(P \smallsetminus Q)$. The induced morphism on coarse moduli spaces $U \to \underline{X}$ is surjective, and U is quasi-projective by [KV04, Lemma 2].

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Beginning with the map $U \to \underline{X}$ and the image of D via $X \to \underline{X}$, repeated applications of Lemma 2.4 yield a closed subscheme $Y \subset U$ such that the map $Y \to \underline{X}$ is finite and surjective, and such that D_Y is a normal crossings divisor whose strata meet the ramification divisor of $Y \to \underline{X}$ properly. We can assume that Y is disjoint from the image of $P \setminus Q$ in U, by dimension reasons. We can thus lift Y to a representable substack of Q, because Q is representable, and get the desired morphism $Y \to X$. □

2.8 Rational and integral points on stacks

We will make use of the following standard observations.

Lemma 2.5. *Let R be a Dedekind domain with fraction field K.*

(1) *Let $f\colon Y \to X$ be a proper representable morphism of algebraic stacks over R. Let $y \in Y(K)$ and $x = f(y)$. Then y extends to a point $\eta \in Y(R)$ if and only if x extends to a point $\xi \in X(R)$.*

(2) *Let X/R be an algebraic stack, Y/R a proper scheme and $Y \to X$ a morphism. If $x \in X(K)$ is the image of $y \in Y(K)$, then it extends to $\xi \in X(R)$.*

(3) *Let X/R be a proper algebraic stack, Y/R a proper scheme and $Y \to X$ a flat surjective morphism of degree M. Let $x \in X(K)$. There are a finite extension L/K, with $[L : K] \leqslant M$ and $R_L \subset L$ the integral closure of R, and a point $\xi \in X(R_L)$ lifting x.*

Proof. (1) Given $\eta \in Y(R)$, we have $f(\eta) = \xi \in X(R)$. If $\xi \in X(R)$, consider the fibered product $Z = \operatorname{Spec} R \times_X Y$ defined by ξ, which is representable and proper over R. Then y gives a point of $Z(K)$, which extends to R by the valuative criterion of properness.

(2) By the valuative criterion for properness, y extends to $\eta \in Y(R)$, whose composition with $Y \to X$ gives $\xi \in X(R)$.

(3) The K-scheme $Z = \operatorname{Spec} K \times_X Y$ is finite of degree M and hence admits a rational point $y \in Z(L)$ with $[L : K] \leqslant M$. The composition $\operatorname{Spec} L \to Z \to Y$ extends to $\eta \in Y(R_L)$ by the valuative criterion for properness, and its composition with f is a point $\xi \in X(R_L)$ lifting x. □

3. Vojta's conjecture for varieties and stacks

We write K_X for the canonical divisor class of a smooth variety or smooth Deligne–Mumford stack X.

Conjecture 3.1 (Vojta [Voj98, Conjecture 2.3]). Let X be a smooth projective variety over a number field K, D a normal crossings divisor on X and H a big line bundle on X. Let r be a positive integer and fix $\delta > 0$. Then there is a proper Zariski-closed subset $Z \subset X$ containing D such that

$$N_K^{(1)}(D, x) + d_K(K(x)) \geqslant h_{K_X(D)}(x) - \delta h_H(x) - O(1)$$

for all $x \in X(\overline{K}) \setminus Z(\overline{K})$ with $[K(x) : K] \leqslant r$.

We note that variants of the conjecture above have been stated, involving the counting function $N_K(D, x) = (1/[K(x) : K]) \sum_{\mathfrak{q}} n_{\mathfrak{q}}(D, x) \log |\kappa(\mathfrak{q})|$ and a different coefficient in front of the discriminant term $d_K(K(x))$. It may be possible to deduce results similar to Theorem B from these variants; we do not do so here.

We shall need a version of Vojta's conjecture for Deligne–Mumford stacks. For a smooth proper Deligne–Mumford stack $\mathscr{X} \to \operatorname{Spec} \mathcal{O}_{K,S}$, we write $X = \mathscr{X}_K$ for the generic fiber, which

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we assume is irreducible, and $\underline{X}$ for the coarse moduli space of X. Similarly, for a normal crossings divisor $\mathscr{D}$ of $\mathscr{X}$, we write D for its generic fiber.

Given a point $x \in \mathscr{X}(\overline{K})$, we take the Zariski closure and normalization of its image, and extend it uniquely to a morphism, denoted $\mathcal{T}_x \to \mathscr{X}$, where $\mathcal{T}_x$ is a normal stack with coarse moduli scheme $\operatorname{Spec}\mathcal{O}_{K(x),S}$. We thus have the relative discriminant $d_K(\mathcal{T}_x)$ defined in § 2.2.

Proposition 3.2 (Vojta for stacks). *Assume that Vojta's conjecture 3.1 holds. Let $\mathscr{X} \to \operatorname{Spec}\mathcal{O}_{K,S}$, X, $\underline{X}$ and D be as above. Suppose that $\underline{X}$ is projective and let H be a big line bundle on it. Let r be a positive integer and fix $\delta > 0$. Then there is a proper Zariski-closed subset $Z \subset X$ containing D such that*

$$N_K^{(1)}(D,x) + d_K(\mathcal{T}_x) \geqslant h_{K_X(D)}(x) - \delta h_H(x) - O(1)$$

for all $x \in X(\overline{K}) \smallsetminus Z(\overline{K})$ with $[K(x):K] \leqslant r$.

Proof. Let $Y \to X$ be the finite cover of X guaranteed by Proposition 2.3. Possibly after enlarging S, we may assume that $Y \to X$ extends to $\pi\colon \mathscr{Y} \to \mathscr{X}$ for some model $\mathscr{Y}$ of Y, so a point $y \in Y(K(y))$ extends to $\operatorname{Spec}\mathcal{O}_{K(y),S} \to \mathscr{Y}$, and composes to $\operatorname{Spec}\mathcal{O}_{K(y),S} \to \mathscr{X}$. We denote $\pi(y) = x$, and its extension as a stack by $\mathcal{T} := \mathcal{T}_x \to \mathscr{X}$.

By Riemann–Hurwitz, we have

$$K_Y + D_Y = (\pi^*K_X + R) + \pi^*D = \pi^*(K_X + D) + R.$$

Thus, for $y \in Y(\overline{K})$ with $\pi(y) = x$ outside a proper Zariski-closed subset of Y, we have

$$h_{K_Y(D_Y)}(y) = h_{K_X(D)}(x) + h_R(y) + O(1).$$

Let $\underline{\pi}\colon Y \to \underline{X}$ be the composition of π with the natural map $X \to \underline{X}$. Let $B = \underline{\pi}^*(H)$; then B is big and, by functoriality of heights, we have

$$h_B(y) = h_H(x) + O(1)$$

for all $y \in Y(\overline{K})$. Let $\mathscr{D}_Y = \pi^*\mathscr{D}$.

