STORAGE AND RELEASE OF MAGNETIC ENERGY IN A FORCE-FREE FIELD

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ABSTRACT. For a 3D force-free field occupying a half-space $D = \{z > 0\}$, we discuss : i) the storage of free magnetic energy when the field evolves quasi-statically as a consequence of motions imposed to its footpoints on the plane $\{z = 0\}$; ii) the release of this energy during a reconnection process implying a rearrangement of the lines which is either local, occuring in the neighbourhood of spontaneously formed current sheets, or global, occuring in an explosive flare-like way.

1. INTRODUCTION

In the solar corona, a large amount of free energy may be stored in the magnetic field which is maintained in a stressed, essentially force-free state by the never-ceasing motions of the subphotospheric plasma in which its footpoints are anchored. Part of this energy may be released : i) in a more or less continuous way if the quasi-static evolution of the field leads to the spontaneous recurrent formation of thin current layers through which the lines may reconnect (thus suffering a local rearrangement) ; ii) in an explosive way if the field is brought into a metastable state which thus effects a reconnecting transition to a lower energy state (this implying a global rearrangement of the lines). The first process may lead to a general heating of the corona (Parker, 1989), while the second one may be related to such well-known phenomena as flares and coronal transients. In this communication, we discuss some theoretical points which pertain to the general processes of energy storage and release. We shall use a simple model in which the corona is represented by the half-space $D = \{z > 0\}$ and the photosphere by the plane $\partial D = \{z = 0\}$.

E. R. Priest and V. Krishan (eds.), Basic Plasma Processes on the Sun, 313–318.

2. SOME GENERAL PROPERTIES OF FORCE-FREE FIELDS IN D

2.1. How much magnetic energy can be stored in D ?

Let us consider in D a finite energy force-free magnetic field $\mathbf{B}$ (thus satisfying $\nabla.\mathbf{B} = 0$ and $(\nabla \times \mathbf{B}) \times \mathbf{B} = 0$) and assume that the function $g(x,y) = B_z(x,y,o)$ is such that $\omega^{2+\varepsilon}\ g = 0(1)$ for some $\varepsilon > 0$ $(\omega^2 = x^2 + y^2)$. Then one has (Aly, 1984)

$$C_o[g] \overset{\text{def}}{=} \int_D B_o^2\, d\mathbf{r} \leq C \overset{\text{def}}{=} \int_D B^2\, d\mathbf{r} \leq C_m[g] \tag{1}$$

where $\mathbf{B}_o$ is the finite energy potential field $(\nabla.\mathbf{B}_o = 0\ ;\ \nabla \times \mathbf{B}_o = 0)$ such that $B_{oz}(x,y,o) = g(x,y)$, and $C_m[g]$ is a number depending only on g and choosen as small as possible (C_m may be explicitly bounded from above). As there are only two force-free fields in D which can be canonically determined from g - i.e. $\mathbf{B}_o$ and $\mathbf{B}_{op}$, which is an open field, potential everywhere but on a current sheet in equilibrium - it is tempting to conjecture that $C_m[g] = C_{op}[g] = \int_D B^2{}_{op}\, d\mathbf{r}$. That this conjecture is true indeed is supported by the fact that one has $C \leq C_{op}[g]$ if g is choosen to belong to a particular class (Aly, 1984) or if $\mathbf{B}$ is a constant-α force-free field (Aly, 1989a) ; and by a general result reported in § 2 below.

2.2. Scaling of the free energy with the current

The force-free equation for $\mathbf{B}$ may be also written : $\nabla \times \mathbf{B} = \alpha\, \mathbf{B}$, where the function α is constant along any field line $(\mathbf{B}.\nabla\, \alpha = 0)$. Clearly, the magnitude of α is a measure of the intensity of the field-aligned current. It may be shown that (Aly, 1984, 1988)

$$|\alpha|_m\ \ell_1 \leq 1 \tag{2}$$

where $|\alpha|_m = \sup_{\partial D} |\alpha| = \sup_{D} |\alpha|$ (quite generally, all the lines of $\mathbf{B}$ are connected to ∂D) and ℓ_1 is a typical length scale depending on g and h, with $h(x,y) = \alpha(x,y,o)/|\alpha|_m$.