Lemma 3.3. *We have $N_K^{(1)}(D_Y,y) \leqslant N_K^{(1)}(D,x)$.*

Proof. Note that $n_q(\mathscr{D}_Y,y) > 0$ if and only if $n_q(\mathscr{D},x) > 0$. Then

$$\begin{aligned}
N_K^{(1)}(D_Y,y) &= \frac{1}{[K(y):K]} \sum_{\substack{\mathfrak{q}\in\operatorname{Spec}\mathcal{O}_{K(y),S} \\ n_\mathfrak{q}(\mathscr{D}_Y,y)>0}} \log|\kappa(\mathfrak{q})| \\
&= \frac{1}{[K(y):K]} \sum_{q:n_q(\mathscr{D},x)>0} \sum_{\mathfrak{q}|q} \log|\kappa(\mathfrak{q})| \\
&\leqslant \frac{1}{[K(y):K]} \sum_{q:n_q(\mathscr{D},x)>0} \sum_{\mathfrak{q}|q} e(\mathfrak{q}\mid q)\log|\kappa(\mathfrak{q})| \\
&= \frac{1}{[K(y):K]} \sum_{q:n_q(\mathscr{D},x)>0} [K(y):K(x)]\log|\kappa(q)| \\
&= \frac{1}{[K(x):K]} \sum_{q:n_q(\mathscr{D},x)>0} \log|\kappa(q)| = N_K^{(1)}(D,x).
\end{aligned}$$

□

Lemma 3.4. *We have*

$$\frac{1}{[K(y):K]}\deg_y \Omega_{\mathcal{O}_{K(y)}/\mathcal{O}_\mathcal{T}} \leqslant h_R(y) + O(1).$$

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Proof. Write $\mathscr{Y}_{\mathcal{T}} = \mathscr{Y} \times_{\mathscr{X}} \mathcal{T}$. The morphism $\mathcal{T} \to \mathscr{X}$ is representable, since it is the normalization of a substack. It follows that $\mathscr{Y}_{\mathcal{T}}$ is a scheme. Also, $\operatorname{Spec} \mathcal{O}_{K(y)} \to \mathscr{Y}_{\mathcal{T}}$ is the normalization of the image subscheme $\overline{\operatorname{Im}(y)}$.

Therefore,

$$\deg_y \Omega_{\mathcal{O}_{K(y)}/\mathcal{T}} \leqslant \deg_y \Omega_{\operatorname{Im}(y)/\mathcal{T}} \leqslant \deg_y \Omega_{\mathscr{Y}_{\mathcal{T}}/\mathcal{T}} \leqslant \deg_y \Omega_{\mathscr{Y}/\mathscr{X}}$$

(since $\deg \Omega$ drops when passing to normalization, subscheme or pullback)

$$= \deg_y \det \Omega_{\mathscr{Y}/\mathscr{X}} = [K(y) : K] \cdot h_R(y) + O(1),$$

as needed. □

Continuing with the proof of Proposition 3.2, Conjecture 3.1 for Y gives

$$N_K^{(1)}(D_Y, y) + d_K(K(y)) \geqslant h_{K_Y + D_Y}(y) - \delta h_B(y) + O_{[K(y):K(x)]}(1) \tag{3.1}$$

for y away from a proper closed subset. By Lemma 2.2, we have

$$d_K(K(y)) = \frac{1}{[K(y) : K]} \deg_y \Omega_{\mathcal{O}_{K(y)}/\mathcal{O}_{\mathcal{T}}} + d_K(\mathcal{T}).$$

By Lemmas 3.3 and 3.4, the left-hand side of (3.1) is majorized by

$$N_K^{(1)}(D, \pi(y)) + h_R(y) + d_K(\mathcal{T}) + O_{[K(y):K(x)]}(1).$$

On the other hand, for the right-hand side of (3.1), we have

$$h_{K_Y(D_Y)}(y) - \delta h_B(y) = h_{K_X(D)}(x) + h_R(y) - \delta h_H(x) + O(1).$$

All together, we obtain

$$N_K^{(1)}(D, x) + h_R(y) + d_K(\mathcal{T}) \geqslant h_{K_X(D)}(x) + h_R(y) - \delta h_H(x) + O_{[K(y):K(x)]}(1),$$

which, after canceling $h_R(y)$, gives

$$N_K^{(1)}(D, x) + d_K(\mathcal{T}) \geqslant h_{K_X(D)}(x) - \delta h_H(x) + O_{[K(y):K(x)]}(1).$$

A point x with $[K(x) : K] \leqslant r$ is the image of a point y with $[K(y) : K] \leqslant r \cdot \deg \pi$. Thus, the proposition for $\mathscr{X}$, $\mathscr{D}$, H, r and δ follows from Conjecture 3.1 applied to Y, $\pi^*\mathscr{D}$, B, $r \cdot \deg \pi$ and δ. □

4. Proof of the main result

4.1 Moduli spaces and toroidal compactifications

We follow the notation of [AV16]. However, we work over $\operatorname{Spec} \mathbb{Z}$:

$\widetilde{\mathcal{A}}_g \subset \overline{\widetilde{\mathcal{A}}}_g$	a toroidal compactification of the moduli *stack* of principally polarized abelian varieties of dimension g,
$\mathcal{A}_g \subset \overline{\mathcal{A}}_g$	the resulting compactification of the moduli *space* of principally polarized abelian varieties of dimension g,
$\widetilde{\mathcal{A}}_g^{[m]} \subset \overline{\widetilde{\mathcal{A}}}_g^{[m]}$	a compatible toroidal compactification of the moduli *stack* of principally polarized abelian varieties of dimension g with full level-m structure,
$\mathcal{A}_g^{[m]} \subset \overline{\mathcal{A}}_g^{[m]}$	the resulting compactification of the moduli space of principally polarized abelian varieties of dimension g with full level-m structure.

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The construction of $\overline{\widetilde{\mathcal{A}}}_g^{[m]}$ by Faltings and Chai [FC90, p. 128] yields a stack smooth over $\operatorname{Spec}\mathbb{Z}[\zeta_m, 1/m]$, where ζ_m is a primitive mth root of unity. Its boundary is a normal crossings divisor. However, their definition of full level-m structure requires a symplectic isomorphism $A[m] \xrightarrow{\sim} (\mathbb{Z}/m\mathbb{Z})^{2g}$. In [FC90, IV, Remark 6.12], they relaxed the requirement that the isomorphism be symplectic, giving a stack smooth over $\operatorname{Spec}\mathbb{Z}[1/m]$; in [FC90, I, Definition 1.8], they also considered full level structures in our sense (albeit still requiring the isomorphism (0.1) to be symplectic). Combining these remarks, we obtain a stack we denote $(\overline{\widetilde{\mathcal{A}}}_g^{[m]})_{\mathbb{Z}[1/m]}$, smooth over $\mathbb{Z}[1/m]$. If $m \geqslant 3$, this stack is a scheme [FC90, IV.6.9].

We extend the construction to $\operatorname{Spec}\mathbb{Z}$ by defining $\overline{\widetilde{\mathcal{A}}}_g^{[m]}$ to be the normalization of $\overline{\widetilde{\mathcal{A}}}_g$ in $(\overline{\widetilde{\mathcal{A}}}_g^{[m]})_{\mathbb{Z}[1/m]}$. The resulting stack is not smooth over primes dividing m, and even the interior of the stack over such primes does not have a modular interpretation. However, the boundary structure of this stack at primes dividing m is described in the Appendix.

The natural morphism $\mathcal{A}_g^{[m]} \to \mathcal{A}_g$ that 'forgets the level structure' is finite and, since we chose compatible compactifications, it extends to a *finite* morphism $\pi_m : \overline{\widetilde{\mathcal{A}}}_g^{[m]} \to \overline{\widetilde{\mathcal{A}}}_g$; see [FC90, Theorem IV.6.7(1)].

4.2 Rational points and covers of bounded degree

The stack $\overline{\widetilde{\mathcal{A}}}_g$ is proper, but a rational point $x \in \overline{\widetilde{\mathcal{A}}}_g$ might not extend to an integral point: it might correspond to an abelian variety with potentially semistable, but not semistable, reduction. In this section, we explain how one can use an integral extension of bounded degree to lift x to a finite cover of $\overline{\widetilde{\mathcal{A}}}_g$ that is a scheme, where the lift of x can be extended to an integral point.

We apply Lemma 2.5(3), which requires a covering $Y \to \overline{\widetilde{\mathcal{A}}}_g$ by a scheme. This can be achieved using [KV04, Theorem 1], but a more explicit construction in our situation is given in the following well-known lemma.

LEMMA 4.1. *Let $m = m_1 m_2$ be a product of two coprime integers each $\geqslant 3$. Then the stack $\overline{\widetilde{\mathcal{A}}}_g^{[m]}$ is a scheme.*

Proof. First, recall that if $d \geqslant 3$ is an integer, the stack $(\overline{\widetilde{\mathcal{A}}}_g^{[d]})_{\mathbb{Z}[1/d]}$ is a scheme. It suffices to show that $(\overline{\widetilde{\mathcal{A}}}_g^{[m]})_{\mathbb{Z}[1/m_1]}$ and $(\overline{\widetilde{\mathcal{A}}}_g^{[m]})_{\mathbb{Z}[1/m_2]}$ are schemes. This in turn follows because for $i = 1$ and 2, the stack $(\overline{\widetilde{\mathcal{A}}}_g^{[m]})_{\mathbb{Z}[1/m_i]}$ is the normalization of the scheme $(\overline{\widetilde{\mathcal{A}}}_g^{[m_i]})_{\mathbb{Z}[1/m_i]}$ in the scheme $(\overline{\widetilde{\mathcal{A}}}_g^{[m]})_{\mathbb{Z}[1/m]}$. □

Since $12 = 3 \cdot 4$ is the product of two relatively prime integers each $\geqslant 3$, it follows that $\overline{\widetilde{\mathcal{A}}}_g^{[12]}$ is a scheme. Let $M = \deg \pi_{12} : \overline{\widetilde{\mathcal{A}}}_g^{[12]} \to \overline{\widetilde{\mathcal{A}}}_g$. We obtain the following result.

PROPOSITION 4.2. *Let R be a Dedekind domain with field of fractions K. Fix a point $y \in \overline{\widetilde{\mathcal{A}}}_g^{[m]}(K)$. There are a finite extension L/K, with $[L : K] \leqslant M$ and $R_L \subset L$ the integral closure of R, and a point $\eta \in \overline{\widetilde{\mathcal{A}}}_g^{[m]}(R_L)$ lifting y.*

Proof. Applying Lemma 2.5(3) to the point $\pi_m(y) \in \overline{\widetilde{\mathcal{A}}}_g$, we have a point $\xi \in \overline{\widetilde{\mathcal{A}}}_g(R_L)$ lifting $\pi_m(y)$. Applying Lemma 2.5(1) to the representable morphism π_m, the point lifts to $\eta \in \overline{\widetilde{\mathcal{A}}}_g^{[m]}(R_L)$. □

4.3 Substacks

Let $X \subseteq (\widetilde{\mathcal{A}}_g)_K$ be a closed substack, let $X' \to X$ be a resolution of singularities and $X' \subset \overline{X}'$ a smooth compactification with $D = \overline{X}' \smallsetminus X'$ a normal crossings divisor. Assume that the rational map $f\colon \overline{X}' \to \overline{\widetilde{\mathcal{A}}}_g$ is a morphism. Let $X'_m = X' \times_{\widetilde{\mathcal{A}}_g} \widetilde{\mathcal{A}}_g^{[m]}$, and let $\overline{X}'_m \to \overline{X}' \times_{\overline{\widetilde{\mathcal{A}}}_g} \overline{\mathcal{A}}_g^{[m]}$ be a resolution of singularities with projections $\pi_m^X\colon \overline{X}'_m \to \overline{X}'$ and $f_m\colon \overline{X}'_m \to \overline{\widetilde{\mathcal{A}}}_g^{[m]}$.

We now spread these objects over $\mathcal{O}_{K,S}$ for a suitable finite set of places S containing the archimedean places. Let $(\mathscr{X}, \mathscr{D})$ be a normal crossings model of $(\overline{X}', D)$ over $\operatorname{Spec} \mathcal{O}_{K,S}$. As above, write $\mathscr{D} = \sum_i \mathscr{D}_i$. Such a model exists, even for Deligne–Mumford stacks, by [Ols06, Proposition 2.2].

Let $X(K)_{[m]}$ be the set of K-rational points of X corresponding to abelian varieties A/K admitting full level-m structure. Define

$$X(K)_{p \geqslant m_0} := \bigcup_{\substack{p \geqslant m_0 \\ p \text{ prime}}} X(K)_{[p]}.$$

4.4 Intersection multiplicities for integral and rational points

Write E for the boundary divisors of $(\overline{\widetilde{\mathcal{A}}}_g)_K$, and $\mathscr{E}$ for its closure in $\overline{\widetilde{\mathcal{A}}}_g$, which is a Cartier divisor. We have an equality of divisors on $\overline{X}'$:

$$f^*E = \sum a_i D_i,$$

where each $a_i > 0$; see [AV16, (4.3)]. This equality extends over $\operatorname{Spec} \mathcal{O}_{K,S}$ to

$$f^*\mathscr{E} = \sum a_i \mathscr{D}_i.$$

By [AV16, Proposition 4.1 or Equation (4.1)], we have that $\pi_m^* E = mE_m$ for some Cartier divisor $E_m \subset (\overline{\widetilde{\mathcal{A}}}_g^{[m]})_K$. Spreading out E_m to $\mathscr{E}_m$ in $\overline{\widetilde{\mathcal{A}}}_g^{[m]}$, we obtain $\pi_m^*\mathscr{E} = m\mathscr{E}_m$; moreover, by Proposition A.4 in the Appendix, $\mathscr{E}_m$ is a Cartier divisor.

Let $q \subset \mathcal{O}_{K,S}$ be a non-zero prime ideal. Assume that there are maps $\xi\colon \operatorname{Spec} \mathcal{O}_{K,q} \to \mathscr{X}$ and $\xi_m\colon \operatorname{Spec} \mathcal{O}_{K,q} \to \mathscr{X}_m$ such that $\xi = \pi_m^X \circ \xi_m$, and write $x \in \mathscr{X}(\mathcal{O}_{K,q})$ and $x_m \in \mathscr{X}_m(\mathcal{O}_{K,q})$ for the respective *integral* points corresponding to ξ and ξ_m. These objects and arrows fit together in the commutative diagram

$$\begin{array}{ccccc}
 & & \mathscr{X}_m & \xrightarrow{f_m} & \overline{\widetilde{\mathcal{A}}}_g[m] \\
 & \overset{\xi_m}{\nearrow} & \big\downarrow{\scriptstyle \pi_m^X} & & \big\downarrow{\scriptstyle \pi_m} \\
\operatorname{Spec} \mathcal{O}_{K,q} & \xrightarrow[\xi]{} & \mathscr{X} & \xrightarrow{f} & \overline{\widetilde{\mathcal{A}}}_g.
\end{array}$$

We have an equality of divisors on $\operatorname{Spec} \mathcal{O}_{K,q}$:

$$\xi^* f^* \mathscr{E} = \xi_m^* f_m^* \pi_m^* \mathscr{E} = m \cdot \xi_m^* f_m^* \mathscr{E}_m,$$