As for the "free energy" $c = C - C_o[g]$ of $\mathbf{B}$, one has (Aly, 1988)

$$(\ell_2\ |\alpha|_m)^2\ C_o \leq c \leq (\ell_3\ |\alpha|_m)^2\ C_o \tag{3}$$

where ℓ_2 is a function of g and h, and ℓ_3 a function of g. When $\mathbf{B}$ and α have the same typical length scale ℓ, one has : $\ell_1 \sim \ell_2 \sim \ell_3 \sim \ell$.

2.3. Linear ideal MHD stability of $\mathbf{B}$

If we assume that D is filled up with a perfectly conducting plasma and that the lines of $\mathbf{B}$ have their feet firmly tied to ∂D, then this field is found to be linearly stable if (sufficient condition) (Aly, 1989b)

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$$[\int_D |\alpha|^3 d\mathbf{r}]^{1/3} + [\int_D |f|^{3/2} d\mathbf{r}]^{1/3} < \frac{\sqrt{3}}{2} \pi^{2/3} \tag{4}$$

$$f(\mathbf{r}) = \int_{\mathcal{L}(\mathbf{r})} |\nabla\alpha|^2 \frac{ds}{B} \cdot \int_{\mathcal{L}(\mathbf{r})} B\, ds \tag{5}$$

where $\mathcal{L}(\mathbf{r})$ denotes the field line passing through $\mathbf{r}$ and ds the element of arc length. For fields for which $\mathbf{B}$ and α have the same typical length scale ℓ, stability is obtained if $|\alpha|_m \ell < O(1)$. A field with $|\alpha|_m \ell \lesssim O(1)$ is stable and from the results of § 2.1 and 2.2, it contains an energy which is near the maximum amount which may be stored in D at g given.

3. QUASI-STATIC EVOLUTION OF $\mathbf{B}$

3.1. Simple topology field

Let us now assume that D is filled up with a perfectly conducting plasma and that a potential field $\mathbf{B}_o$ having a simple topology (i.e. whose lines do not form separatrix surfaces) is made evolving quasi-statically through a sequence of force-free configurations as a consequence of a smooth stationary velocity field $\mathbf{v}$ (with $v_z = 0$ and $|\mathbf{v}|$ vanishing fast at infinity) imposed on ∂D. The field $\mathbf{B}_t$ at time t may be written as $\mathbf{B}_t [\mathbf{r} (\mathbf{r}_o,t)] = J^{-1} (\mathbf{r}_o,t) \cdot \mathbf{B}_o(\mathbf{r}_o) \cdot \nabla_o \mathbf{r} (\mathbf{r}_o,t)$, where $\mathbf{r} (\mathbf{r}_o,t)$ is the position at time t of a plasma element located at $\mathbf{r}_o$ at t = 0, and $J = \det(\nabla_o \mathbf{r})$. The function $\mathbf{r}$ is determined at each time t as a stationary point of

$$C [\mathbf{r}] = \int_D B^2 d\mathbf{r} = \int_D [\mathbf{B}_o(\mathbf{r}_o).\nabla_o \mathbf{r}]^2 \frac{d\mathbf{r}_o}{J} \tag{6}$$

defined over the set of all the functions $\mathbf{r} (\mathbf{r}_o)$ which take on ∂D the values $\mathbf{r} (\mathbf{r}_o,t)$ associated with the given velocity field $\mathbf{v}$. Note that this problem looks similar to a problem of anisotropic elasticity ; it turns out to be more complicated, however, because of the obvious possibility to construct a field $\mathbf{B}$ at t from $\mathbf{B}_o$ by an infinite number of admissible mappings $\mathbf{r}$.

We have recently undertaken a study of the functional (6). However, so for we have only been able to develop arguments indicating that (6) must admit an absolute minimizer when $|\mathbf{r} (r_o,t) - \mathbf{r}_o|$ is not too large on ∂D, the associated magnetic field being smooth (i.e. no current sheets do form ; compare with Parker, 1989).

Assuming the existence of $\mathbf{B}_t$ for $0 \leq t < \infty$, we have also shown shown that, when $t \to \infty$, $\mathbf{B}_t$ approaches asymptotically an open field, all the currents concentrating into a current sheet (we suppose here to simplify the presentation that B_z is kept invariant by $\mathbf{v}$). This result strongly supports our conjecture of § 2.1.