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which translates to

$$\sum a_i \xi^* \mathscr{D}_i = m \cdot \xi_m^* f_m^* \mathscr{E}_m.$$

The divisor on the left has multiplicity $\sum a_i n_q(\mathscr{D}_i, x)$. If $x \in \mathscr{X}(\mathcal{O}_{K,q})$, then the intersection multiplicities $n_q(\mathscr{D}_i, x)$ are integers, and we deduce that

$$m \Bigm| \sum a_i n_q(\mathscr{D}_i, x).$$

If the quantity $\sum a_i n_q(\mathscr{D}_i, x)$ is non-zero, then $m \leqslant \sum a_i n_q(\mathscr{D}_i, x)$, and thus

$$m \leqslant \max\{a_i\} \sum n_q(\mathscr{D}_i, x) = \max\{a_i\} n_q(\mathscr{D}, x);$$

in other words,

$$n_q(\mathscr{D}, x) \geqslant \frac{m}{\max\{a_i\}}.$$

Given a *rational* point $x \in \mathscr{X}(K)$, we apply Proposition 4.2 and obtain an extension field L/K with $[L : K] \leqslant M$ and an integral extension $\mathcal{O}_{L,q}$ with a point $\xi \in \mathscr{X}(\mathcal{O}_{L,q})$ lifting x. Since for any $\mathfrak{q} \mid q$ we have $e(\mathfrak{q} \mid q) \leqslant M$, (2.3) gives

$$n_q(\mathscr{D}, x) \geqslant \frac{m}{M \max\{a_i\}}.$$

We summarize this discussion in the following proposition.

Proposition 4.3. *With notation as in § 4.3, write $\alpha(X) := (M \cdot \max\{a_i\})^{-1} > 0$, which depends X, but not on x. Let $x_m \in X'_m(K)$ be a rational point in X'_m with image $x \in X'(K)$. Suppose that $n_q(\mathscr{D}, x) > 0$. Then*

$$n_q(\mathscr{D}, x) \geqslant m\alpha(X). \tag{4.1}$$

4.5 Proof of Theorem B

Lemma 4.4. *Fix $\epsilon' > 0$. Then there is an integer $m_0 := m_0(\epsilon', K, X)$ such that for all primes $p \geqslant m_0$ and $x \in X(K)_{[p]}$ we have*

$$d_K(\mathcal{T}_x) \leqslant h_{\epsilon' D}(x) + O(1).$$

Proof. Let A/K be the abelian variety of dimension g associated with $x \in X(K)_{[p]}$. Since A has full level-p structure, we know that $\#A[p](K) \geqslant p^g$. Thus, if $\mathfrak{q}$ is a prime ideal of K that does not divide p, then $\#A[p](\kappa(\mathfrak{q})) \geqslant p^g$ (see [HS00, C.1.4]). We choose $m_0 \geqslant 8$, so $p \neq 2$, freeing us to pick $\mathfrak{q} \mid 2$. This implies that $\kappa(\mathfrak{q}) = 2^{f(\mathfrak{q}|q)} \leqslant 2^{[K:\mathbb{Q}]}$.

We follow Flexor–Oesterlé [FO90, Théorème 3] and Silverberg [Sil92, Theorem 3.3]; see also Kamienny [Kam82, § 6(2a)]. Suppose now that A has good reduction at $\mathfrak{q}$, so that, by the Lang–Weil estimates, we have

$$\#A(\kappa(\mathfrak{q})) \leqslant (1 + \kappa(\mathfrak{q})^{1/2})^{2g} \leqslant (1 + 2^{[K:\mathbb{Q}]/2})^{2g}.$$

Thus, if A has good reduction at $\mathfrak{q} \mid 2$, we have

$$p \leqslant (1 + 2^{[K:\mathbb{Q}]/2})^2 := \gamma.$$

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In other words, if $p > \gamma$, then A must have bad reduction at primes $\mathfrak{q} \mid 2$, so $n_{\mathfrak{q}}(\mathscr{D}, x) > 0$. By Proposition 4.3, the stronger inequality (4.1) holds with $m = p$. We use this to see that if $p > \gamma$, then as in the estimate (2.5) we have

$$h_{\epsilon' D}(x) + O(1) \geqslant \epsilon' \sum_{\mathfrak{q}} n_{\mathfrak{q}}(D, x) \log |\kappa(\mathfrak{q})| \geqslant \epsilon' \sum_{\mathfrak{q}|2} p\alpha(X) \log |\kappa(\mathfrak{q})| \geqslant \epsilon'(\alpha(X) \log 2) \cdot p,$$

so $h_{\epsilon' D}(x)$ grows at least linearly in p.

Now we crudely bound $d_K(\mathcal{T}_x)$ from above. Note that x is an integral point away from p. As in § 4.4, passing to a cover of finite *bounded* degree $\leqslant M = M(g)$, we may replace x with an integral point y in such a way that $[K(y) : K] \leqslant M$. The discriminant ideal of $\mathcal{T}_x$ divides the discriminant ideal of the extension $K(y)/K$; we compare their factors at p. Let $d_K(K(y))_p$ denote the contribution at p of $d_K(K(y))$; ignoring negative terms coming from the discriminant of $\mathcal{O}_K$, we have the estimate

$$d_K(\mathcal{T}_x) \leqslant d_K(K(y))_p \leqslant \frac{v_p(|\mathrm{Disc}(\mathcal{O}_{K(y)})|)}{[K(y) : K]} \cdot \log p,$$

where v_p denotes the usual p-adic valuation. By [Neu99, Proof of III.2.13], we have

$$v_p(|\mathrm{Disc}(\mathcal{O}_{K(y)})|) \leqslant [K(y) : K](1 + [K(y) : K]).$$

Hence,

$$d_K(\mathcal{T}_x) \leqslant (1 + [K(y) : K]) \cdot \log p := \beta \cdot \log p$$

grows at most linearly in $\log p$, and the result follows. □

Lemma 4.5. *Fix $\epsilon' > 0$. Then there is an integer $m_0 := m_0(\epsilon', K, X)$ such that for all primes $p \geqslant m_0$, if $x \in X(K)_{[p]}$, then*

$$N_K^{(1)}(D, x) \leqslant h_{\epsilon' D}(x) + O(1).$$

Proof. If $x \in X(K)_{[p]}$, then whenever $n_q(\mathscr{D}, x) > 0$, Proposition 4.3 implies that the stronger inequality (4.1) holds. Hence,

$$\begin{aligned} p\alpha(X) N_K^{(1)}(D, x) &= \sum_{n_q(\mathscr{D},x)>0} p\alpha(X) \log |\kappa(q)| \\ &\leqslant \sum_{n_q(\mathscr{D},x)>0} n_q(\mathscr{D}, x) \log |\kappa(q)| \\ &\leqslant h_D(x) + O(1), \end{aligned}$$

where in the last inequality we use the estimate (2.5). Taking $m_0 > 1/(\epsilon' \alpha(X))$, we have $p\alpha(X) > 1/\epsilon'$ and hence $N_K^{(1)}(D, x) \leqslant h_{\epsilon' D}(x) + O(1)$. □

Proof of Theorem B. We proceed by Noetherian induction. For each integer $i \geqslant 1$, let

$$W_i = \overline{\widetilde{\mathcal{A}}_g(K)_{p \geqslant i}}.$$

Note that W_i is a closed subset of $\mathcal{A}_g$, and that $W_i \supseteq W_{i+1}$ for every i. The chain of W_i must stabilize by the Noetherian property of the Zariski topology of $\mathcal{A}_g$. Say $W_n = W_{n+1} = \cdots$.

We claim that W_n has dimension $\leqslant 0$. Suppose not, and let $X \subseteq W_n$ be an irreducible component of positive dimension. Fix $\epsilon > 0$ so that $K_X + (1-\epsilon)D$ is big: such an ϵ exists by [AV16, Corollary 1.10]. Next, choose a $\mathbb{Q}$-ample divisor H such that $K_X + (1-\epsilon)D - H$ is effective, and apply Proposition 3.2, with $r = 1$, to conclude that there is a Zariski-closed proper subset $Z \subset X$ such that if $x \in X(K) \smallsetminus Z(K)$, then

$$N_K^{(1)}(D, x) + d_K(\mathcal{T}_x) \geqslant h_{K_X(D)}(x) - \delta h_H(x) - O(1).$$

By Lemma 4.5, for all primes $p > m_0$, any $x \in X(K)_{[p]}$ satisfies $N_K^{(1)}(D, x) \leqslant h_{(\epsilon/2)D}(x) + O(1)$. On the other hand, Lemma 4.4 guarantees that, after possibly enlarging m_0, for all primes $p \geqslant m_0$, any $x \in X(K)_{[p]}$ satisfies $h_{(\epsilon/2)D}(x) + O(1) \geqslant d_K(\mathcal{T}_x)$. If also $x \notin Z(K)$, we deduce that

$$h_{\epsilon D}(x) \geqslant h_{K_X(D)}(x) - \delta h_H(x) - O(1).$$

By our choice of H and [HS00, Theorem B.3.2(e)], we obtain

$$O(1) \geqslant (1-\delta) h_{K_X((1-\epsilon)D)}(x).$$

Using [HS00, Theorem B.3.2(e,g)], we conclude that the set of $x \in X(K)_{p \geqslant m_0}$ outside $Z(K)$ is not dense, and thus $X(K)_{p \geqslant m_0}$ is contained in a Zariski-closed proper subset of X. On the other hand, if $m_0 > n$, then $W_{m_0} = W_n$, so X is also an irreducible component of W_{m_0} and hence $\overline{X(K)_{p \geqslant m_0}} = X$, which is a contradiction. This proves that $\dim W_n \leqslant 0$.

Finally, if W_n is a finite set of points, then it is well known that the full level structures that can possibly appear in any of the corresponding finitely many geometric isomorphism classes are bounded. Indeed, if $\mathfrak{q} \in M_K^0$ is a fixed prime of potential good reduction, all twists with full level-p structure with $p > 2, \mathfrak{q} \nmid p$ have good reduction at $\mathfrak{q}$. Since the p-torsion points inject modulo $\mathfrak{q}$, we have $p \leqslant (1 + N\mathfrak{q}^{1/2})^2$. Alternatively, following Manin [Man69, § 3], there are only finitely many isomorphism classes over $K_{\mathfrak{q}}$ and, for each, the torsion subgroup is finite. □

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Appendix. Compactifications with full level structure

Keerthi Madapusi Pera

The purpose of this appendix is to lay out certain facts about toroidal compactifications of the moduli of principally polarized abelian varieties with full level structure at 'bad' primes. This is a straightforward extension of the theory of [FC90] and could possibly also be extracted from the work of Lan [Lan16].

A.1 Equip $\mathbb{Z}^{2g} = \mathbb{Z}^g \oplus \mathbb{Z}^g$ with the standard non-degenerate symplectic pairing

$$\psi\colon ((u_1, v_1), (u_2, v_2)) \mapsto u_1 v_2^t - u_2 v_1^t.$$

For every integer $m \in \mathbb{Z}_{>0}$, equip $(\mathbb{Z}/m\mathbb{Z})^{2g}$ with the non-degenerate pairing ψ_m inherited from ψ. A symplectic level-m structure on a principally polarized abelian scheme (A, λ) over a base S will consist of a pair (η, ϕ), where

$$\eta\colon \underline{(\mathbb{Z}/m\mathbb{Z})}^{2g} \xrightarrow{\simeq} A[m]; \quad \phi\colon \mu_m \xrightarrow{\simeq} \underline{\mathbb{Z}/m\mathbb{Z}}$$

are isomorphisms of group schemes over S such that η carries the pairing ψ_m to the pairing $\phi \circ e_\lambda$ on $A[m]$. Here

$$e_\lambda\colon A[m] \times A[m] \to \mu_m$$

is the symplectic Weil pairing induced by the polarization λ.

A.2 Let $\widetilde{\mathcal{A}}_g$ be the algebraic stack over $\mathbb{Z}$ parameterizing principally polarized abelian varieties of dimension g. Over $\mathbb{Z}[1/m]$, we have a finite étale morphism of algebraic stacks

$$\widetilde{\mathcal{A}}_{g,m}[1/m] \to \widetilde{\mathcal{A}}_g[1/m]$$

parameterizing symplectic level-m structures on the universal abelian scheme over $\widetilde{\mathcal{A}}_g[1/m]$. By a classical argument of Serre, points of $\widetilde{\mathcal{A}}_{g,m}[1/m]$ have trivial automorphism schemes as soon as $m \geqslant 3$; see e.g. [FC90, ch. IV, Remark 6.2(c) or Corollary 6.9].

Fix any toroidal compactification $\overline{\widetilde{\mathcal{A}}}_g$ of $\widetilde{\mathcal{A}}_g$ (see [FC90, ch. IV]). We now obtain an open immersion

$$\widetilde{\mathcal{A}}_{g,m} \hookrightarrow \overline{\widetilde{\mathcal{A}}}_{g,m}$$

of algebraic stacks over $\mathbb{Z}$ by taking the normalization of the open immersion

$$\widetilde{\mathcal{A}}_g \hookrightarrow \overline{\widetilde{\mathcal{A}}}_g$$

in $\widetilde{\mathcal{A}}_{g,m}[1/m]$.

The stack $\widetilde{\mathcal{A}}_{g,m}$ has no obvious moduli interpretation over $\mathbb{Z}$, and we know little about the singularities of its fibers over primes dividing m. However, this is not an obstruction to studying its general structure at the boundary. For this, we will need some information about the stratification of the boundary.

A.3 We direct the reader to [Mad15, § 1] for the notion of a principally polarized 1-motif (Q, λ) over a base S. Here we will note that it consists of a 1-motif Q, that is, a two-term complex $u\colon X \to J$, where J is a semi-abelian scheme over S that is an extension of an abelian scheme by a torus, and X is a locally constant sheaf of finite free abelian groups, and an isomorphism $\lambda\colon Q \xrightarrow{\simeq} Q^\vee$ to its dual 1-motif $Q^\vee$.

We will say that (Q, λ) is of type (r, s) for $r, s \in \mathbb{Z}_{\geqslant 0}$ if the abelian part of J has dimension s and if $X = \underline{\mathbb{Z}}^r$. The polarization λ then canonically identifies the toric part of J with $\mathbb{G}_m^r$.

Suppose that (Q, λ) is of type (r, s), and set $g = r + s$. Given $m \in \mathbb{Z}_{>0}$, one has the m-torsion $Q[m]$ of the 1-motif Q: this is a finite flat group scheme over S of rank $2g$, and the polarization equips it with a non-degenerate Weil pairing e_λ with values in μ_m.