3.2. Complex topology fields

If the lines of the initial field $\underset{\sim}{B}_o$ have a complex topology (i.e. if separatrices do exist), then the problem of quasi-static evolution driven by $\underset{\sim}{v}$ on ∂D may still be formulated as above as a problem of minimizing (or more generally of finding the stationary points) of the functional (6) over an appropriate set of functions at each time t. In that case, however, we are sure a priori that the solutions are not going to be smooth : current-sheets have to develop indeed along the separatrices because the "connectivity" of the lines is discontinuous accross these surfaces (Aly, 1987 ; Antiochos, 1989).

Physically, such current sheets are unsatisfactory features, as any tiny amount of resistivity would imply their dissipation, and then, to get a meaningful formulation of the problem of evolution in that case, one needs to add a prescription describing the ineluctable reconnection of the field accross them (this is still to be done). Anyway, we may conclude that deformation of a complex topology configuration leads necessarily to plasma heating and thus is a potentially interesting mechanism for heating the corona.

4. LARGE-SCALE RECONNECTION

The asymptotic result reported in § 3.1 strongly suggests that, if we allow the plasma to have a small resistivity, the field must become unstable with respect to reconnection when the shear becomes large enough. Up to now, we have been able to gain some insight into this conjecture only in the particular situation where $\underset{\sim}{B}$ is an arcade which is axisymmetric around $\hat{z}$. In that case, we may reasonably admit that 2D reconnection, acting on a time scale much smaller than the diffusion time scale of the field, can change the topology of the lines, but not the "distribution of the magnetic fluxes" : i.e. the range of values of the "flux function" u, as well as the amount of toroidal flux between any two surfaces Σ_{u1} and Σ_{u2}, are conserved during a reconnecting transition. Then reconnection is possible in an arcade configuration only if there exists a field related to it by these flux constraints, but having a more complex topology and a lower energy. We have shown that such fields start existing at some critical time t_c when the shear exceeds a critical value. For $t > t_c$ the <u>arcade is metastable with respect to reconnection</u> and is able to effect an <u>explosive transition</u> (flare) towards a lower energy state under the action of a finite perturbation.

5. CONCLUSION

We have been able to extend to a 3D situation some of the analytical results previously obtained in a study of the quasi-static evolution of a 2D force-free field (Aly, 1989c). In particular, we have shown that a 3D field may approach asymptotically an open field. But much is still to be done, especially on the problem of the existence and smoothness of the solutions.

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DISCUSSION

PRIEST: (i) Have you proved the existence of your open fields?
(ii) As you mentioned, the field may lose equilibrium or stability before reaching the maximum energy. Hood and Priest (1980) considered the stability of simple arcades and flux tubes. They could not find an instability for simple arcades, but flux tubes or arcades with flux tubes (which would have $\alpha\ell > 1$) become unstable when the twist is too great.

ALY: (i) For a given flux distribution $B_z(x,y,0)$, it is simple to prove that you can construct one and only one totally open field of finite energy. What is difficult to prove (and this has not yet been done) is the existence of a sequence of (closed) force-free fields approaching this open field when they are indefinitely sheared and twisted.
(ii) In 3D, a flux tube (isolated or embedded in an arcade) has to be tied to the boundary, no region magnetically disconnected from $\{z = 0\}$ generally exists. For a tube for which the current and the field have the same typical length-scale ℓ, my calculations imply existence and stability for $|\alpha|_m \ell \lesssim 1$, non-existence for $|\alpha|_m \ell \gtrsim 1$, and there seems to be not much room for an instability to exist. But they allow (but not imply!) the possibility of a tube existing and being unstable if the current is concentrated in a region of transverse scale $\ell_\alpha << \ell_B$.

FORBES: (i) Is the result you have for the maximum energy storage possible for a force-free field being twice the potential energy completely general?
(ii) Does it still hold if there are x-lines or x-points in the configuration?

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ALY: (i) The maximum energy you can store in a force-free field is equal to the energy of the open field having the same B_Z on $\{z = 0\}$. That is the exact result. The fact that this energy is about twice the energy of the potential field is just an estimate, which may be checked to be correct in most cases of interest.
(ii) To get my result, you have just to assume that the field is force-free, no other assumptions being required. The field may have neutral points or current sheets. The upper bound on the energy still holds true.