Let B be the abelian part of J. Then there is a natural ascending three-step filtration

$$0 = W_{-3}Q[m] \subset W_{-2}Q[m] = \mu_m^r \subset W_{-1}Q[m] \subset W_0 Q[m] = Q[m],$$

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where $W_{-2}Q[m]$ is isotropic for the Weil pairing, $W_{-1}Q[m]$ is its orthogonal complement, $\mathrm{gr}^W_{-1}Q[m]$ is identified with $B[m]$, compatibly with Weil pairings, and $\mathrm{gr}^W_0 Q[m]$ is identified with $(\mathbb{Z}/m\mathbb{Z})^r$. The induced pairing

$$(\mathbb{Z}/m\mathbb{Z})^r \times \mu_m^r = \mathrm{gr}^W_0 Q[m] \times W_{-2}Q[m] \xrightarrow{e_\lambda} \mu_m$$

is the canonical one.

Let $I_r \subset \mathbb{Z}^{2g}$ be the isotropic subspace spanned by the first r basis vectors of the first copy of $\mathbb{Z}^g$. We have identifications

$$\mathbb{Z}^r = I_r; \quad \mathbb{Z}^r = \mathbb{Z}^{2g}/I_r^\perp,$$

so that the induced non-degenerate pairing

$$\mathbb{Z}^r \times \mathbb{Z}^r = I_r \times \mathbb{Z}^{2g}/I_r^\perp \xrightarrow{\psi} \mathbb{Z}$$

is the standard symmetric pairing $(u, v) \mapsto uv^t$.

A symplectic level-m structure on (Q, λ) is a pair (η, ϕ), where

$$\eta\colon \underline{(\mathbb{Z}/m\mathbb{Z})}^{2g} \xrightarrow{\simeq} Q[m]; \phi\colon \mu_m \xrightarrow{\simeq} \underline{\mathbb{Z}/m\mathbb{Z}}$$

are isomorphisms of group schemes over S such that η carries the pairing ψ_m to the pairing $\phi \circ e_\lambda$ on $Q[m]$ and the subspace I_r/mI_r onto $W_{-2}Q[m]$, so that the induced isomorphism

$$(\mathbb{Z}/m\mathbb{Z})^r = I_r/mI_r \xrightarrow{\simeq} W_{-2}Q[m] = \mu_m^r \xrightarrow[\phi^{-1}]{\simeq} (\mathbb{Z}/m\mathbb{Z})^r$$

is the identity.

We now obtain a moduli stack $\widetilde{\mathcal{Y}}_{r,s}$ over $\mathbb{Z}$ of principally polarized 1-motifs, and a finite étale cover

$$\widetilde{\mathcal{Y}}_{r,s,m}[1/m] \to \widetilde{\mathcal{Y}}_{r,s}[1/m]$$

over $\mathbb{Z}[1/m]$, parameterizing symplectic level-m structures on the universal principally polarized 1-motif.

A.4 Consider the moduli stack $\widetilde{\mathcal{Y}}_{r,0}$: this parameterizes principally polarized 1-motifs of the form $u\colon \mathbb{Z}^r \to \mathbb{G}_m^r$. Alternatively, it parameterizes symmetric pairings $\mathbb{Z}^r \times \mathbb{Z}^r \to \mathbb{G}_m$. As such, it is represented over $\mathbb{Z}$ by the torus with character group $\mathsf{S}_r = \mathrm{Sym}^2\mathbb{Z}^r$.

Similarly, by the discussion in [FC90, ch. IV, § 6.5], the morphism

$$\widetilde{\mathcal{Y}}_{r,0,m}[1/m] \to \widetilde{\mathcal{Y}}_{r,0}[1/m]$$

parameterizes lifts $(1/m)\mathbb{Z}^r \to \mathbb{G}_m^r$ of the universal homomorphism $\mathbb{Z}^r \to \mathbb{G}_m^r$, and so is represented over $\mathbb{Z}[1/m]$ by the torus with character group $(1/m)\mathsf{S}_r$. The natural map

$$\widetilde{\mathcal{Y}}_{r,0,m}[1/m] \to \widetilde{\mathcal{Y}}_{r,0}[1/m]$$

corresponds to the map of tori induced by the inclusion $\mathsf{S}_r \hookrightarrow (1/m)\mathsf{S}_r$ of character groups.

Therefore, the normalization $\widetilde{\mathcal{Y}}_{r,0,m}$ of $\widetilde{\mathcal{Y}}_{r,0}$ in $\widetilde{\mathcal{Y}}_{r,0,m}[1/m]$ is represented over $\mathbb{Z}$ by the torus with character group $(1/m)\mathsf{S}_r$, and is in particular smooth over $\mathbb{Z}$.

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A.5 When $s > 0$, $\widetilde{\mathcal{Y}}_{r,s}$ permits a similar, but slightly more elaborate, description. We have the obvious map $\widetilde{\mathcal{Y}}_{r,s} \to \widetilde{\mathcal{A}}_s$ assigning to a polarized 1-motif (Q, λ) of type (r, s) the abelian part of the semi-abelian scheme J.

There is a natural action of the torus $\mathcal{Y}_{r,0}$ on $\mathcal{Y}_{r,s}$: given a polarized 1-motif (Q_0, λ_0) of type $(r, 0)$ associated with a homomorphism $u_0 \colon \underline{\mathbb{Z}}^r \to \mathbb{G}_m^r$ and a polarized 1-motif (Q, λ) of type (r, s) associated with $u \colon \underline{\mathbb{Z}}^r \to J$, the product $u_0 \cdot u \colon \underline{\mathbb{Z}}^r \to J$ corresponds to another principally polarized 1-motif of type (r, s).

The quotient of $\widetilde{\mathcal{Y}}_{r,s}$ by this action is naturally identified with the abelian scheme $\widetilde{\mathcal{C}}_{r,s} \to \widetilde{\mathcal{A}}_s$ that parameterizes homomorphisms

$$v \colon \mathbb{Z}^r \to B,$$

where B is the universal abelian scheme over $\widetilde{\mathcal{A}}_s$. So, we obtain a tower of algebraic stacks:

$$\widetilde{\mathcal{Y}}_{r,s} \to \widetilde{\mathcal{C}}_{r,s} \to \widetilde{\mathcal{A}}_s, \tag{A.1}$$

where the first morphism is a $\widetilde{\mathcal{Y}}_{r,0}$-torsor, and the second is an abelian scheme.

From the discussion in [FC90, ch. IV, § 6.5], we find that the stack $\widetilde{\mathcal{Y}}_{r,s,m}[1/m]$ admits a compatible tower structure:

$$\widetilde{\mathcal{Y}}_{r,s,m}[1/m] \to \widetilde{\mathcal{C}}_{r,s,m}[1/m] \to \widetilde{\mathcal{A}}_{s,m}[1/m]. \tag{A.2}$$

Here $\widetilde{\mathcal{C}}_{r,s,m}[1/m]$ parameterizes homomorphisms

$$v_m \colon \frac{1}{m}\mathbb{Z}^r \to B,$$

where B is the universal abelian scheme over $\widetilde{\mathcal{A}}_{s,m}[1/m]$, and $\widetilde{\mathcal{Y}}_{r,s,m}[1/m]$ parameterizes homomorphisms

$$u_m \colon \frac{1}{m}\mathbb{Z}^r \to J$$

lifting v_m, where J is the universal semi-abelian scheme over $\widetilde{\mathcal{C}}_{r,s,m}[1/m]$ parameterized by the homomorphism

$$m \cdot v_m \colon \mathbb{Z}^r \to B \xrightarrow{\simeq} B^\vee.$$

It is therefore naturally a $\widetilde{\mathcal{Y}}_{r,0,m}[1/m]$-torsor over $\widetilde{\mathcal{C}}_{r,s,m}[1/m]$.

From this description, it is clear that the normalization of the tower (A.1) in the tower (A.2) gives us a tower

$$\widetilde{\mathcal{Y}}_{r,s,m} \to \widetilde{\mathcal{C}}_{r,s,m} \to \widetilde{\mathcal{A}}_{s,m},$$

where

$$\widetilde{\mathcal{C}}_{r,s,m} \to \widetilde{\mathcal{A}}_{s,m}$$

is still an abelian scheme parameterizing homomorphisms $v_m \colon (1/m)\mathbb{Z}^r \to B$ (with B the universal abelian scheme over $\widetilde{\mathcal{A}}_{s,m}$), and

$$\widetilde{\mathcal{Y}}_{r,s,m} \to \widetilde{\mathcal{C}}_{r,s,m}$$

is once again a $\widetilde{\mathcal{Y}}_{r,0,m}$-torsor parameterizing lifts $u_m \colon (1/m)\mathbb{Z}^r \to J$ of v_m, where J is still classified by $v = m \cdot v_m$.

In particular, the morphism

$$\widetilde{\mathcal{Y}}_{r,s,m} \to \widetilde{\mathcal{Y}}_{r,s} \times_{\widetilde{\mathcal{C}}_{r,s}} \widetilde{\mathcal{C}}_{r,s,m} \tag{A.3}$$

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is obtained via pushforward of torsors along the morphism

$$\widetilde{\mathcal{Y}}_{r,0,m} \to \widetilde{\mathcal{Y}}_{r,0}$$

of tori, which is of course canonically isomorphic to the multiplication-by-m map

$$\widetilde{\mathcal{Y}}_{r,0} \xrightarrow{[m]} \widetilde{\mathcal{Y}}_{r,0}.$$

A.6 Fix a rational polyhedral cone $\sigma \subset (\mathsf{S}_r)_{\mathbb{Q}}$: this gives us twisted toric embeddings

$$\widetilde{\mathcal{Y}}_{r,s} \hookrightarrow \widetilde{\mathcal{Y}}_{r,s}(\sigma); \widetilde{\mathcal{Y}}_{r,s,m} \hookrightarrow \widetilde{\mathcal{Y}}_{r,s,m}(\sigma).$$

The complements of these embeddings admit a natural stratification with a unique closed stratum, which we denote by $\mathcal{Z}_{r,s}(\sigma)$ and $\mathcal{Z}_{r,s,m}(\sigma)$, respectively.

Let $\hat{\widetilde{\mathcal{Y}}}_{r,s}(\sigma)$ and $\hat{\widetilde{\mathcal{Y}}}_{r,s}(\sigma)$ be the formal completions of $\widetilde{\mathcal{Y}}_{r,s}(\sigma)$ and $\widetilde{\mathcal{Y}}_{r,s,m}(\sigma)$, respectively, along their closed strata. By abuse of notation, write $\hat{\widetilde{\mathcal{Y}}}_{r,s,m}(\sigma)[1/m]$ for the completion of $\widetilde{\mathcal{Y}}_{r,s,m}(\sigma)[1/m]$ along its closed stratum.

Note that the morphism

$$\widetilde{\mathcal{Y}}_{r,s,m}(\sigma) \to \widetilde{\mathcal{Y}}_{r,s}(\sigma) \times_{\widetilde{\mathcal{C}}_{r,s}} \widetilde{\mathcal{C}}_{r,s,m} \tag{A.4}$$

is obtained via contraction along the multiplication-by-m map on $\widetilde{\mathcal{Y}}_{r,0}$.

A.7 Let $\Gamma(\sigma) \subset \mathrm{GL}_r(\mathbb{Z})$ be the stabilizer of σ, and let $\Gamma_m(\sigma) \leqslant \Gamma(\sigma)$ be the subgroup of matrices that are trivial mod m: these are both finite groups, and $\Gamma_m(\sigma)$ is trivial as soon as $m \geqslant 3$.

By the main results of [FC90, ch. IV], the toroidal compactification $\overline{\mathcal{A}}_g$ admits a stratification by locally closed substacks $\mathcal{Z}(r,\sigma)$ equipped with an isomorphism to $\Gamma(\sigma)\backslash\mathcal{Z}_{r,g-r}(\sigma)$ for some $r \leqslant g$ and some $\sigma \subset (\mathsf{S}_r)_{\mathbb{Q}}$, and such that this isomorphism extends to one of formal completions

$$(\overline{\mathcal{A}}_g)^{\wedge}_{\mathcal{Z}(r,\sigma)} \xrightarrow{\simeq} \Gamma(\sigma)\backslash\hat{\widetilde{\mathcal{Y}}}_{r,g-r}(\sigma).$$

Faltings and Chai use the language of degeneration data. For a formulation using our language of 1-motifs, we guide the reader to [Str10, § 3.1.5].

The main idea is that, on every formally étale affine chart

$$\mathrm{Spf}(R,I) \to \hat{\widetilde{\mathcal{Y}}}_{r,g-r}(\sigma),$$

one obtains a principally polarized 1-motif (Q,λ) of type $(r,g-r)$ over the fraction field $K(R)$ of R associated with a semi-abelian scheme $J \to \mathrm{Spec}\, R$, and a period map $u\colon \mathbb{Z}^r \to J(K(R))$. This period map 'degenerates' along $\mathrm{Spec}\, R/I$, and a construction of Mumford, explained in [FC90, ch. III], now gives us a principally polarized abelian scheme (A,ψ) over $K(R)$ with semi-abelian degeneration over R, and equipped with a canonical symplectic identification $Q[m] \xrightarrow{\simeq} A[m]$ for every integer m. The pair (A,ψ) now gives a map $\mathrm{Spec}\, K(R) \to \mathcal{A}_g$, which extends to a map

$$\mathrm{Spec}\, R \to \overline{\mathcal{A}}_g,$$

which in turn induces a map

$$\mathrm{Spf}(R,I) \to (\overline{\mathcal{A}}_g)^{\wedge}_{\mathcal{Z}(r,\sigma)}$$

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of formal algebraic stacks. These maps are now glued together to give the inverse of the desired isomorphism of formal neighborhoods.

Similarly, $\overline{\widetilde{\mathcal{A}}}_{g,m}[1/m]$ admits a compatible stratification by locally closed substacks $\mathcal{Z}_m(r,\sigma)[1/m]$ equipped with an isomorphism to $\Gamma(\sigma)\backslash\mathcal{Z}_{r,g-r,m}(\sigma)[1/m]$, and such that this isomorphism extends to one of formal completions

$$(\overline{\widetilde{\mathcal{A}}}_{g,m}[1/m])^{\wedge}_{\mathcal{Z}_m(r,\sigma)[1/m]} \xrightarrow{\simeq} \Gamma_m(\sigma)\backslash\widehat{\widetilde{\mathcal{Y}}}_{r,g-r,m}(\sigma)[1/m].$$

Proposition A.1. *The stratification on* $\overline{\widetilde{\mathcal{A}}}_{g,m}[1/m]$ *extends to one of* $\overline{\widetilde{\mathcal{A}}}_{g,m}$ *by substacks* $\mathcal{Z}_m(r,\sigma)$ *equipped with an isomorphism to* $\mathcal{Z}_{r,g-r,m}(\sigma)$, *extending to an isomorphism*

$$(\overline{\widetilde{\mathcal{A}}}_{g,m})^{\wedge}_{\mathcal{Z}_m(r,\sigma)} \xrightarrow{\simeq} \Gamma_m(\sigma)\backslash\widehat{\widetilde{\mathcal{Y}}}_{r,g-r,m}(\sigma).$$

Proof. Let

$$\operatorname{Spf}(R,I) \to \widehat{\widetilde{\mathcal{Y}}}_{r,g-r,m}(\sigma)$$

be a formally étale affine chart. The tautological principally polarized 1-motif (Q,λ) over $\operatorname{Spec} K(R)$ is now equipped with a canonical symplectic level-m structure, which in turn also equips the principally polarized abelian scheme (A,ψ), obtained from it via Mumford's construction, with a symplectic level-m structure.

This implies that the associated map $\operatorname{Spec} K(R) \to \widetilde{\mathcal{A}}_g$ has a canonical lift

$$\operatorname{Spec} K(R) \to \widetilde{\mathcal{A}}_{g,m},$$

which then extends to a map $\operatorname{Spec} R \to \overline{\widetilde{\mathcal{A}}}_{g,m}$.

Assume now that R is a complete local ring of $\widetilde{\mathcal{Y}}_{r,g-r,m}(\sigma)$ with maximal ideal I and algebraically closed residue field, and let R' be the complete local ring of $\overline{\widetilde{\mathcal{A}}}_{g,m}$ at the image of the geometric closed point of $\operatorname{Spec} R$. We claim that the induced map $R' \to R$ is an isomorphism. This follows from two observations: first, it is a *finite* map of normal local rings. Second, by the description of the stratification in characteristic 0, if p is the residue characteristic of R, then, for any maximal ideal $\mathfrak{m}' \subset R'[1/p]$, the ideal $\mathfrak{m} = \mathfrak{m}'R[1/p]$ is once again maximal, and the induced map

$$\widehat{R'[1/p]}_{\mathfrak{m}'} \to \widehat{R[1/p]}_{\mathfrak{m}}$$

is an isomorphism. The second assertion shows, via faithfully flat descent, that every element of R is contained in $R'[1/p]$, and the first shows that it must already be contained in R'.

Let $\eta_m : \overline{\widetilde{\mathcal{A}}}_{g,m} \to \overline{\widetilde{\mathcal{A}}}_g$ be the natural finite map. Combining the previous paragraph with Artin approximation, we find that η_m is étale locally isomorphic to the finite map

$$\mathcal{Y}_{r,g-r,m}(\sigma) \to \mathcal{Y}_{r,g-r}(\sigma) \tag{A.5}$$

for varying choices of r and σ.

We claim that the reduced stack $\mathcal{Z}_m(r,\sigma)$ underlying the locally closed substack $\eta_m^{-1}(\mathcal{Z}(r,\sigma)) \subset \overline{\widetilde{\mathcal{A}}}_{g,m}$ is normal. This can be checked on complete local rings using the observation that the reduced substack underlying the pre-image of $\mathcal{Z}_{r,g-r}(\sigma)$ under the map (A.5) is normal.

Moreover, from this and the fact that the locally closed substacks $\mathcal{Z}(r,\sigma)$ stratify $\overline{\widetilde{\mathcal{A}}}_g$, one can deduce that the locally closed substacks $\mathcal{Z}_m(r,\sigma)$ stratify $\overline{\widetilde{\mathcal{A}}}_{g,m}$.

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By normality of the target, the map

$$\mathcal{Z}_{r,g-r,m}(\sigma)[1/m] \xrightarrow{\simeq} \mathcal{Z}_m(r,\sigma)[1/m] \hookrightarrow \overline{\widetilde{\mathcal{A}}}_{g,m}$$

extends uniquely to a map

$$\mathcal{Z}_{r,g-r,m}(\sigma) \to \overline{\widetilde{\mathcal{A}}}_{g,m}$$

lifting the composition

$$\mathcal{Z}_{r,g-r,m}(\sigma) \to \mathcal{Z}_{r,g-r}(\sigma) \to \overline{\widetilde{\mathcal{A}}}_{g}.$$

This extension necessarily factors through a finite map

$$\Gamma_m(\sigma)\backslash\mathcal{Z}_{r,g-r,m} \to \mathcal{Z}_m(r,\sigma),$$

which is an isomorphism in the generic fiber. By looking at complete local rings, it is seen to be a finite étale map and hence an isomorphism.

The last assertion about the formal completions now follows from [Mad15, (A.3.2)]. □

From this and the explicit nature of the map (A.4), we immediately obtain the following result.

PROPOSITION A.2. *Let $\eta_m\colon \overline{\widetilde{\mathcal{A}}}_{g,m} \to \overline{\widetilde{\mathcal{A}}}_{g}$ be the natural finite map, and let $\mathcal{D}_m \subset \overline{\widetilde{\mathcal{A}}}_{g,m}$ be the complement of $\widetilde{\mathcal{A}}_{g,m}$, equipped with its reduced scheme structure.*

Then $\mathcal{D}_m$ is a relative Cartier divisor over $\mathbb{Z}$. Moreover, if $\mathcal{D} \subset \overline{\widetilde{\mathcal{A}}}_{g}$ is the boundary divisor with its reduced scheme structure, then we have an equality of Cartier divisors $\eta_m^*\mathcal{D} = m\cdot\mathcal{D}_m$.

Remark A.3. Note that the above proposition remains true if we replace $\widetilde{\mathcal{A}}_{g,m}$ and its compactification with the normalizations of their base change over $\mathcal{O}_K$, for any number field $K/\mathbb{Q}$.

PROPOSITION A.4. *Let $\widetilde{\mathcal{A}}_g^{[m]}$ and $\overline{\widetilde{\mathcal{A}}}_g^{[m]}$ be as in § 4. Let $\pi_m\colon \overline{\widetilde{\mathcal{A}}}_g^{[m]} \to \overline{\widetilde{\mathcal{A}}}_g$ be the natural finite map, and let $\mathcal{D}^{[m]} \subset \overline{\widetilde{\mathcal{A}}}_g^{[m]}$ be the complement of $\widetilde{\mathcal{A}}_g^{[m]}$, equipped with its reduced scheme structure.*

Then $\mathcal{D}^{[m]}$ is a relative effective Cartier divisor over $\mathbb{Z}$.[2] *Moreover, we have $\pi_m^*\mathcal{D} = m\cdot\mathcal{D}^{[m]}$.*

Proof. Over $\mathbb{Z}[1/m,\mu_m]$, $\widetilde{\mathcal{A}}_g^{[m]}$ and $\overline{\widetilde{\mathcal{A}}}_g^{[m]}$ can be identified with a disjoint union of copies of $(\widetilde{\mathcal{A}}_{g,m})_{\mathbb{Z}[1/m,\mu_m]}$ and $(\overline{\widetilde{\mathcal{A}}}_{g,m})_{\mathbb{Z}[1/m,\mu_m]}$, respectively. So, the result is true over $\mathbb{Z}[1/m]$. Moreover, by Proposition A.2 and Remark A.3, it is true after a change of scalars to $\mathbb{Z}[\mu_m]$ followed by normalization. Combining the two, we find that the result is already true over $\mathbb{Z}$. □

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